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A Study on f-Rectifying Curves in Euclidean n-Space

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Abstract

A rectifying curve in Euclidean n-space $\mathbb{E}^n$ is defined as an arc-length parametrized curve γ in $\mathbb{E}^n$ such that its position vector always lies in its rectifying space (i.e., the orthogonal complement of its principal normal vector field) in $\mathbb{E}^n$. In this paper, in analogy to this, we introduce the notion of an f-rectifying curve in $\mathbb{E}^n$ as a curve γ in $\mathbb{E}^n$ parametrized by its arc-length s such that its f-position vector field γ_f, defined by $\gamma_f(s) = \int f(s)d\gamma$, always lies in its rectifying space in $\mathbb{E}^n$, where f is a nowhere vanishing real-valued integrable function in parameter s. The main purpose is to characterize and classify such curves in $\mathbb{E}^n$.

1. Introduction

Let $\mathbb{E}^3$ denote the Euclidean 3-space (i.e., the three-dimensional real vector space $\mathbb{R}^3$ endowed with the *standard inner product* $\langle \cdot, \cdot \rangle$). Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit-speed curve (i.e., a curve in $\mathbb{E}^3$ parametrized by *arc length function* s) of class at least $\mathscr{C}^3$ (i.e., possessing continuous derivatives at least up to third order). Needless to mention, I denotes a non-trivial interval in $\mathbb{R}$, i.e., a connected set in $\mathbb{R}$ containing at least two points. We consider the *Frenet apparatus* $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$ for the curve γ which is defined as follows: $T_\gamma = \gamma'$ is the unit *tangent vector field* along γ; N_γ is the unit *principal normal vector field* along γ obtained by normalizing the acceleration vector field T'_γ; $B_\gamma = T_\gamma \times N_\gamma$ is the unit *binormal vector field* along γ and it is the unique vector field along γ orthogonal to both T_γ and N_γ so that the *dynamic Frenet frame* $\{T_\gamma, N_\gamma, B_\gamma\}$ is positive definite along γ having the same orientation as that of $\mathbb{E}^3$; κ_γ is the *curvature* and τ_γ is the *torsion* of γ. If γ is of class at least $\mathscr{C}^5$, then its curvature κ_γ and torsion τ_γ are at least twice differentiable. Moreover, γ reduces to a *tortuous curve* in $\mathbb{E}^3$ if it has nowhere vanishing curvature κ_γ and torsion τ_γ (cf. [1] or [2]).

At each point $\gamma(s)$ on γ, the planes spanned by $\{T_\gamma(s), N_\gamma(s)\}$, $\{T_\gamma(s), B_\gamma(s)\}$ and $\{N_\gamma(s), B_\gamma(s)\}$ are respectively called the *osculating plane*, *rectifying plane* and *normal plane* of γ ([1,2]). It is well known from elementary *Differential Geometry* that a space curve γ lies in a *plane* in $\mathbb{E}^3$ if its position vector field always lies in its osculating planes, and it lies on a *sphere* in $\mathbb{E}^3$ if its position vector field always lies in its normal planes. In this point of view, it is natural to inquire the geometric question: *Does there exist a space curve $\gamma : I \longrightarrow \mathbb{E}^3$ whose position vector field always lies in its rectifying planes?* The existence of such space curves was introduced by B.Y. Chen in his paper [3] and named as *rectifying curves*. Thus, the position vector field of a rectifying curve $\gamma : I \longrightarrow \mathbb{E}^3$ parametrized by arc length function s satisfies the equation

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s)$$

for some smooth functions $\lambda, \mu : I \longrightarrow \mathbb{R}$. In [3], B.Y. Chen explored some characterizations of rectifying curves in $\mathbb{E}^3$ in terms of distance functions, tangential, normal and binormal components of their position vector field and also in terms of ratios of their curvature and torsion. Also, he attempted for a classification of such curves in $\mathbb{E}^3$ based on a sort of dilation applied on unit-speed curves on the unit sphere $\mathbb{S}^2(1)$.

In [4], B.Y. Chen and F. Dillen observed that rectifying curves can be viewed as *centrodes* and *extremal curves* in $\mathbb{E}^3$. Moreover, they found a relation between rectifying curves and centrodes which performs a significant role in defining the curves of constant procession in *Differential Geometry* as well as in *Kinematics* or, in general, *Mechanics*. Thereafter, several characterizations of rectifying curves in

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Euclidean spaces were evolved in [5–8]. Meanwhile, the notion of rectifying curves were extended to several ambient spaces, e.g., 3D sphere $\mathbb{S}^3(r)$ [9], 3D hyperbolic space $\mathbb{H}^3(-r)$ [10], Minkowski 3-space $\mathbb{E}^3_1$ [11,12], Minkowski space-time $\mathbb{E}^4_1$ [13–15]. Furthermore, a new kind of curves were studied in $\mathbb{E}^3$ which generalizes rectifying curves and helices [16]. Also, some characterizations and classification of non-null and null f-rectifying curves (which are a sort of generalization of rectifying curves) were investigated in *Minkowski* 3-*space* $\mathbb{E}^3_1$ [17,18], *Minkowski space-time* $\mathbb{E}^4_1$ [19] and Euclidean 4-space [20].

In section 2, we give requisite preliminaries and then, in section 3, we introduce the notion of f-rectifying curves in $\mathbb{E}^n$. Thereafter, section 4 is devoted to investigate some simple geometric characterizations of f-rectifying curves in $\mathbb{E}^n$. Afterwards, section 5 is dedicated to classify f-rectifying curves in terms of their f-position vectors in $\mathbb{E}^n$. Finally, we conclude our study in section 6. This is how the paper is organised.

2. Preliminaries

The *Euclidean n-space* $\mathbb{E}^n$ is the n-dimensional real vector space $\mathbb{R}^n$ equipped with the *standard inner product* $\langle\cdot,\cdot\rangle$ defined by

$$\langle x,y\rangle := \sum_{i=1}^{n} x_i y_i$$

for all tangent vectors $x=(x_1,x_2,\dots,x_n), y=(y_1,y_2,\dots,y_n)$ to $\mathbb{R}^n$. As usual, the *norm* or *length* of a tangent vector $x=(x_1,x_2,\dots,x_n)$ to $\mathbb{R}^n$ is denoted and defined by

$$\|x\| := \sqrt{\langle x,x\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}\,.$$

Let $\gamma: J \longrightarrow \mathbb{E}^n$ be an arbitrary differentiable curve parametrized by t and γ' denotes its velocity vector field in $\mathbb{E}^n$. Also, we assume that γ is regular, i.e., its velocity vector field γ' is nowhere vanishing. If we change the parameter t by arc-length function $s: J \longrightarrow I$ based at t_0 given by

$$s(t) = \int_{t_0}^{t} \left\|\gamma'(u)\right\| du$$

such that $\|\gamma'(s)\| = \sqrt{\langle\gamma'(s),\gamma'(s)\rangle} = 1$, i.e., $\langle\gamma'(s),\gamma'(s)\rangle = 1$, then $\gamma: I \longrightarrow \mathbb{E}^n$ is referred to as an *arc-length parametrized* or a *unit-speed* curve in $\mathbb{E}^n$. We may consider that γ is of class at least $\mathscr{C}^4$. Now, let T_γ, N_γ denote respectively the unit *tangent vector field* and the unit *principal normal vector field* of γ and for each $i \in \{1,2,\dots,n-2\}$, let B_{γ_i} denote the unit i-th *binormal vector field* of γ so that $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2},\dots,B_{\gamma_{n-2}}\}$ forms the positive definite *dynamic Frenet frame* along γ having the same orientation as that of $\mathbb{E}^n$. Also, for each $i \in \{1,2,\dots,n-1\}$, let κ_{γ_i} denote the i-th *curvature* of γ. Then the Frenet-Serret formulae for the curve γ are given by ([21,22])

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_{\gamma_1} \\ B'_{\gamma_2} \\ \vdots \\ B'_{\gamma_{n-2}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{\gamma_1} & 0 & 0 & \cdots & 0 & 0 \\ -\kappa_{\gamma_1} & 0 & \kappa_{\gamma_2} & 0 & \cdots & 0 & 0 \\ 0 & -\kappa_{\gamma_2} & 0 & \kappa_{\gamma_3} & \cdots & 0 & 0 \\ 0 & 0 & -\kappa_{\gamma_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \kappa_{\gamma_{n-1}} & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_{\gamma_1} \\ B_{\gamma_2} \\ \vdots \\ B_{\gamma_{n-2}} \end{pmatrix}. \tag{2.1}$$

From the above formulae, it follows that $\kappa_{\gamma_{n-1}} \not\equiv 0$ if and only if the curve γ lies wholly in $\mathbb{E}^n$. This is equivalent to saying that $\kappa_{\gamma_{n-1}} \equiv 0$ if and only if the curve γ lies wholly in a hypersurface in $\mathbb{E}^n$ (cf. [21,22]). We recall that the hypersurface in $\mathbb{E}^n$ defined by

$$\mathbb{S}^{n-1}(1) := \{x \in \mathbb{E}^n : \langle x,x\rangle = 1\}$$

is called the *unit sphere* with centre at the origin in $\mathbb{E}^n$. We also recall that the *rectifying space* of the curve γ in $\mathbb{E}^n$ is the orthogonal complement ${N_\gamma}^\perp$ of its principal normal vector field N_γ in $\mathbb{E}^n$ defined by

$${N_\gamma}^\perp := \left\{x \in \mathbb{E}^n : \langle x, N_\gamma\rangle = 0\right\}.$$

3. Notion of f-rectifying curves in $\mathbb{E}^n$

Let $\gamma: I \longrightarrow \mathbb{E}^n$ be a unit-speed curve (parametrized by arc length s) with Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2},\dots,B_{\gamma_{n-2}}, \kappa_{\gamma_1},\kappa_{\gamma_2},\dots,\kappa_{\gamma_{n-1}}\}$. As found in [8], γ is a rectifying curve in $\mathbb{E}^n$ if and only if its position vector field always lies in its rectifying space, i.e., if and only if its position vector field satisfies

$$\gamma(s) = \lambda(s) T_\gamma(s) + \sum_{i=1}^{n-2} \mu_i(s) B_{\gamma_i}(s)$$

for some differentiable functions $\lambda, \mu_1, \mu_2, \dots, \mu_{n-2} : I \longrightarrow \mathbb{R}$. Now, let $f: I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function. Then the *f-position vector field* of γ is denoted by γ_f and is defined by

$$\gamma_f(s) = \int f(s)\, d\gamma.$$

Here, the integral sign is used in this sense that on differentiation of previous equation, one finds

$$\gamma'_f(s) = f(s) T_\gamma(s)$$

so that γ_f is an *integral curve* of the vector field fT_γ along γ in $\mathbb{E}^n$. Using this concept of f-position vector field of a curve in $\mathbb{E}^n$, we define an f-rectifying curve in $\mathbb{E}^n$ as follows:

Definition 3.1. *Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve with Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}, \ldots, B_{\gamma_{n-2}}, \kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}\}$ and $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in arc-length parameter s of γ with at least $(n-2)$-times differentiable primitive function F. Then γ is referred to as an f-rectifying curve in $\mathbb{E}^n$ if its f-position vector field γ_f always lies in its rectifying space in $\mathbb{E}^n$, i.e., if its f-position vector field γ_f satisfies the equation*

$$\gamma_f(s) = \lambda(s)T_\gamma(s) + \sum_{i=1}^{n-2} \mu_i(s)B_{\gamma_i}(s) \tag{3.1}$$

for some differentiable functions $\lambda, \mu_1, \mu_2, \ldots, \mu_{n-2} : I \longrightarrow \mathbb{R}$.

Remark 3.2. *In particular, if the function f is a non-zero constant on I, then, up to isometries (rigid motions) of $\mathbb{E}^n$, an f-rectifying curve $\gamma : I \longrightarrow \mathbb{E}^n$ is congruent to a rectifying curve in $\mathbb{E}^n$ and the study coincides with the same incorporated in [8].*

4. Some geometric characterizations of f-rectifying curves in $\mathbb{E}^n$

In this section, we present some geometrical characterizations of unit-speed f-rectifying curves in $\mathbb{E}^n$ in terms of the norm functions, tangential components, normal components, binormal components of their f-position vector field.

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve (parametrized by arc length s) having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$ and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F. If γ is a f-rectifying curve in $\mathbb{E}^n$, then the following statements are true:*

1. *The norm function $\rho = \|\gamma_f\|$ is given by $\rho(s) = \sqrt{F^2(s) + c^2}$, where c is a non-zero constant.*
2. *The tangential component $\langle \gamma_f, T_\gamma \rangle$ of γ_f is given by $\langle \gamma_f(s), T_\gamma(s) \rangle = F(s)$.*
3. *The normal component $\gamma_f^{N_\gamma}$ of γ_f has a constant length and the norm function ρ is non-constant.*
4. *The first binormal component $\langle \gamma_f, B_{\gamma_1} \rangle$ and the second binormal component $\langle \gamma_f, B_{\gamma_2} \rangle$ of γ_f are respectively given by*

$$\langle \gamma_f(s), B_{\gamma_1}(s) \rangle = \frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s), \quad \langle \gamma_f(s), B_{\gamma_2}(s) \rangle = \frac{1}{\kappa_{\gamma_3}(s)} \frac{d}{ds}\left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s)\right)$$

and for each $i \in \{2, 3, \ldots, n-3\}$, the $(i+1)$-th binormal component $\left\langle \gamma_f, B_{\gamma_{i+1}} \right\rangle$ of γ_f is given by

$$\left\langle \gamma_f(s), B_{\gamma_{i+1}}(s) \right\rangle = \frac{1}{\kappa_{\gamma_{i+2}}(s)} \left[\kappa_{\gamma_{i+1}}(s) \left\langle \gamma_f(s), B_{\gamma_{i-1}}(s) \right\rangle + \langle \gamma_f(s), B_{\gamma_i}(s) \rangle\right].$$

Conversely, if $\gamma : I \longrightarrow \mathbb{E}^n$ is a unit-speed curve having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$, and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F such that any one of the statements (1), (2), (3) or (4) is true, then γ is an f-rectifying curve in $\mathbb{E}^n$.

Proof. First, for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ with at least $(n-2)$-times differentiable primitive function F, let $\gamma : I \longrightarrow \mathbb{E}^n$ be an f-rectifying curve in $\mathbb{E}^n$ having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$. Then for some differentiable functions $\lambda, \mu_1, \mu_2, \ldots, \mu_{n-2} : I \longrightarrow \mathbb{R}$, the f-position vector field γ_f of γ satisfies

$$\gamma_f(s) = \lambda(s)T_\gamma(s) + \sum_{i=1}^{n-2} \mu_i(s)B_{\gamma_i}(s). \tag{4.1}$$

Differentiating (4.1) and then applying the Frenet-Serret formulae (2.1), we obtain

$$\begin{aligned} f(s)T_\gamma(s) \quad = \quad & \lambda'(s)T_\gamma(s) + \left(\lambda(s)\kappa_{\gamma_1}(s) - \mu_1(s)\kappa_{\gamma_2}(s)\right)N_\gamma(s) + \left(\mu_1'(s) - \mu_2(s)\kappa_{\gamma_3}(s)\right)B_{\gamma_1}(s) \\ & + \sum_{i=2}^{n-3}\Big(\mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s) + \mu_i'(s) - \mu_{i+1}(s)\kappa_{\gamma_{i+2}}(s)\Big)B_{\gamma_i}(s) + \Big(\mu_{n-3}(s)\kappa_{\gamma_{n-1}}(s) + \mu_{n-2}'(s)\Big)B_{\gamma_{n-2}}(s) \end{aligned}$$

which gives the following set of relations

$$\begin{cases} \lambda'(s) = f(s), \\ \lambda(s)\kappa_{\gamma_1}(s) - \mu_1(s)\kappa_{\gamma_2}(s) = 0, \\ \mu_1'(s) - \mu_2(s)\kappa_{\gamma_3}(s) = 0, \\ \mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s) + \mu_i'(s) - \mu_{i+1}(s)\kappa_{\gamma_{i+2}}(s) = 0 \quad \text{for } i \in \{2, 3, \ldots, n-3\}, \\ \mu_{n-3}(s)\kappa_{\gamma_{n-1}}(s) + \mu_{n-2}'(s) = 0. \end{cases} \tag{4.2}$$

From the first $n-1$ relations of (4.2), we find

$$\begin{cases} \lambda(s) = F(s), \\ \mu_1(s) = \dfrac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s), \\ \mu_2(s) = \dfrac{1}{\kappa_{\gamma_3}(s)} \dfrac{d}{ds}\left(\dfrac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s)\right), \\ \mu_{i+1}(s) = \dfrac{1}{\kappa_{\gamma_{i+2}}(s)}\left[\mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s) + \mu_i'(s)\right] \quad \text{for } i \in \{2, 3, \ldots, n-3\}. \end{cases} \tag{4.3}$$

On the other hand, from the last $n-2$ relations of (4.2), we get

$$\mu_1(s)\left(\mu_1'(s)-\mu_2(s)\kappa_{\gamma_3}(s)\right)+\sum_{i=2}^{n-3}\mu_i(s)\left(\mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s)+\mu_i'(s)-\mu_{i+1}(s)\kappa_{\gamma_{i+2}}(s)\right)+\mu_{n-2}(s)\left(\mu_{n-2}'(s)+\mu_{n-3}(s)\kappa_{\gamma_{n-1}}(s)\right)=0$$

which reduces to

$$\sum_{i=1}^{n-2}\mu_i(s)\mu_i'(s)=0. \tag{4.4}$$

Integrating (4.4), we obtain

$$\sum_{i=1}^{n-2}\mu_i^2(s)=c^2, \tag{4.5}$$

where c is an arbitrary non-zero constant. Using (4.1), (4.3) and (4.5), the norm function $\rho=\|\gamma_f\|$ is given by

$$\rho^2(s)=\left\|\gamma_f(s)\right\|^2=\left\langle\gamma_f(s),\gamma_f(s)\right\rangle=F^2(s)+\sum_{i=1}^{n-2}\mu_i^2(s)=F^2(s)+c^2.$$

This proves the statement (1). Again, using (4.1) and (4.3), the tangential component $\left\langle\gamma_f,T_\gamma\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),T_\gamma(s)\right\rangle=\lambda(s)=F(s).$$

This proves the statement (2). Now, for each $s\in I$, $\gamma_f(s)$ can be decomposed as

$$\alpha_f(s)=\nu(s)T_\gamma(s)+\alpha_f^{N_\gamma}(s)$$

for some differentiable function $\nu: I\longrightarrow\mathbb{R}$, where $\gamma_f^{N_\gamma}$ denotes the normal component of γ_f. Thus, in view of (4.1), $\gamma_f^{N_\gamma}$ is given by

$$\gamma_f^{N_\gamma}(s)=\sum_{i=1}^{n-2}\mu_i(s)B_{\gamma_i}(s).$$

Therefore, we have

$$\left\|\gamma_f^{N_\gamma}(s)\right\|=\sqrt{\left\langle\gamma_f^{N_\gamma}(s),\gamma_f^{N_\gamma}(s)\right\rangle}=\sqrt{\sum_{i=1}^{n-2}\mu_i^2(s)}. \tag{4.6}$$

Now, by using (4.5) in (4.6), we find $\|\gamma_f^{N_\gamma}(s)\|=c$. This proves the statement (3). Finally, using (4.1) and (4.3), the first binormal component $\left\langle\gamma_f,B_{\gamma_1}\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),B_{\gamma_1}(s)\right\rangle=\mu_1(s)=\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)}F(s),$$

the second binormal component $\left\langle\gamma_f,B_{\gamma_2}\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),B_{\gamma_2}(s)\right\rangle=\mu_2(s)=\frac{1}{\kappa_{\gamma_3}(s)}\frac{d}{ds}\left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)}F(s)\right)$$

and for each $i\in\{2,3,\dots,n-3\}$, the $(i+1)$-th binormal component $\left\langle\gamma_f,B_{\gamma_{i+1}}\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),B_{\gamma_{i+1}}(s)\right\rangle=\mu_{i+1}(s)=\frac{1}{\kappa_{\gamma_{i+2}}(s)}\left[\kappa_{\gamma_{i+1}}(s)\left\langle\gamma_f(s),B_{\gamma_{i-1}}(s)\right\rangle+\left\langle\gamma_f(s),B_{\gamma_i}(s)\right\rangle\right].$$

Thus the statement (4) is proved.

Conversely, let $\gamma: I\longrightarrow\mathbb{E}^n$ be a unit-speed curve having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1},\kappa_{\gamma_2},\dots,\kappa_{\gamma_{n-1}}$, and $f: I\longrightarrow\mathbb{R}$ be a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F such that the statement (1) or the statement (2) is true. Then, in either case, we must have

$$\left\langle\gamma_f(s),T_\gamma(s)\right\rangle=F(s). \tag{4.7}$$

Differentiating (4.7) and then using the Frenet-Serret formulae (2.1), we finally obtain

$$\left\langle\gamma_f(s),N_\gamma(s)\right\rangle=0.$$

This implies that γ_f lies in the rectifying space of γ and hence γ is an f-rectifying curve in $\mathbb{E}^n$.

Next, we assume that the statement (3) is true. Then $\|\gamma_f^{N_\gamma}\|=$ a constant $=c$, say. Again, the normal component $\gamma_f^{N_\gamma}$ is given by

$$\gamma_f(s)=F(s)T_\gamma(s)+\gamma_f^{N_\gamma}(s)$$

and hence we have

$$\langle \gamma_f(s), \gamma_f(s) \rangle = \langle \gamma_f(s), T_\gamma(s) \rangle^2 + c^2. \tag{4.8}$$

Differentiating (4.8) and then applying the Frenet-Serret formulae (2.1), we obtain

$$\langle \gamma_f(s), N_\gamma(s) \rangle = 0.$$

This proves that γ_f lies in the rectifying space of γ and hence γ is an f-rectifying curve in $\mathbb{E}^n$.

Finally, we assume that the statement (4) is true. Then the first binormal component and the second binormal component of γ_f are respectively given by

$$\langle \gamma_f(s), B_{\gamma_1}(s) \rangle = \frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s), \tag{4.9}$$

$$\langle \gamma_f(s), B_{\gamma_2}(s) \rangle = \frac{1}{\kappa_{\gamma_3}(s)} \frac{d}{ds} \left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right). \tag{4.10}$$

Differentiating (4.9) and by using the Frenet-Serret formulae (2.1), we obtain

$$-\kappa_{\gamma_2}(s) \langle \gamma_f(s), N_\gamma(s) \rangle + \kappa_{\gamma_3}(s) \langle \gamma_f(s), B_{\gamma_2}(s) \rangle = \frac{d}{ds} \left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right). \tag{4.11}$$

Combining (4.10) and (4.11), we find

$$\langle \gamma_f(s), N_\gamma(s) \rangle = 0.$$

Consequently, γ_f lies in the rectifying space of γ and hence γ is an f-rectifying curve in $\mathbb{E}^n$. □

5. Classification of f-rectifying curves in $\mathbb{E}^n$

In many papers (e.g., [3], [7], [8], [11] etc.), several interesting results were found primarily attempting towards the classification of rectifying curves which are mostly based on their parametrizations. In this section, we attempt for the same in $\mathbb{E}^n$ and this classification is totally based on the parametrizations of their f-position vector field.

Theorem 5.1. *Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve (parametrized by arc-length s) having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$ and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F. Then γ is an f-rectifying curve in $\mathbb{E}^n$ if and only if, up to a parametrization, its f-position vector field γ_f is given by*

$$\gamma_f(t) = c \sec \left(t + \arctan \left(\frac{F(s_0)}{c} \right) \right) \beta(t),$$

where c is a positive constant, $s_0 \in I$ and $\beta : J \longrightarrow \mathbb{S}^{n-1}(1)$ is a unit-speed curve having $t : I \longrightarrow J$ as arc length function based at s_0.

Proof. First, for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ with at least $(n-2)$-times differentiable primitive function F, let $\gamma : I \longrightarrow \mathbb{E}^n$ be an f-rectifying curve having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$. Then by Theorem 4.1, the norm function $\rho = \|\gamma_f\|$is given by

$$\rho(s) = \sqrt{F^2(s) + c^2}, \tag{5.1}$$

where we may choose c as a positive constant. Now, we define a curve $\beta : I \longrightarrow \mathbb{E}^n$ by

$$\beta(s) := \frac{1}{\rho(s)} \gamma_f(s). \tag{5.2}$$

Then we find

$$\langle \beta(s), \beta(s) \rangle = 1. \tag{5.3}$$

Therefore, β is a curve in the unit-sphere $\mathbb{S}^{n-1}(1)$. Differentiating (5.3), we get

$$\langle \beta(s), \beta'(s) \rangle = 0. \tag{5.4}$$

Now, from (5.1) and (5.2), we obtain

$$\gamma_f(s) = \beta(s) \sqrt{F^2(s) + c^2}. \tag{5.5}$$

Again, differentiating (5.5), we obtain

$$f(s) T_\gamma(s) = \beta'(s) \sqrt{F^2(s) + c^2} + \frac{\beta(s) f(s) F(s)}{\sqrt{F^2(s) + c^2}}. \tag{5.6}$$

Using (5.3), (5.4) and (5.6), we obtain

$$\left\langle \beta'(s), \beta'(s) \right\rangle = \frac{c^2 f^2(s)}{\left(F^2(s)+c^2\right)^2}. \tag{5.7}$$

Therefore, we get

$$\left\|\beta'(s)\right\| = \sqrt{\langle \beta'(s), \beta'(s)\rangle} = \frac{c\, f(s)}{F^2(s)+c^2}. \tag{5.8}$$

Now, for some $s_0 \in I$, let $t : I \longrightarrow J$ be arc-length parameter of β given by

$$t = \int_{s_0}^{s} \left\|y'(u)\right\| du. \tag{5.9}$$

Then we have

$$\begin{aligned} t &= \int_{s_0}^{s} \frac{c\, f(u)}{F^2(u)+c^2}\, du \\ \Longrightarrow \quad t &= \arctan\left(\frac{F(s)}{c}\right) - \arctan\left(\frac{F(s_0)}{c}\right) \\ \Longrightarrow \quad s &= F^{-1}\left(c \tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\right). \end{aligned} \tag{5.10}$$

Substituting (5.10) in (5.5), we obtain the f-position vector field of γ as follows:

$$\gamma_f(t) = c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\beta(t).$$

Conversely, let γ be a unit-speed curve in $\mathbb{E}^n$ such that, up to a parametrization, its f-position vector field γ_f is defined by

$$\gamma_f(t) := c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\beta(t), \tag{5.11}$$

where c is a positive constant and $\beta : J \longrightarrow \mathbb{S}^{n-1}(1)$ is a unit-speed curve having $t : I \longrightarrow J$ as arc length function based at s_0. Differentiating (5.11), we obtain

$$\gamma_f{}'(t) = c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\left[\tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\beta(t) + 1\right]\beta'(t). \tag{5.12}$$

Since β is a unit-speed curve in the unit-sphere $\mathbb{S}^{n-1}(1)$, we have $\langle \beta'(t), \beta'(t)\rangle = 1$, $\langle \beta(t), \beta(t)\rangle = 1$ and consequently $\langle \beta(t), \beta'(t)\rangle = 0$. Therefore, from (5.11) and (5.12), we have

$$\left\langle \gamma_f(t), \gamma_f(t) \right\rangle = c^2 \sec^2\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right), \tag{5.13}$$

$$\left\langle \gamma_f(t), \gamma_f{}'(t) \right\rangle = c^2 \sec^2\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right), \tag{5.14}$$

$$\left\langle \gamma_f{}'(t), \gamma_f{}'(t) \right\rangle = c^2 \sec^4\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right). \tag{5.15}$$

Now, if we put

$$t = \arctan\left(\frac{F(s)}{c}\right) - \arctan\left(\frac{F(s_0)}{c}\right),$$

then s becomes arc length parameter of γ and equations (5.13), (5.14), (5.15) reduce to

$$\left\langle \gamma_f(s), \gamma_f(s) \right\rangle = c^2 \sec^2\left(\frac{F(s)}{c}\right), \tag{5.16}$$

$$\left\langle \gamma_f(s), \gamma_f{}'(s) \right\rangle = c^2 \sec^2\left(\frac{F(s)}{c}\right)\tan\left(\frac{F(s)}{c}\right), \tag{5.17}$$

$$\left\langle \gamma_f{}'(s), \gamma_f{}'(s) \right\rangle = c^2 \sec^4\left(\frac{F(s)}{c}\right). \tag{5.18}$$

Again, the normal component $\gamma_f^{N_\gamma}$ of γ_f is given by

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle = \left\langle \gamma_f(s), \gamma_f(s) \right\rangle - \frac{\left\langle \gamma_f(s), \gamma_f{}'(s)\right\rangle^2}{\left\langle \gamma_f{}'(s), \gamma_f{}'(s)\right\rangle}.$$

Then substituting (5.16), (5.17)) and (5.18) in the previous equation, we obtain

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle = \left\|\gamma_f^{N_\gamma}(s)\right\|^2 = c^2.$$

This implies that the normal component $\gamma_f^{N_\gamma}$ of γ_f has a constant length. Also, the norm function $\rho = \|\gamma_f\|$ is given by

$$\rho(s) = \sqrt{\left\langle \gamma_f(s), \gamma_f(s)\right\rangle} = c \sec\left(\frac{F(s)}{c}\right)$$

and it is non-constant. Therefore, by applying the Theorem 4.1, we conclude that γ is an f-rectifying curve in $\mathbb{E}^n$. □

6. Conclusion

It goes without saying that f-rectifying curves in Euclidean spaces are a sort of generalizations of rectifying curves therein. In this paper, we presented a study on f-rectifying curves in Euclidean n-space $\mathbb{E}^n$. Predominantly, we explored two main theorems demonstrating some necessary and sufficient conditions for a regular curve to be an f-rectifying curve in $\mathbb{E}^n$. The first theorem portrays some geometric characterizations of f-rectifying curves in $\mathbb{E}^n$ in connection with norm functions, tangential, normal and $n-2$ binormal components of their f-position vector field. Whereas the second theorem classifies such curves based on parametrization of their f-position vector field. Moreover, it yields an important characterization: namely, the f-position vector field of an f-rectifying curve in $\mathbb{E}^n$ is a dilation of a unit-speed curve in the unit $(n-1)$-sphere $\mathbb{S}^{n-1}(1)$ with dilation factor $c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)$ for some constants $c > 0$ and s_0. Extensions of such study to other ambient spaces may be considered as problems of interest.

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