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Global Behavior of a System of Second-Order Rational Difference Equations

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Abstract
In this paper, we consider the following system of rational difference equations

$$x_{n+1} = \frac{a + x_n}{b + cy_n + dx_{n-1}}, \; y_{n+1} = \frac{\alpha + y_n}{\beta + \gamma x_n + \eta y_{n-1}}, \; n = 0, 1, 2, \ldots$$

where $a, b, c, d, \alpha, \beta, \gamma, \eta \in (0, \infty)$ and the initial values $x_{-1}, x_0, y_{-1}, y_0 \in (0, \infty)$. Our main aim is to investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system.

Keywords: Equilibrium points, Global behavior, Local stability, Positive solutions, Rate of convergence.
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1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so forth [5, 7, 14, 15]. Recently, there has been a lot of works concerning the global behaviors of positive solutions of rational difference equations and positive solutions of systems of rational difference equations [1, 2, 4, 8, 9, 12, 13]. It is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations and solutions of systems of rational difference equations, although they have very simple forms. One can refer to [1]-[22] and the references cited therein to illustrate this. Therefore, the study of rational difference equations and systems of rational difference equations is worth further consideration.

In [1] M.R.S. Kulenović and M. Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{Ax_n y_n}{1 + y_n}, \; y_{n+1} = \frac{Bx_n y_n}{1 + x_n}, \; n = 0, 1, 2, \ldots,$$

where $A, B \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In [2] S. Kalabusić and M.R.S. Kulenović considered two systems of difference equations

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n}, \; y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{y_n}, \; n = 0, 1, 2, \ldots,$$

and

$$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{1+x_n},\ y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{1+y_n},\ n=0,1,2,\ldots,$$

where $\alpha_1,\alpha_2,\beta_2,\gamma_1\in(0,\infty)$ and x_0,y_0 are positive numbers.

In [3], Q. Din et al. investigated behavior of the competitive system of difference equations

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_{n-1}}{a_1+b_1 y_n},\ y_{n+1}=\frac{\alpha_2+\beta_2 y_{n-1}}{a_2+b_2 x_n},\ n=0,1,2,\ldots,$$

where $a_i,b_i,\alpha_i,\beta_i\in(0,\infty)$ for $i\in\{1,2\}$ and initial conditions x_{-1},x_0,y_{-1},y_0 are positive numbers.

In [4], the author investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system

$$x_{n+1}=\frac{ax_n-bx_ny_n}{1+cx_n+dy_n},\ y_{n+1}=\frac{\alpha x_ny_n-\beta y_n}{1+\gamma x_n+\eta y_n},\ n=0,1,2,\ldots,$$

where $a,b,c,d,\alpha,\beta,\gamma,\eta\in(0,\infty)$ and the initial values $(x_0,y_0)\in(0,\infty)$.

Motivated by these above papers, in this paper we will consider the following system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+cy_n+dx_{n-1}},\ y_{n+1}=\frac{\alpha+y_n}{\beta+\gamma x_n+\eta y_{n-1}},\ n=0,1,2,\ldots, \tag{1.1}$$

where $a,b,c,d,\alpha,\beta,\gamma,\eta\in(0,\infty)$ and the initial values $x_{-1},x_0,y_{-1},y_0\in(0,\infty)$. More precisely, we investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system (1.1) which converge to its unique positive equilibrium point.

2. Boundedness and persistence

In the first result we will establish the boundedness and persistence of every positive solution of the system (1.1).

Theorem 2.1. *Assume that $b>1$, $d<1$, $\beta>1$ and $\gamma<1$ then every positive solution $\{(x_n,y_n)\}$ of the system (1.1) is bounded and persists.*

Proof. For any positive solution $\{(x_n,y_n)\}$ of the system (1.1), we have

$$x_{n+1}\le\frac{a}{b}+\frac{1}{b}x_n,\ y_{n+1}\le\frac{\alpha}{\beta}+\frac{1}{\beta}y_n,\ n=0,1,2,\ldots. \tag{2.1}$$

Consider the following linear difference equations:

$$u_{n+1}=\frac{a}{b}+\frac{1}{b}u_n,\ v_{n+1}=\frac{\alpha}{\beta}+\frac{1}{\beta}v_n,\ n=0,1,2,\ldots. \tag{2.2}$$

We can see the solutions of (2.2) have the forms

$$u_n=\frac{a}{b-1}+C_1(\frac{1}{b})^n,\ v_n=\frac{\alpha}{\beta-1}+C_2(\frac{1}{\beta})^n,\ n=1,2,\ldots, \tag{2.3}$$

where C_1,C_2 depend on initial conditions u_0,v_0.

Assume that $b>1$ and $\beta>1$ then the sequences $\{u_n\}$ and $\{v_n\}$ are bounded. Suppose that $u_0=x_0$ and $v_0=y_0$ then by comparison we have

$$x_n\le\frac{a}{b-1}=U_1,\ y_n\le\frac{\alpha}{\beta-1}=U_2,\ n=1,2,\ldots. \tag{2.4}$$

Also, from (1.1) and (2.4), we infer

$$\begin{aligned}x_{n+1}&\ge\frac{a}{b+cy_n+dx_{n-1}}\ge\frac{a}{b+cU_2+dx_{n-1}},\\ y_{n+1}&\ge\frac{\alpha}{\beta+\gamma x_n+\eta y_{n-1}}\ge\frac{\alpha}{\beta+\gamma U_1+\eta y_{n-1}},\end{aligned}\quad n=1,2,\ldots. \tag{2.5}$$

Consider the following linear difference equations:

$$s_{n+1} = \delta + ds_{n-1},\ t_{n+1} = \theta + \eta t_n,\ n = 0,1,2,\ldots, \tag{2.6}$$

where $\delta = b + cU_2, \theta = \beta + \gamma U_1$.
We can see the solutions of (2.6) have the forms

$$\begin{aligned} s_n &= \frac{\delta}{1-d} + C_3(\sqrt{d})^n + C_4(-\sqrt{d})^n, \\ t_n &= \frac{\theta}{1-\gamma} + C_5(\sqrt{\gamma})^n + C_6(-\sqrt{\gamma})^n, \end{aligned} \quad n = 1,2,\ldots, \tag{2.7}$$

where C_3, C_4 depend on initial conditions s_{-1}, s_0 and C_5, C_6 depend on initial conditions t_{-1}, t_0.

Assume that $d < 1$ and $\gamma < 1$ then the sequences $\{s_n\}$ and $\{t_n\}$ are bounded. Suppose that $s_{-1} = x_{-1}, s_0 = x_0$ and $t_{-1} = y_{-1}, t_0 = y_0$ then by comparison we have

$$\begin{aligned} x_n &\geq \frac{a}{\delta/(1-d)} = \frac{a(1-d)}{b+cU_2} = \frac{a(1-d)}{b+c\frac{\alpha}{\beta-1}} = \frac{a(1-d)(\beta-1)}{b(\beta-1)+c\alpha} = L_1, \\ y_n &\geq \frac{\alpha}{\theta/(1-\gamma)} = \frac{\alpha(1-\gamma)}{\beta+\gamma U_1} = \frac{\alpha(1-\gamma)}{\beta+\gamma\frac{a}{b-1}} = \frac{\alpha(1-\gamma)(b-1)}{\beta(b-1)+\gamma a} = L_2, \end{aligned} \quad n = 1,2,\ldots. \tag{2.8}$$

From (2.4) and (2.5), we have

$$L_1 \leq x_n \leq U_1,\ L_2 \leq y_n \leq U_2,\ n = 1,2,\ldots. \tag{2.9}$$

Hence, the proof is completed. □

Lemma 2.2. *Let $\{(x_n, y_n)\}$ be a positive solution of the system (1.1). Then, $[L_1, U_1] \times [L_2, U_2]$ is an invariant set for system (1.1).*

Proof. The proof follows by induction. □

3. Global behavior

In the following, we state some main definitions used in this paper.

Let I, J be some intervals of real numbers and let

$$f : I^2 \times J^2 \longrightarrow I \text{ and } g : I^2 \times J^2 \longrightarrow J \tag{3.1}$$

are continuously differentiable functions. Then, for all initial values $(x_{-1}, x_0, y_{-1}, y_0) \in I^2 \times J^2$, the system of difference equations

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}),\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}),\ n = 0,1,2,\ldots, \tag{3.2}$$

has a unique solution $\{(x_n, y_n)\}_{n=1}^{\infty}$.

Definition 3.1. *A point $(\bar{x}, \bar{y})$ is called an equilibrium point of the system (3.2) if*

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}),\ \bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}). \tag{3.3}$$

Definition 3.2. *[3, 5] Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (3.2).*

1. *An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial point $(x_i, y_i), i \in \{-1, 0\}$ if $\sum_{i=-1}^{0} \|(x_i, y_i) - (\bar{x}, \bar{y})\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$. An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable (the Euclidean norm in $\mathbb{R}^2$ given by $\|(x,y)\| = \sqrt{x^2+y^2}$ is denoted by $\|.\|$).*

2. *An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\sum_{i=-1}^{0} \|(x_i, y_i) - (\bar{x}, \bar{y})\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.*

3. *An equilibrium point $(\bar{x},\bar{y})$ is called a global attractor if $(x_n,y_n) \to (\bar{x},\bar{y})$ as $n \to \infty$.*

4. *An equilibrium point $(\bar{x},\bar{y})$ is called an asymptotic global attractor if it is global attractor and stable.*

Definition 3.3. *[3, 5] Let $(\bar{x},\bar{y})$ be an equilibrium point of a map $F = (f,x_n,g,y_n)$, where f and g are continuously differentiable functions at $(\bar{x},\bar{y})$. The linearized system of (3.2) about the equilibrium point $(\bar{x},\bar{y})$ is given by*

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ *and F_J is a Jacobian matrix of the system (3.2) about the equilibrium point $(\bar{x},\bar{y})$.*

In order to corresponding linearized form of system (1.1) we consider the following transformation:

$$(x_n,x_{n-1},y_n,y_{n-1}) \longrightarrow (f,g,f_1,g_1), \tag{3.4}$$

where $f = x_{n+1}, g = y_{n+1}, f_1 = x_n, g_1 = y_n$. The linearized system of (1.1) about $(\bar{x},\bar{y})$ is given by

$$Y_{n+1} = F_J(\bar{x},\bar{y})Y_n, \tag{3.5}$$

where $Y_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and the Jacobian matrix of the system (1.1) about the equilibrium point $(\bar{x},\bar{y})$ is given by

$$F_J(\bar{x},\bar{y}) = \begin{pmatrix} \frac{1}{b+c\bar{y}+d\bar{x}} & \frac{-c\bar{x}}{b+c\bar{y}+d\bar{x}} & \frac{-d\bar{x}}{b+c\bar{y}+d\bar{x}} & 0 \\ \frac{-\gamma\bar{y}}{\beta+\gamma\bar{x}+\eta\bar{y}} & \frac{1}{\beta+\gamma\bar{x}+\eta\bar{y}} & 0 & \frac{-\eta\bar{y}}{\beta+\gamma\bar{x}+\eta\bar{y}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.6}$$

The following results will be useful in the sequel.

Lemma 3.4. *[3] Assume that $X_{n+1} = F(X_n), n = 0,1,2,\ldots,$ is a system of difference equations such that $\bar{X}$ is a fixed point of F. If all eigenvalues of Jacobian matrix F_J about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Lemma 3.5. *[6] Assume that $q_0,q_1,\ldots,q_k$ are real numbers such that*

$$|q_0| + |q_1| + \ldots + |q_k| < 1.$$

Then all roots of the equation

$$\lambda^{k+1} + q_0\lambda^k + \ldots + q_{k-1}\lambda + q_k = 0$$

lie inside the unit disk.

The next theorem will show the existence and uniqueness of positive equilibrium point of the system (1.1).

Theorem 3.6. *Assume that $b > 1, \beta > 1$ and the following conditions are satisfied:*

$$-(c^2\alpha + cd\gamma)U_1^2 - cd(\beta-1)U_1 + d^2\eta < 0, \tag{3.7}$$

and

$$\begin{aligned} &(cd\gamma - d^2\eta)L_1^4 + [(b-1)c\gamma + cd(\beta-1) - 2(b-1)d\eta]L_1^3 \\ &+ [2ad\eta + c^2\alpha + (b-1)c(\beta-1) - ac\gamma - \eta(b-1)^2]L_1^2 \\ &+ [2a(b-1)\eta - ac(\beta-1)]L_1 - a^2\eta > 0, \end{aligned} \tag{3.8}$$

or

$$-(c^2\alpha+cd\gamma)U_1^2-cd(\beta-1)U_1+d^2\eta>0, \tag{3.9}$$

and

$$\begin{aligned}&(cd\gamma-d^2\eta)L_1^4+[(b-1)c\gamma+cd(\beta-1)-2(b-1)d\eta]L_1^3\\&+[2ad\eta+c^2\alpha+(b-1)c(\beta-1)-ac\gamma-\eta(b-1)^2]L_1^2\\&+[2a(b-1)\eta-ac(\beta-1)]L_1-a^2\eta<0,\end{aligned} \tag{3.10}$$

and

$$U_1<\frac{2\sqrt{\alpha\eta}}{\gamma} \tag{3.11}$$

Then there exists unique positive equilibrium point of the system (1.1) in $[L_1,U_1]\times[L_2,U_2]$.

Proof. Firstly, we consider the following system of algebraic equations

$$x=\frac{a+x}{b+cy+dx},\ y=\frac{\alpha+y}{\beta+\gamma x+\eta y}. \tag{3.12}$$

From (3.12), it follows that

$$\begin{aligned}y=&\frac{a+x-bx-dx^2}{cx}=\frac{a}{cx}-\frac{d}{c}x-\frac{b-1}{c},\\x=&\frac{\alpha+y-\beta y-\eta y^2}{\gamma y}=\frac{\alpha}{\gamma y}-\frac{\eta}{\gamma}y-\frac{\beta-1}{\gamma}.\end{aligned} \tag{3.13}$$

Set

$$f(x)=\frac{a}{cx}-\frac{d}{c}x-\frac{b-1}{c}, \tag{3.14}$$

and

$$F(x)=\frac{\alpha}{\gamma f(x)}-\frac{\eta}{\gamma}f(x)-\frac{\beta-1}{\gamma}-x. \tag{3.15}$$

We have

$$\begin{aligned}f(U_1)=&\frac{a}{cU_1}-\frac{d}{c}U_1-\frac{b-1}{c}\\=&\frac{a}{c}\frac{b-1}{a}-\frac{d}{c}\frac{a}{b-1}-\frac{b-1}{c}\\=&-\frac{d}{c}\frac{a}{b-1}=-\frac{d}{cU_1},\end{aligned} \tag{3.16}$$

$$\begin{aligned}F(U_1)=&\frac{\alpha}{\gamma f(U_1)}-\frac{\eta}{\gamma}f(U_1)-\frac{\beta-1}{\gamma}-U_1\\=&-\frac{c\alpha U_1}{d\gamma}+\frac{d\eta}{c\gamma U_1}-\frac{\beta-1}{\gamma}-U_1\\=&\frac{-(c^2\alpha+cd\gamma)U_1^2-cd(\beta-1)U_1+d^2\eta}{cd\gamma U_1},\end{aligned} \tag{3.17}$$

$$f(L_1)=\frac{a}{cL_1}-\frac{d}{c}L_1-\frac{b-1}{c}=\frac{-dL_1^2-(b-1)L_1+a}{cL_1}, \tag{3.18}$$

$$
\begin{aligned}
F(L_1) =& \frac{\alpha}{\gamma f(L_1)} - \frac{\eta}{\gamma} f(L_1) - \frac{\beta - 1}{\gamma} - L_1 \\
=& \frac{-\eta[-dL_1^2 - (b-1)L_1 + a]^2 - c\gamma L_1^2[-dL_1^2 - (b-1)L_1 + a]}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
&+ \frac{-c(\beta-1)L_1[-dL_1^2 - (b-1)L_1 + a] + c^2\alpha L_1^2}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
=& \frac{(cd\gamma - d^2\eta)L_1^4 + [(b-1)c\gamma + cd(\beta-1) - 2(b-1)d\eta]L_1^3}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
&+ \frac{[2ad\eta + c^2\alpha + (b-1)c(\beta-1) - ac\gamma - \eta(b-1)^2]L_1^2}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
&+ \frac{[2a(b-1)\eta - ac(\beta-1)]L_1 - a^2\eta}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]}.
\end{aligned} \tag{3.19}
$$

From (3.15), we have

$$
F(x) = \frac{\alpha}{\gamma} . \frac{cx}{-dx^2 - (b-1)x + a} - \frac{\eta}{\gamma} . \frac{-dx^2 - (b-1)x + a}{cx} - \frac{\beta - 1}{\gamma} - x. \tag{3.20}
$$

It follows that

$$
\begin{aligned}
F'(x) =& \frac{c\alpha}{\gamma} . \frac{-dx^2 - (b-1)x + a - x(-2dx - b + 1)}{[-dx^2 - (b-1)x + a]^2} \\
&- \frac{\eta}{c\gamma} . \frac{x(-2dx - b + 1) - [-dx^2 - (b-1)x + a]}{x^2} - 1 \\
=& \frac{c\alpha}{\gamma} . \frac{dx^2 + a}{[-dx^2 - (b-1)x + a]^2} + \frac{\eta}{c\gamma} . \frac{dx^2 + a}{x^2} - 1 \\
\geq& 2\frac{\sqrt{\alpha\eta}}{\gamma} . \frac{dx^2 + a}{x[-dx^2 - (b-1)x + a]} - 1 > 2\frac{\sqrt{\alpha\eta}}{\gamma} . \frac{1}{x} - 1.
\end{aligned} \tag{3.21}
$$

Assume that condition (3.11) is satisfied, then we have $F'(x) > 0$. Hence, $F(x) = 0$ has a unique positive solution in $[L_1, U_1]$. $\square$

Theorem 3.7. *The unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (1.1) is locally asymptotically stable if the following condition holds*

$$
\frac{1 + dU_1}{b + dL_1 + cL_2} + \frac{1 + \eta U_2}{\beta + \gamma L_1 + \eta L_2} + \frac{1 + dU_1 + \eta U_2 + (c\gamma + d\eta)U_1U_2}{(b + dL_1 + cL_2)(\beta + \gamma L_1 + \eta L_2)} < 1. \tag{3.22}
$$

Proof. The characteristic polynomial of Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(\bar{x}, \bar{y})$ is given by

$$
P(\lambda) = \lambda^4 - (A + B)\lambda^3 + (d\bar{x}A + \eta\bar{y}B + AB - c\gamma\bar{x}\bar{y}AB)\lambda^2 - (d\bar{x}AB + \eta\bar{y}AB)\lambda + d\eta\bar{x}\bar{y}AB, \tag{3.23}
$$

where $A = \dfrac{1}{b + d\bar{x} + c\bar{y}}, B = \dfrac{1}{\beta + \gamma\bar{x} + \eta\bar{y}}$.
We have

$$
\begin{aligned}
&|A + B| + |d\bar{x}A + \eta\bar{y}B + AB - c\gamma\bar{x}\bar{y}AB| + |d\bar{x}AB + \eta\bar{y}AB| + |d\eta\bar{x}\bar{y}AB| \\
&< (1 + d\bar{x})A + (1 + \eta\bar{y})B + (1 + d\bar{x} + \eta\bar{y} + c\gamma\bar{x}\bar{y} + d\eta\bar{x}\bar{y})AB \\
&< \frac{1 + dU_1}{b + dL_1 + cL_2} + \frac{1 + \eta U_2}{\beta + \gamma L_1 + \eta L_2} + \frac{1 + dU_1 + \eta U_2 + (c\gamma + d\eta)U_1U_2}{(b + dL_1 + cL_2)(\beta + \gamma L_1 + \eta L_2)} < 1.
\end{aligned} \tag{3.24}
$$

By using Lemma 3.5, we can see that all the roots of (3.23) satisfy $|\lambda| < 1$, and it follows from Lemma 3.4 that the unique positive equilibrium point $(\bar{x}, \bar{y})$ of the system (1.1) is locally asymptotically stable. Hence, the proof is completed. $\square$

Theorem 3.8. *The unique positive equilibrium point* $(\bar{x},\bar{y})$ *of system* (1.1) *is globally asymptotically stable if the following condition holds*

$$a+U_1 < L_1(b+cL_2+dL_1), \alpha+U_2 < L_2(\beta+\gamma L_1+\eta L_2). \tag{3.25}$$

Proof. Arguing as in Theorem 1.1 of [11], we consider the following Lyapunov function:

$$V_n = \bar{x}g(\frac{x_n}{\bar{x}})+\bar{y}g(\frac{y_n}{\bar{y}}), \tag{3.26}$$

where

$$g(x) = x-1-\ln x \geq 0, \forall x>0. \tag{3.27}$$

It is easy to see that V_n is nonnegative function.
Consider

$$\begin{aligned} V_{n+1}-V_n =&\bar{x}(\frac{x_{n+1}}{\bar{x}}-1-\ln\frac{x_{n+1}}{\bar{x}})+\bar{y}(\frac{y_{n+1}}{\bar{y}}-1-\ln\frac{y_{n+1}}{\bar{y}}) \\ &-\bar{x}(\frac{x_n}{\bar{x}}-1-\ln\frac{x_n}{\bar{x}})+\bar{y}(\frac{y_n}{\bar{y}}-1-\ln\frac{y_n}{\bar{y}}) \\ =&\bar{x}(\frac{x_{n+1}-x_n}{\bar{x}}+\ln\frac{x_n}{x_{n+1}})+\bar{y}(\frac{y_{n+1}-y_n}{\bar{y}}+\ln\frac{y_n}{y_{n+1}}). \end{aligned} \tag{3.28}$$

Furthermore, from (3.27) we have

$$\ln\frac{x_n}{x_{n+1}} \leq \frac{x_n}{x_{n+1}}-1, \ln\frac{y_n}{y_{n+1}} \leq \frac{y_n}{y_{n+1}}-1. \tag{3.29}$$

Then, from (3.28) and (3.29) we have

$$\begin{aligned} V_{n+1}-V_n \leq&\bar{x}\left(\frac{x_{n+1}-x_n}{\bar{x}}+\frac{x_n-x_{n+1}}{x_{n+1}}\right)+\bar{y}\left(\frac{y_{n+1}-y_n}{\bar{y}}+\frac{y_n-y_{n+1}}{y_{n+1}}\right) \\ =&\frac{(x_{n+1}-x_n)(x_{n+1}-\bar{x})}{x_{n+1}}+\frac{(y_{n+1}-y_n)(y_{n+1}-\bar{y})}{y_{n+1}} \\ \leq&(U_1-L_1)(1-\frac{\bar{x}}{x_{n+1}})+(U_2-L_2)(1-\frac{\bar{y})}{y_{n+1}}) \\ =&(U_1-L_1)\frac{[a+x_n-\bar{x}(b+cy_n+dx_{n-1})]}{a+x_n} \\ &+(U_2-L_2)\frac{[\alpha+y_n-\bar{y}(\beta+\gamma x_n+\eta y_{n-1})]}{\alpha+y_n} \\ \leq&\frac{(U_1-L_1)[a+U_1-L_1(b+cL_2+dL_1)]}{a+L_1} \\ &+\frac{(U_2-L_2)[\alpha+U_2-L_2(\beta+\gamma L_1+\eta L_2)]}{\alpha+L_2}. \end{aligned} \tag{3.30}$$

By using condition (3.25), we have $V_{n+1}-V_n \leq 0$ for all $n \geq 0$, so that $V_n \geq 0$ is monotonically decreasing sequence. It follows that $\lim_{n\to\infty} V_n$ exists and is nonnegative. Hence, we imply that

$$\lim_{n\to\infty}(V_{n+1}-V_n) = 0. \tag{3.31}$$

Then it follows that $\lim_{n\to\infty} x_{n+1} = \bar{x}$ and $\lim_{n\to\infty} y_{n+1} = \bar{y}$. Furthermore, $V_n \leq V_0$ for all $n \geq 0$, which gives that $(\bar{x},\bar{y}) \in [L_1,U_1]\times[L_2,U_2]$ is uniformly stable. Hence, unique positive equilibrium point $(\bar{x},\bar{y}) \in [L_1,U_1]\times[L_2,U_2]$ of system (1.1) is globally asymptotically stable. □

4. Rate of convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x},\ \bar{y})$ of the systems (1.1) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [16] and [17].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n \tag{4.1}$$

where $\mathbf{x}_n$ is a k-dimensional vector, $A \in \mathbb{C}^{k\times k}$ is a constant matrix, and $B:\ \mathbb{Z}^+ \longrightarrow \mathbb{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to\ \infty, \tag{4.2}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm; $\|.\|$ also denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|\mathbf{x}\| = \|(x,\ y)\| = \sqrt{x^2 + y^2}. \tag{4.3}$$

Theorem 4.1. *([18]) Assume that condition (4.2) holds. If $\mathbf{x}_n$ is a solution of system (4.1), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 4.2. *([18]) Assume that condition (4.2) holds. If $\mathbf{x}_n$ is a solution of system (4.1), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 4.3. *Assume that $\{(x_n, y_n)\}$ is a positive solution of the system (1.1) such that $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1], \bar{y} \in [L_2, U_2]$. Then the error vector $\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \\ x_{n-1} - \bar{x} \\ y_{n-1} - \bar{y} \end{pmatrix}$ of every solution $(x_n, y_n) \neq (\bar{x}, \bar{y})$ of (1.1) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_F(\bar{x}, \bar{y}))| \textit{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_F(\bar{x}, \bar{y}))| \textit{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

where $|\lambda_i(J_F(\bar{x}, \bar{y}))|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $(\bar{x}, \bar{y})$.

Proof. Let $\{(x_n, y_n)\}$ be an arbitrary positive solution of the system (1.1) such that $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1], \bar{y} \in [L_2, U_2]$. Firstly, we will find a system satisfied by the error terms, which are given as

$$\begin{aligned} x_{n+1} - \bar{x} =& \frac{a + x_n}{b + cy_n + dx_{n-1}} - \frac{a + \bar{x}}{b + c\bar{y} + d\bar{x}} \\ =& \frac{1}{(b + cy_n + dx_{n-1})}(x_n - \bar{x}) \\ &- \frac{c(a + \bar{x})}{(b + cy_n + dx_{n-1})(b + c\bar{y} + d\bar{x})}(y_n - \bar{y}) \\ &- \frac{d(a + \bar{x})}{(b + cy_n + dx_{n-1})(b + c\bar{y} + d\bar{x})}(x_{n-1} - \bar{x}), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
y_{n+1}-\bar{y} &= \frac{\alpha+y_n}{\beta+\gamma x_n+\eta y_{n-1}}-\frac{\alpha+\bar{y}}{\beta+\gamma\bar{x}+\eta\bar{y}} \\
&= -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}(x_n-\bar{x}) \\
&\quad+\frac{1}{(\beta+\gamma x_n+\eta y_{n-1})}(y_n-\bar{y}) \\
&\quad-\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}(y_{n-1}-\bar{y}).
\end{aligned} \tag{4.5}$$

Let $e_n^1 = x_n-\bar{x}$ and $e_n^2 = y_n-\bar{y}$, then from (4.4) and (4.5) we have:

$$\begin{aligned}
e_{n+1}^1 &= p_n e_n^1 + q_n e_n^2 + r_n e_{n-1}^1, \\
e_{n+1}^2 &= g_n e_n^1 + h_n e_n^2 + w_n e_{n-1}^2,
\end{aligned}$$

where

$$\begin{aligned}
p_n &= \frac{1}{(b+cy_n+dx_{n-1})}, \\
q_n &= -\frac{c(a+\bar{x})}{(b+cy_n+dx_{n-1})(b+c\bar{y}+d\bar{x})}, \\
r_n &= -\frac{d(a+\bar{x})}{(b+cy_n+dx_{n-1})(b+c\bar{y}+d\bar{x})}, \\
g_n &= -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}, \\
h_n &= \frac{1}{(\beta+\gamma x_n+\eta y_{n-1})}, \\
w_n &= -\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}.
\end{aligned}$$

Taking the limmits of $p_n,\ q_n,\ r_n, g_n, h_n$ and w_n as $n\to\infty$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} p_n = \frac{1}{(b+c\bar{y}+d\bar{x})},\ \lim_{n\to\infty} q_n = -\frac{c(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2},\ \lim_{n\to\infty} r_n = -\frac{d(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2}, \\
&\lim_{n\to\infty} g_n = -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2},\ \lim_{n\to\infty} h_n = \frac{1}{(\beta+\gamma\bar{x}+\eta\bar{y})},\ \lim_{n\to\infty} w_n = -\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2}.
\end{aligned}$$

that is

$$\begin{aligned}
&p_n = \frac{1}{(b+c\bar{y}+d\bar{x})}+\alpha_n,\ q_n = -\frac{c(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2}+\beta_n,\ r_n = -\frac{d(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2}+\gamma_n, \\
&g_n = -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2}+\delta_n,\ h_n = \frac{1}{(\beta+\gamma\bar{x}+\eta\bar{y})}+\eta_n,\ w_n = -\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2}+\theta_n.
\end{aligned}$$

where $\alpha_n\to 0,\ \beta_n\to 0,\ \gamma_n\to 0, \delta_n\to 0, \eta_n\to 0$ and $\theta_n\to 0$ as $n\to\infty$.

Now, we have system of the form (4.1):

$$\mathbf{e}_{n+1} = (A+B(n))\mathbf{e}_n,$$

where

$$A = \begin{pmatrix}
\frac{1}{(b+c\bar{y}+d\bar{x})} & -\frac{c(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2} & -\frac{d(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2} & 0 \\
-\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2} & \frac{1}{(\beta+\gamma\bar{x}+\eta\bar{y})} & 0 & -\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},$$

$$B(n) = \begin{pmatrix} \alpha_n & \beta_n & \gamma_n & 0 \\ \delta_n & \eta_n & 0 & \theta_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\|B(n)\| \to 0 \text{ as } n \to \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^1_n \\ e^2_n \end{pmatrix} = A \begin{pmatrix} e^1_n \\ e^2_n \\ e^1_{n-1} \\ e^2_{n-1} \end{pmatrix}.$$

The system is exactly linearized system of (1.1) evaluated at the equilibrium $E = (\bar{x},\ \bar{y})$. Then Theorem 4.1 and Theorem 4.2 imply the result. □

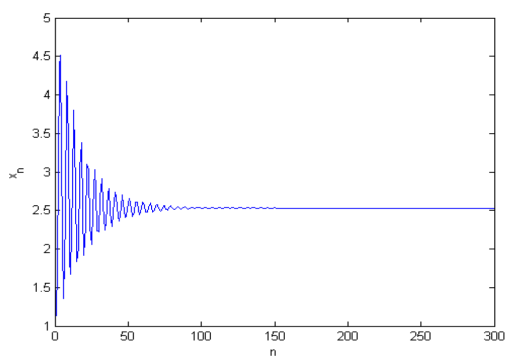
5. Examples

In order to verify our theoretical results and to support our theoretical discussion, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems (1.1). All plots in this section are drawn with Matlab.

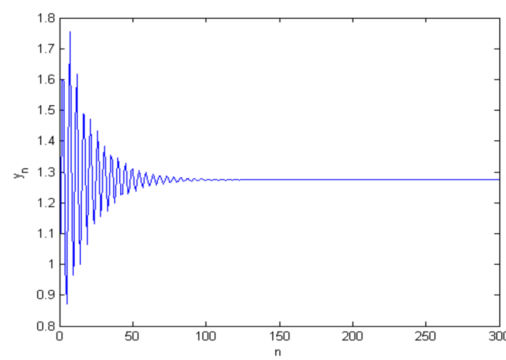
Example 5.1. *Let* $a = 3, b = 1.045, c = 0.09, d = 0.8, \alpha = 4, \beta = 1.5, \gamma = 0.69, \eta = 0.7$. *The system* (1.1) *can be written as*

$$x_{n+1} = \frac{3 + x_n}{1.045 + 0.09y_n + 0.8x_{n-1}},\ y_{n+1} = \frac{4 + y_n}{1.5 + 0.69x_n + 0.7y_{n-1}}, \tag{5.1}$$

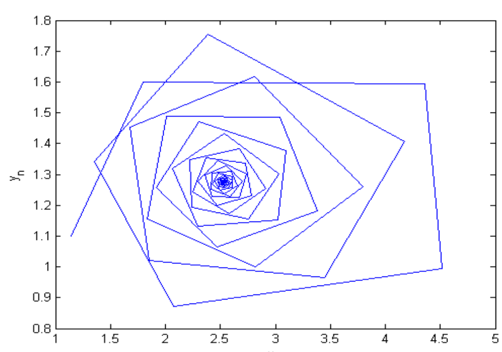
with initial conditions $x_{-1} = 1.14$, $x_0 = 1.8$, $y_{-1} = 1.1$ *and* $y_0 = 1.6$.



(a) Plot of x_n for the system (5.1)



(b) Plot of y_n for the system (5.1)



(c) An attractor of the system (5.1)

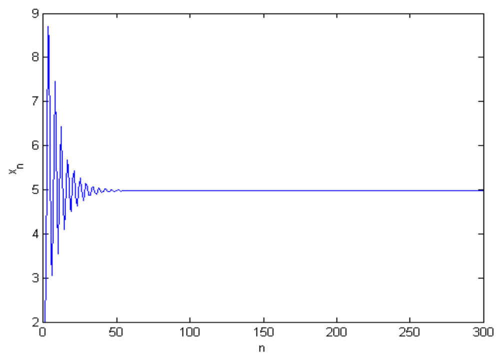
Figure 5.1. Plots for the system (5.1)

In this case, the unique positive equilibrium point of the system (1.1) is global attractor. In Figure 5.1, the plot of x_n is shown in Figure 5.1 (a), the plot of y_n is shown in Figure 5.1 (b), and a phase portrait of the system (5.1) is shown in Figure 5.1 (c).

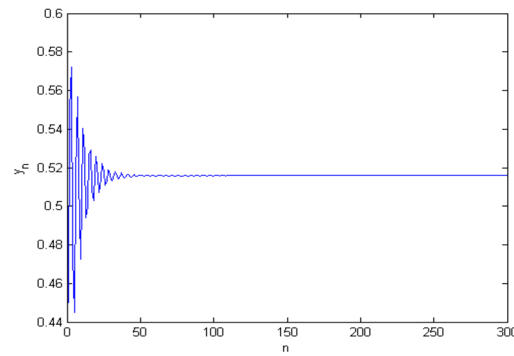
Example 5.2. *Let* $a = 20, b = 1.002, c = 0.07, d = 0.8, \alpha = 0.8, \beta = 2, \gamma = 0.09, \eta = 0.2$. *The system* (1.1) *can be written as*

$$x_{n+1} = \frac{20 + x_n}{1.002 + 0.07y_n + 0.8x_{n-1}}, \; y_{n+1} = \frac{0.8 + y_n}{2 + 0.09x_n + 0.2y_{n-1}}, \tag{5.2}$$

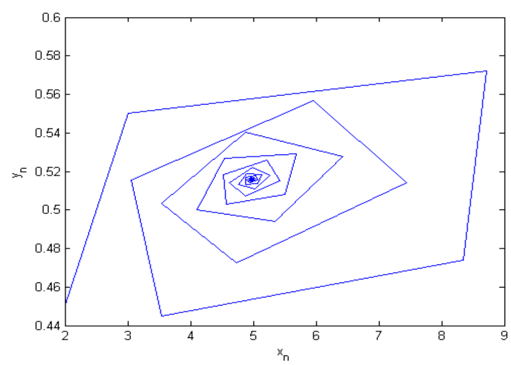
with initial conditions $x_{-1} = 2$, $x_0 = 3$, $y_{-1} = 0.45$ *and* $y_0 = 0.55$.



(a) Plot of x_n for the system (5.2)



(b) Plot of y_n for the system (5.2)



(c) An attractor of the system (5.2)

Figure 5.2. Plots for the system (5.2)

In this case, the unique positive equilibrium point of the system (1.1) is global attractor. In Figure 5.2, the plot of x_n is shown in Figure 5.2 (a), the plot of y_n is shown in Figure 5.2 (b), and a phase portrait of the system (5.2) is shown in Figure 5.2 (c).

6. Conclusion

This work is related to qualitative behavior of the system of second-order rational difference equations. We have investigated the existence and uniqueness of positive equilibrium of system (1.1). Under certain parametric conditions the boundedness and persistence of positive solutions is proved. Moreover, we have shown that unique positive equilibrium point of system (1.1) is locally as well as globally asymptotically stable under certain parametric conditions. Furthermore, the rate of convergence of positive solutions of (1.1) which converge to its unique positive equilibrium point is demonstrated. Finally, numerical examples are established to support our theoretical results.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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