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ON SPHERICAL INDICATRICES AND THEIR SPHERICAL IMAGE OF NULL CURVES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we investigate the spherical images of null curves and null helixes in Minkowski 3-space. We provide the spherical indicatrices of null curves in Minkowski 3-space with their causal characteristics. We also show the conditions of spherical indicatrices of null curves to be a curve lying on pseudo-sphere in Minkowski 3-space. In addition, we give the properties of spherical indicatrices of null curves satisfying generalized helices and lying on pseudo-sphere in Minkowski 3-space.

1. Introduction

Since the second mid of 20th-century mathematicians and physicist have actively studied about differential geometry of Riemannian manifold and its applications. It is because theories in differential geometry connect mathematics with real problems, especially physics. Many topics in classical differential geometry of Riemannian space are then extended into those of Lorentz-Minkowski space since its important use in physics especially related to general relativity theory. Some literatures providing an explanation about differential geometry in Riemannian space can be seen in [2, 12, 13] while the theory of differential geometry in the semi-Riemannian manifold can be seen in [5].

One theory of differential geometry in Riemannian space that can be extended to Lorentz-Minkowski space is the spherical indicatrix of curves. The idea has been existed for a long time ago to the tie of Gauss. The idea is essentially simple. if there is some group of the set of lines in space in some organized relationship with another, one might construct and examine the relevant spherical indicatrix [14]. The theory of spherical indicatrix of curves in Riemannian space can be found in [1, 15, 17]while in the case of Lorentz-Minkowski space can be seen in [16].

In Lorentz-Minkowski space, a curve can locally be timelike, spacelike or null depending on the casual character of the tangent vector along the curves. Some studies about the theory of curves in Minkowski space and its applications have been studied by [3, 4, 6]. In Lorentzian geometry, the properties of spacelike curves and

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timelike curves can be studied by approaches similar to those in Riemann geometry. However, it does not work for null curves or it can be said that the theory of null curves has many results which have no Riemannian analogues. It is because, in the case of the null curves, the arc length vanishes so that it is impossible to normalize the tangent vector in the usual way as in spacelike and timelike curve cases.

In the mathematical study of relativity theory, it is known that a lightlike particle is a future-pointing null geodesic in spacetime which is a connected and time-oriented 3-dimensional Lorentzian manifold [5]. The study of null curves also plays an important role in the physical theories that the classical relativistic string is a surface or world-sheet in Minkowski space which satisfies the Lorentzian analogue of the minimal surface equations [10]. In another finding, Nersian and Ramos [11] show that there exists a geometrical particle model based entirely on the geometry of null curves in Minkowski 4-dimensional spacetime.

Since its important roles both in mathematics and physics, many mathematicians and physicist are interested in studying the theory of null curves. For instance, Duggal and Jin [8] write a comprehensive book related to the theory of null from its introduction, properties until its applications. Inoguchi and Lee also explained the theory of null curves comprehensively in another article [3].

In this paper, we study the spherical indicatrices of null curves parametrized by distinguished parameter in Minkowski 3-space. In this work, we assume that the null curve is a space curve such that its curvature and torsion are not vanish. After the preliminary section, we give the Frenet frames of the spherical indicatrices of a null curve in term of the Frenet frame of the null curve. We also provide the curvatures and torsions of the spherical indicatrices. We also then show the conditions of spherical indicatrices of null curves to be a curve lying on pseudo-sphere in Minkowski 3-space. In addition, we give the properties of spherical indicatrices of null curves satisfying generalized helices and lying on pseudo-sphere in Minkowski 3-space.

2. Preliminaries

Minkowski space $\mathbf{E}_1^3$ is the real vector space $\mathbf{R}^3$ equipped with the standard indefinite Lorentzian metric g defined by

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3 \tag{2.1}$$

for any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbf{E}_1^3$. The cross product in Minkowski 3-space is defined by

$$x \times y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_3). \tag{2.2}$$

In Minkowski 3-space, v is timelike if $g(v, v) < 0$, spacelike if $g(v, v) > 0$ or $v = 0$, or null (lightlike) if $g(v, v) = 0$ and $v \neq 0$. The norm of a vector in $\mathbf{E}_1^3$ is defined by $||v|| = \sqrt{|g(v, v)|}$.

Let $\alpha : I \to \mathbf{E}_1^3$ be a curve in Minkowski 3-space. Locally, α can be timelike, spacelike or null if its tangent vectors along the curve are timelike, spacelike or null, respectively. For non null curves, the arc length s is defined by $s = \int_0^t \sqrt{|g(\alpha', \alpha')|}dt$. If $g(\alpha', \alpha) = \pm 1$, the non null curves are called curves parametrized by arc length. For a null curve, since $g(\alpha', \alpha') = 0$ then the pseudo-arc length s is defined by $s = \int_0^t g(\alpha'', \alpha'')^{\frac{1}{4}}dt$ and if $g(\alpha'', \alpha'') = 1$ the the null curve is parametrized by pseudo-arc length.

Let $\{T, N, B\}$ is the Frenet frame of α in $\mathbf{E}_1^3$. T, N and B are called tangent vector, principal normal vector and binormal vector of α, respectively.

If α is a non null curve with non null normals parametrized by arch length, then the Frenet equations of α are given by [21]

$$T' = \kappa N, \quad N' = -\varepsilon_0\varepsilon_1\kappa T + \tau B, \quad B' = -\varepsilon_1\varepsilon_2\tau N \tag{2.3}$$

where

$$\begin{aligned} &g(T,T) = \varepsilon_0 = \pm 1, \quad g(N,N) = \varepsilon_1 = \pm 1, \quad g(B,B) = \varepsilon_2 = \pm 1, \\ &g(T,N) = g(T,B) = g(N,B) = 0. \end{aligned}$$

The vector products of Frenet vectors of α in Minkowski 3-space are given by

$$T \times N = B, \quad N \times B = -\epsilon_1 T, \quad B \times T = -\varepsilon_0 N. \tag{2.4}$$

If α is a pseudo null curve, that is α is a spacelike curve with a null principal normal N, then the Frenet equations of α are given by [20]

$$T' = \kappa N, \quad N' = \tau N, \quad B' = -\kappa T - \tau B \tag{2.5}$$

where

$$g(T,T) = g(N,B) = 1, \quad g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0$$

and

$$T \times N = N, \quad N \times B = T, \quad T \times B = -B. \tag{2.6}$$

Here, κ can only two values:, $\kappa = 0$ if α is a straight line and $\kappa = 0$, otherwise.

If α is a null curve parametrized by distingushed parameter, then the Frenet equations of α are given by [18]

$$T' = \kappa N, \quad N' = \tau T - \kappa B, \quad B' = -\tau N \tag{2.7}$$

where

$$g(T,T) = g(B,B) = 0, \quad g(T,B) = g(N,B) = 0, \quad g(N,N) = g(T,B) = 1$$

and

$$T \times B = N, \quad T \times N = -T, \quad N \times B = -B. \tag{2.8}$$

Here, κ and τ are called the curvature and the torsion if α is a timelike curve or a spacelike curve with non null Frener frame. In case α is a pseudo null curve or a null curve then τ is called pseudo torsion.

Let $C : I \to \mathbf{E}_1^3$ be a null curve paremetrized by pseudo arc length s. A curve $\alpha : I \to \mathbf{E}_1^3$ generated by the unit tangent vector along a curve $C(s)$, i.e., $\alpha(s) = T(s)$ on the sphere of radius 1 about the origin is called tangent indicatrix of $C(s)$. Similarly, $\alpha(s) = N(s)$ and $\alpha(s) = B(s)$ are called the principal indicatrix and binormal indicatrix of $C(s)$

3. Spherical Indicatrices of Null Curves

In this section, we provide the causal characteristics of spherical indicatrices of null curves in Minkowski 3-space. In this section we assume that the null curve is not a straight line so that the null curve has non null curvature anywhere.

3.1. **Tangent indicatrix.**

Theorem 3.1. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of a null curve parametrized by distinguished parameter s. Then α is a spacelike curve.*

Proof. From equation (2.7), we have $\alpha'(s) = \kappa(s)N(s)$. Therefore, $g(\alpha', \alpha') = \kappa^2(s) > 0$. It implies that α is a spacelike curve. □

Theorem 3.2. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of null curves parametrized by distinguished parameter s. If the null curve is not a plane curve, then $\alpha(s)$ is a spacelike curve with non null Frenet frame satisfying*

$$\begin{bmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{-\kappa}{\sqrt{|2\kappa\tau|}} \\ \frac{\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{\kappa}{\sqrt{|2\kappa\tau|}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{3.1}$$

Proof. Let $\bar{s}$ be arc length of the $\alpha(s)$. Then, since α is spacelike curve with non null Frenet frame, then by taking derivative of α with respect to the pseudo arc length s using equations (2.5) and (2.7), we have

$$\frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = \kappa N \Rightarrow \bar{T} \cdot \frac{d\bar{s}}{ds} = \kappa N. \tag{3.2}$$

Taking the norm of equation (3.2), we have $\frac{d\bar{s}}{ds} = \pm\kappa$. Take $\frac{d\bar{s}}{ds} = \kappa$ so that

$$\bar{T} = N. \tag{3.3}$$

Differentiating equation (3.3), yields

$$\frac{d\bar{T}}{d\bar{s}}\frac{d\bar{s}}{ds} = \tau T - \kappa B \Rightarrow \bar{\kappa}\bar{N}\kappa = \tau T - \kappa B. \tag{3.4}$$

Since the null curve is not a straight line and not a plane curve then $\kappa \neq 0$ and $\tau \neq 0$, by taking the norm of equation (3.4), we have

$$\bar{\kappa}\kappa = \sqrt{|-2\kappa\tau|} = \sqrt{|2\kappa\tau|}. \tag{3.5}$$

Therefore, from equation (3.4), we find

$$\bar{N} = \frac{\tau T - \kappa B}{\sqrt{|2\kappa\tau|}}. \tag{3.6}$$

Consequently, $\bar{N}$ is timelike or spacelike if $\kappa\tau > 0$ or $\kappa\tau < 0$, respectively. Therefore, from equations (2.6) and (2.8), we have

$$\begin{aligned} \bar{B} &= \cdot\, \bar{T} \times \bar{N} \\ &= \cdot\, N \times \left(\frac{\tau T - \kappa B}{\sqrt{|2\kappa\tau|}}\right) \\ &= \frac{\tau T + \kappa B}{\sqrt{|2\kappa\tau|}}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3.3. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of a null curve parametrized by distinguished parameter s. If $\alpha(s) = T(s)$ has non null Frenet frame then the curvature and torsion of $\alpha(s)$ are respectively given by*

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\kappa} \quad \textit{and} \quad \bar{\tau} = -\frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}. \tag{3.7}$$

Proof. It is clear from equation (3.5) that

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\kappa}.$$

Taking derivative of $\bar{B}$ in equation (3.1) with respect to the pseudo arc length s yields

$$\begin{aligned}
\frac{d\bar{B}}{d\bar{s}}\frac{d\bar{s}}{ds} =& \frac{\tau' T + \kappa\tau N + \kappa' B - \kappa\tau N}{|-2\kappa\tau|^{\frac{1}{2}}} - \frac{(-2\kappa'\tau - 2\kappa\tau')(\tau T + \kappa B)}{2|-2\kappa\tau|^{\frac{3}{2}}}\\
\frac{d\bar{B}}{d\bar{s}}\kappa =& \frac{(-2\kappa\tau)(\tau' T + \kappa' B) + (\kappa'\tau + \kappa\tau')(\tau T + \kappa B)}{|-2\kappa\tau|^{\frac{3}{2}}}\\
\frac{d\bar{B}}{d\bar{s}} =& \frac{\tau(\kappa'\tau - \kappa\tau')T - \kappa(\kappa'\tau - \kappa\tau')B}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}\\
=& \frac{(\kappa'\tau - \kappa\tau')(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}.
\end{aligned}$$

By applying equations (2.5) and (2.7), we get

$$\begin{aligned}
\bar{\tau} =& \cdot g\left(\frac{d\bar{B}}{d\bar{s}}, \bar{N}\right)\\
=& \cdot g\left(\frac{(\kappa'\tau - \kappa\tau')(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}, \frac{\tau T - \kappa B}{|-2\kappa\tau|^{\frac{1}{2}}}\right)\\
=& \frac{(\kappa'\tau - \kappa\tau')(-2\kappa\tau)}{\kappa|-2\kappa\tau|^{2}}\\
=& -\frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}.
\end{aligned}$$

□

3.2. **Principal Normal Indicatrix.**

Theorem 3.4. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of a null curve parametrized by pseudo arc length s. Then if $\alpha(s)$ is not a plane curve then it is a spacelike or a timelike curve and if $\alpha(s)$ is a plane curve then it is a null curve.*

Proof. From equation (2.7), we have $\alpha'(s) = -\tau(s)T(s) + \kappa(s)B(s)$. Therefore, $g(\alpha'(s), \alpha'(s)) = 2\kappa(s)\tau(s)$. As a consequence, if $\alpha(s)$ is not a plane curve, then it is a spacelike or a timelike curve if κ and τ have different sign or same sign, respectively. If $\alpha(s)$ is a plane curve then $\tau(s) = 0$ which implies $\alpha(s)$ is a null curve. □

Theorem 3.5. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of a non plane null curve parametrized by pseudo arc length s. Then the Frenet frame of $\alpha(s)$ is given by*

$$\begin{bmatrix}\bar{T}\\ \bar{N}\\ \bar{B}\end{bmatrix} = \begin{bmatrix} \frac{\tau}{\sqrt{|\lambda|}} & 0 & \frac{-\kappa}{\sqrt{|\lambda|}}\\ \frac{\tau\mu}{\sqrt{\mu^2\lambda+\lambda^4}} & \frac{-\lambda^2}{\sqrt{\mu^2\lambda+\lambda^4}} & \frac{\kappa\mu}{\sqrt{\mu^2\lambda+\lambda^4}}\\ \frac{\tau\lambda}{\sqrt{\mu^2+\lambda^2}} & \frac{\mu}{\sqrt{\mu^2+\lambda^2}} & \frac{\kappa\lambda}{\sqrt{\mu^2+\lambda^2}} \end{bmatrix} \begin{bmatrix}T\\ N\\ B\end{bmatrix} \tag{3.8}$$

where $\lambda = 2\kappa\tau$ and $\mu = \kappa'\tau - \kappa\tau'$.

Proof. Let $\bar{s}$ be the arc length of the curve $\alpha(s)$. Taking derivative of α with respect to the pseudo arc length s, we have

$$\frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = \tau T - \kappa B \Rightarrow \bar{T}\frac{d\bar{s}}{ds} = \tau T - \kappa B. \tag{3.9}$$

Taking the inner product of equation (3.9), we get

$$\frac{d\bar{s}}{ds} = \sqrt{|-2\kappa\tau|}. \tag{3.10}$$

Therefore,

$$\bar{T} = \frac{\tau T - \kappa B}{\sqrt{|-2\kappa\tau|}}. \tag{3.11}$$

Differentiating equation (3.11) and using equation (3.10), we have

$$\begin{aligned}
\frac{d\bar{T}}{d\bar{s}}\frac{d\bar{s}}{ds} =& \frac{\tau' T + \kappa\tau N - \kappa' B + \kappa\tau N}{|-2\kappa\tau|^{\frac{1}{2}}} - \frac{(-2\kappa'\tau - 2\kappa\tau')(\tau T - \kappa B)}{2|-2\kappa\tau|^{\frac{3}{2}}}\\
\bar{\kappa}\bar{N} =& \frac{(-2\kappa\tau)(\tau' T + 2\kappa\tau N - \kappa' B) + (\kappa'\tau + \kappa\tau')(\tau T - \kappa B)}{|-2\kappa\tau|^2}\\
=& \frac{(-2\kappa\tau\tau' + \kappa'\tau^2 + \kappa\tau\tau')T - 4\kappa^2\tau^2 N + (2\kappa\kappa'\tau - \kappa'\kappa\tau - \kappa^2\tau')B}{|-2\kappa\tau|^2}\\
=& \frac{\tau(\kappa'\tau - \kappa\tau')T - 4\kappa^2\tau^2 N + \kappa(\kappa'\tau - \kappa\tau')B}{|-2\kappa\tau|^2}\\
=& \frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - 4\kappa^2\tau^2 N}{4\kappa^2\tau^2}.
\end{aligned}$$

Taking the norm of the equation above yields

$$\bar{\kappa} = \frac{|2\kappa\tau(\kappa'\tau - \kappa\tau')^2 + 16\kappa^4\tau^4|^{\frac{1}{2}}}{4\kappa^2\tau^2}. \tag{3.12}$$

Therefore,

$$\bar{N} = \frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - 4\kappa^2\tau^2 N}{|2\kappa\tau(\kappa'\tau - \kappa\tau')^2 + 16\kappa^4\tau^4|^{\frac{1}{2}}}. \tag{3.13}$$

As a consequence,

$$\begin{aligned}
\bar{B} =& \bar{T} \times \bar{N}\\
=& \left(\frac{\tau T - \kappa B}{\sqrt{|-2\kappa\tau|}}\right) \times \left(\frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - 4\kappa^2\tau^2 N}{4\kappa^2\tau^2}\right)\\
=& \frac{\kappa\tau(\kappa'\tau - \kappa\tau')(T \times B) - 4\kappa^2\tau^3(T \times N) - \kappa\tau(\kappa'\tau - \kappa\tau')(B \times T) + 4\kappa^3\tau^2(B \times N)}{2\kappa\tau|(\kappa'\tau - \kappa\tau')^2 + 8\kappa^3\tau^3)|^{\frac{1}{2}}}\\
=& \frac{2\kappa\tau(\kappa'\tau - \kappa\tau')N + 4\kappa^2\tau^2(\tau T + \kappa B)}{2\kappa\tau|(\kappa'\tau - \kappa\tau')^2 + 8\kappa^3\tau^3)|^{\frac{1}{2}}}\\
=& \frac{(\kappa'\tau - \kappa\tau')N + 2\kappa\tau(\tau T + \kappa B)}{|(\kappa'\tau - \kappa\tau')^2 + 8\kappa^3\tau^3)|^{\frac{1}{2}}}.
\end{aligned}$$

Setting $\lambda = 2\kappa\tau$ and $\mu = \kappa'\tau - \kappa\tau'$ completes the proof. $\square$

Theorem 3.6. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of a null curve parametrized by distinguished parameter s. If the null curve is not a plane curve, then the curvature and torsion of α are given by*

$$\bar{\kappa} = \frac{|\lambda\mu^2 + \lambda^4|^{\frac{1}{2}}}{\lambda^2} \tag{3.14}$$

and

$$\begin{aligned}\bar{\tau} =& \frac{-(\mu^2+\lambda^3)(\lambda^2\kappa'\tau + \lambda^2\kappa\tau' + 2\kappa\tau\lambda\lambda') + (2\mu\mu' + 3\lambda^2\lambda')\kappa\tau\lambda^2}{(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}} \\ &+ \frac{(\mu^2\lambda^3)\mu^2\mu' - \mu^3(2\mu\mu' + 3\lambda^2\lambda')}{2(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}}.\end{aligned} \tag{3.15}$$

Proof. From equation (3.11), we have

$$\bar{\kappa} = \frac{|2\kappa\tau(\kappa\tau' - \kappa'\tau)^2 + 16\kappa^4\tau^4|^{\frac{1}{2}}}{4\kappa^2\tau^2} = \frac{|\lambda\mu^2 + \lambda^4|^{\frac{1}{2}}}{\lambda^2}.$$

Taking derivative of $\bar{B}$ in equation (3.8) with respect to the pseudo arc length s, we have

$$\begin{aligned}\frac{d\bar{B}}{d\bar{d}}\frac{d\bar{s}}{ds} =& \frac{2(\mu^2+\lambda^3)(\tau'\lambda + \tau\lambda' + \mu\tau) - (2\mu\mu' + 3\lambda^2\lambda')\tau\lambda}{2(\mu^2+\lambda^3)^{\frac{3}{2}}}T \\ &+ \frac{2(\mu^2+\lambda^3)\mu' - (2\mu\mu' + 3\lambda\lambda')\mu}{2(\mu^2+\lambda^3)^{\frac{3}{2}}}N \\ &\frac{2(\mu^2+\lambda^3)(\kappa'\lambda + \kappa\lambda' - \kappa\mu) - (2\mu\mu' + 3\lambda^2\lambda')\kappa\lambda}{2(\mu^2+\lambda^3)^{\frac{3}{2}}}B \\ \frac{d\bar{B}}{ds} =& \frac{2(\mu^2+\lambda^3)(\tau'\lambda + \tau\lambda' + \mu\tau) - (2\mu\mu' + 3\lambda^2\lambda')\tau\lambda}{2\lambda^{\frac{1}{2}}(\mu^2+\lambda^3)^{\frac{3}{2}}}T \\ &+ \frac{2(\mu^2+\lambda^3)\mu' - (2\mu\mu' + 3\lambda\lambda')\mu}{2\lambda^{\frac{1}{2}}(\mu^2+\lambda^3)^{\frac{3}{2}}}N \\ &\frac{2(\mu^2+\lambda^3)(\kappa'\lambda + \kappa\lambda' - \kappa\mu) - (2\mu\mu' + 3\lambda^2\lambda')\kappa\lambda}{2\lambda^{\frac{1}{2}}(\mu^2+\lambda^3)^{\frac{3}{2}}}B.\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\tau} =& -g\left(\frac{d\bar{B}}{d\bar{s}}, \bar{N}\right) \\ =& \frac{-(\mu^2+\lambda^3)(\lambda^2\kappa'\tau + \lambda^2\kappa\tau' + 2\kappa\tau\lambda\lambda') + (2\mu\mu' + 3\lambda^2\lambda')\kappa\tau\lambda^2}{(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}} \\ &+ \frac{(\mu^2\lambda^3)\mu^2\mu' - \mu^3(2\mu\mu' + 3\lambda^2\lambda')}{2(\lambda\mu^2+\lambda^4)^{\frac{3}{2}}(\mu+\lambda)^{\frac{1}{2}}}.\end{aligned}$$

□

3.3. **Binormal Indicatrix.**

Theorem 3.7. *Let $\alpha(s) = B(s)$ be a binormal indicatrix of a null curve parametrized by distinguished parameter s. Then α is a spacelike or a null curve.*

Proof. From equation (2.7), we have $\alpha'(s) = -\tau(s)N(s)$. Therefore, $g(\alpha', \alpha') = \tau^2(s) > 0$. It implies that α is a spacelike curve if the null curve is not a plane curve and α is a null curve is the null curve is a plane curve. □

Theorem 3.8. *Let $\alpha(s) = B(s)$ be a binormal indicatrix of a non plane null curves parametrized by distinguished parameter s. If α is a spacelike curve with non null Frenet frames, then the Frenet frame of $\alpha(s)$ is given by*

$$\begin{bmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{-\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{\kappa}{\sqrt{|2\kappa\tau|}} \\ \frac{-\tau}{\sqrt{|2\kappa\tau|}} & 0 & \frac{-\kappa}{\sqrt{|2\kappa\tau|}} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.16}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ when α is a spacelike curve or timelike principal normal, respectively.

Proof. Let $\bar{s}$ be arc length of the $\alpha(s)$. Then, since α is a spacelike curve with non null Frenet frame, then by taking derivative of α with respect to the pseudo arc length s using equations (2.5) and (2.7), we have

$$\frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = -\tau N \Rightarrow \bar{T} \cdot \frac{d\bar{s}}{ds} = -\tau N. \tag{3.17}$$

Taking the norm of equation (3.17), we have $\frac{d\bar{s}}{ds} = \pm\tau$. Take $\frac{d\bar{s}}{ds} = \tau$ so that

$$\bar{T} = -N. \tag{3.18}$$

Differentiating equation (3.18), yields

$$\frac{d\bar{T}}{d\bar{s}}\frac{d\bar{s}}{ds} = -\tau T + \kappa B \Rightarrow \bar{\kappa}\bar{N}\tau = -\tau T + \kappa B. \tag{3.19}$$

Since α is not a straight line and not a plane curve then $\kappa \neq 0$ and $\tau \neq 0$, by taking the norm of equation (3.19), we have

$$\bar{\kappa}\tau = \sqrt{|-2\kappa\tau|} = \sqrt{|2\kappa\tau|}. \tag{3.20}$$

Therefore, from equation (3.19), we find

$$\bar{N} = \frac{-\tau T + \kappa B}{\sqrt{|2\kappa\tau|}}. \tag{3.21}$$

Consequently, $\bar{N}$ is timelike or spacelike if $\kappa\tau > 0$ or $\kappa\tau < 0$, respectively. Therefore, from equations (2.6) and (2.8), we have

$$\begin{aligned} \bar{B} &= \cdot\, \bar{T} \times \bar{N} \\ &= \cdot\, N \times \left(\frac{-\tau T + \kappa B}{\sqrt{|2\kappa\tau|}} \right) \\ &= \frac{-\tau T - \kappa B}{\sqrt{|2\kappa\tau|}}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3.9. *Let $\alpha(s) = B(s)$ be a tangent indicatrix of a null curve parametrized by distinguished parameter s. If $\alpha(s) = B(s)$ is not a plane curve then the curvature and torsion of $\alpha(s)$ are respectively given by*

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\tau} \quad \textit{and} \quad \bar{\tau} = \frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}. \tag{3.22}$$

Proof. It is clear from equation (3.20) that

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\tau}.$$

Taking derivative of $\bar{B}$ in equation (3.1) with respect to the pseudo arc length s yields

$$\begin{aligned}
\frac{d\bar{B}}{d\bar{s}}\frac{d\bar{s}}{ds} =& \frac{-\tau' T - \kappa\tau N - \kappa' B + \kappa\tau N}{|-2\kappa\tau|^{\frac{1}{2}}} - \frac{(-2\kappa'\tau - 2\kappa\tau')(-\tau T - \kappa B)}{2|-2\kappa\tau|^{\frac{3}{2}}} \\
\frac{d\bar{B}}{d\bar{s}}\kappa =& \frac{(-2\kappa\tau)(-\tau' T - \kappa' B) + (\kappa'\tau + \kappa\tau')(-\tau T - \kappa B)}{|-2\kappa\tau|^{\frac{3}{2}}} \\
\frac{d\bar{B}}{d\bar{s}} =& \frac{\tau(\kappa\tau' - \kappa'\tau)T - \kappa(\kappa\tau' - \kappa'\tau)B}{\kappa|-2\kappa\tau|^{\frac{3}{2}}} \\
=& \frac{(\kappa\tau' - \kappa'\tau)(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}.
\end{aligned}$$

By applying equations (2.5) and (2.7), we get

$$\begin{aligned}
\bar{\tau} =& \cdot\, g\left(\frac{d\bar{B}}{d\bar{s}}, \bar{N}\right) \\
=& \cdot\, g\left(\frac{(\kappa\tau' - \kappa'\tau)(\tau T - \kappa B)}{\kappa|-2\kappa\tau|^{\frac{3}{2}}}, \frac{\tau T - \kappa B}{|-2\kappa\tau|^{\frac{1}{2}}}\right) \\
=& \frac{(\kappa\tau' - \kappa'\tau)(-2\kappa\tau)}{\kappa|-2\kappa\tau|^{2}} \\
=& \frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau}.
\end{aligned}$$

□

4. Spherical Image of Spherical Indicatrices

In this section, we provide the properties of spherical indicatrices of null curves on pseudo sphere in Minkowski 3-space. In this section we assume the null curve is neither a plane curve nor a straight line.

Definition 4.1. [5] Pseudo sphere in semi-Riemannian space of center c and radius r is defined by

$$\mathbf{S}_1^2 = \{\alpha \in \mathbf{E}_1^3 : g(\alpha - c, \alpha - c) = r^2\}. \tag{4.1}$$

Theorem 4.2. *Let $\alpha(s) = T(s)$ be a unit speed tangent indicatrix of a null curve. If α lies on the pseudo sphere of center c and radius r, then*

$$\alpha - c = \rho\bar{N} + \sigma\bar{B}, \tag{4.2}$$

where $\rho = -\frac{1}{\bar{\kappa}}$ and $\sigma = \frac{\bar{\kappa}'\bar{\tau}}{\bar{\kappa}^2}$.

Proof. Let $\alpha(s) = T(s)$ is a unit speed curve lying on a pseudo sphere $\mathbf{S}_1^2$ of center c and radius r. Therefore, it satisfies $g(\alpha - c, \alpha - c) = r^2$. Differentiating this equation yields

$$g(\bar{T}, \alpha - c) = 0. \tag{4.3}$$

Taking the derivative of equation (4.3)

$$g(\bar{T},\bar{T})+\bar{\kappa}g(\alpha-c,\bar{N})=0\Rightarrow g(\alpha-c,\bar{N})=-\frac{1}{\bar{\kappa}}. \tag{4.4}$$

Differentiating equation (4.4) and using the fact that $\alpha-c$ is perpendicular to $\bar{T}$, we have

$$\begin{aligned} g(\bar{T},\bar{N})+g(\alpha-c,-\varepsilon_0\varepsilon_1\bar{\kappa}\bar{T}+\bar{\tau}\bar{B}) &= \frac{\bar{\kappa}'}{\bar{\kappa}^2}\\ \bar{\tau}g(\alpha-c,\bar{B}) &= \frac{\bar{\kappa}'}{\bar{\kappa}^2}\\ g(\alpha-c,\bar{B}) &= \frac{\bar{\kappa}'\bar{\tau}}{\bar{\kappa}^2}. \qquad (4.5)\end{aligned}$$

On the other hand, since $\alpha-c$ is perpendicular to $\bar{T}$, then we can express

$$\alpha-c=\rho\bar{N}+\sigma\bar{B}$$

where $\rho=g(\alpha-c,\bar{N})$ and $\sigma=g(\alpha-c,B)$. Consequently, by equation (4.4) and (4.5), we find equation (4.2). Hence the proof is completed □

Corollary 4.1. Let $\alpha(s)=T(s)$ be unit tangent indicatrix of a space null curve. If α lies on the pseudo sphere, then the center c and the radius r f the curve α are respectively given by

$$c=\alpha+\frac{1}{\bar{\kappa}}\bar{N}-\frac{\bar{\kappa}'\bar{\tau}}{\bar{\kappa}^2}\bar{B} \quad \textit{and} \quad r=\frac{1}{\bar{\kappa}^2}\sqrt{|\bar{\kappa}^2+\epsilon_0(\bar{\kappa}'\bar{\tau})^2|}. \tag{4.6}$$

Theorem 4.3. *Let $\alpha(s)=T(s)$ be unit speed tangent indicatrix of a space null curve. If α lies on pseudo sphere of center c, radius r and positive curvature, then α has curvature $\bar{\kappa}\geq\frac{1}{r}$.*

Proof. Let α lies on the pseudo sphere of center c and radius r. Then, we have $g(\alpha-c,\alpha-c)=r^2$. From equation (4.4) we have

$$\bar{\kappa}=\frac{1}{g(\alpha-c,\bar{N}}.$$

By Schwarz inequality, $||g(\alpha-c,\bar{N}||\leq||\alpha-c||||\bar{N}||=a$, we have $\bar{\kappa}\geq\frac{1}{r}$ and the proof is completed. □

Remark 4.4. The theorem 4.2 and 4.3 is similar in case α is the principal normal indicatrix or binormal indicatrix of a space null curve.

Definition 4.5. [7] A null curve $\alpha: I\to\mathbf{E}_1^3$ is called generalized helix is there exist a non-zero vector v in $\mathbf{E}_1^3$ such that $g(\alpha',v)=constant$.

Theorem 4.6. [7] *A non-geodesic null Frenet curve is a null generalized helix if and only if its slope $\frac{\tau}{\kappa}$ is constant.*

Theorem 4.7. *Let $C: I\to\mathbf{E}_1^3$ be a null generalized helix in $\mathbf{E}_1^3$. Then the tangent indicatrix $\alpha(s)=T(s)$ of the generalized null helix C lies on the osculating plane of radius $r=\frac{1}{2}\sqrt{\frac{2\kappa}{\tau}}$ and center $c=\bar{T}+r\bar{N}$.*

Proof. Let $C : I \to \mathbf{E}_1^3$ be a null generalized helix, then $\frac{\tau}{\kappa} = constant$. From equation (3.7), we have

$$\bar{\kappa} = \frac{\sqrt{|2\kappa\tau|}}{\kappa} = constant$$

and

$$\bar{\tau} = -\frac{\kappa'\tau - \kappa\tau'}{2\kappa^2\tau} = -\left[\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)'\right]\frac{1}{2\tau} = 0.$$

Therefore, $\alpha(s) = T(s)$ is a circle in $\mathbf{E}_1^3$ which lies on the plane spanned by $\{\bar{T}, \bar{N}\}$ or osculating plane. From equation (4.6), the radius and the center of α are given by

$$r = \frac{1}{\bar{\kappa}} = \frac{1}{2}\sqrt{\frac{2\kappa}{\tau}}$$

and

$$c = \alpha + \frac{1}{\bar{\kappa}}\bar{N} = \alpha + r\bar{N}.$$

□

Remark 4.8. The theorem 4.7 is similar in case α is the principal normal indicatrix or binormal of indicatrix lying on pseudo sphere.

Example 4.9. Define a null curve $C : I \to \mathbf{E}_1^3$ parametrized by distinguished parameter s defined by

$$\alpha(s) = (s, \cos s, \sin s).$$

With simple calculation, we have

$$T = (1, -\sin s, \cos s), \quad N = (0, -\cos s, -\sin s), \quad B = \left(-\frac{1}{2}, -\frac{\sin s}{2}, \frac{\cos s}{2}\right)$$

and

$$\kappa = 1 \quad and \quad \tau = -\frac{1}{2}.$$

Since $\frac{\tau}{\kappa} = -\frac{1}{2} = constant$, the curve C is a helix in $\mathbf{E}_1^3$.
1. Tangent indicatrix
The curve $\alpha(s) = T(s) = (1, -\sin s, \cos s)$ is the tangent indicatrix of C. By using equations (3.1) and (3.7) we get

$$\bar{T} = (0, -\cos s, -\sin s), \quad \bar{N} = (0, \sin s, -\cos s), \quad \bar{B} = (-1, 0, 0)$$

and

$$\bar{\kappa} = 1, \quad \bar{\tau} = 0.$$

It can be seen that α is a spacelike helix with spacelike principal normal vector. Furthermore, by theorem 4.7, α lies on $\mathbf{S}_1^2$ with radius 1 and centered in (1, 0, 0).
2. Principal normal indicatrix
The curve $\alpha(s) = N(s) = (0, -\cos s, -\sin s)$ is the principal normal of C. By using equations (3.8), (3.14) and (3.15) we get

$$\bar{T} = (0, \sin s, -\cos s), \quad \bar{N} = (0, \cos s, -\sin s), \quad \bar{B} = (-1, 0, 0)$$

and

$$\bar{\kappa} = 1, \quad \bar{\tau} = 0.$$

It can be seen that α is a spacelike helix with spacelike principal normal vector. Furthermore, by theorem 4.7, α lies on $\mathbf{S}_1^2$ of radius 1 and centered in $(1, 0, 0)$.
3. Binormal indicatrix
The curve $\alpha(s) = B(s) = \left(-\frac{1}{2}, -\frac{\sin s}{2}, \frac{\cos s}{2}\right)$ is the binormal indicatrix of C. By using equations (3.16) and (3.22) we get

$$\bar{T} = (0, \cos s, \sin s), \quad \bar{N} = (0, -\sin s, \cos s), \quad \bar{B} = (1, 0, 0)$$

and

$$\bar{\kappa} = 1, \quad \bar{\tau} = 0.$$

It can be seen that α is a spacelike helix with spacelike principal normal vector. Furthermore, by theorem 4.7, α lies on $\mathbf{S}_1^2$ with radius 1 and centered in (1, 0, 0).

5. Conclusion

Spherical indicatrices of a space null curve in Minkowski 3-space are spacelike curves with non null Frenet frame. Sphaerical indicatrices lying on the pseudo sphere of a space null curve with positive curvature has a curvature $\bar{\kappa} \geq \frac{1}{r}$ where r is the radius of the pseudo sphere.

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