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APPROXIMATION BY SZÁSZ-MIRAKJAN-DURRMEYER OPERATORS BASED ON SHAPE PARAMETER λ

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Abstract. In this work, we study several approximation properties of Szász-Mirakjan-Durrmeyer operators with shape parameter $\lambda \in [-1, 1]$. Firstly, we obtain some preliminaries results such as moments and central moments. Next, we estimate the order of convergence in terms of the usual modulus of continuity, for the functions belong to Lipschitz type class and Peetre's K-functional, respectively. Also, we prove a Korovkin type approximation theorem on weighted spaces and derive a Voronovskaya type asymptotic theorem for these operators. Finally, we show the comparison of the convergence of these newly defined operators to certain functions with some graphics and an error of approximation table.

1. Introduction

One of the famous linear positive operators in the theory of approximation, Szász [29] and Mirakjan [18] introduced following operators

$$S_m(\mu; y) = \sum_{j=0}^{\infty} s_{m,j}(y)\mu\left(\frac{j}{m}\right), \tag{1}$$

where $m \in \mathbb{N}$, $y \geq 0$, $\mu \in C[0, \infty)$ and Szász-Mirakjan basis functions $s_{m,j}(y)$ are defined as below:

$$s_{m,j}(y) = e^{-my}\frac{(my)^j}{j!}. \tag{2}$$

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In 1985, Mazhar and Totik [17] proposed Durrmeyer type integral modifications of operators (1) as follows:

$$D_m(\mu;y) = m\sum_{j=0}^{\infty} s_{m,j}(y)\int_0^{\infty} s_{m,j}(t)\mu(t)\,dt,\ \ y\in[0,\infty), \tag{3}$$

where $s_{m,j}(y)$ given as in (2).

Recently, some various approximation properties of operators (3) have been introduced by several authors. We refer the readers some papers on this direction [1, 3, 11–15].

A short time ago, the Bézier basis with shape parameter $\lambda \in [-1, 1]$ which is presented by Ye et al. [30], has attracted attention by some authors. Firstly, Cai et al. [7] introduced $\lambda-$Bernstein operators and obtained various approximation theorems, namely, Korovkin type convergence, local approximation and Voronovskaya-type asymptotic. Acu et al. [2] proposed the Kantorovich type $\lambda-$Bernstein operators and established some approximation features such as order of convergence, in connection with the Ditzian-Totik modulus of smoothness and Grüss-Voronovskaya type theorems. In 2019, Qi et al. [25] introduced a new generalization of Szász-Mirakjan operators based on shape parameter $\lambda \in [-1, 1]$ as below:

$$S_{m,\lambda}(\mu;y) = \sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;y)\mu\left(\frac{j}{m}\right), \tag{4}$$

where Szász-Mirakjan basis functions $\widetilde{s}_{m,j}(\lambda;y)$ with shape parameter $\lambda \in [-1, 1]$:

$$\begin{aligned}\widetilde{s}_{m,0}(\lambda;y) &= s_{m,0}(y) - \frac{\lambda}{m+1}s_{m+1,1}(y);\\ \widetilde{s}_{m,i}(\lambda;y) &= s_{m,i}(y) + \lambda\left(\frac{m-2i+1}{m^2-1}s_{m+1,i}(y)\right.\\ &\left.\quad -\frac{m-2i-1}{m^2-1}s_{m+1,i+1}(y)\right)\ (i=1,2,...,\infty,\ y\in[0,\infty)).\end{aligned} \tag{5}$$

For the operators defined by (4), they studied some theorems such as Korovkin type convergence, local approximation, Lipschitz type convergence, Voronovskaja and Grüss-Voronovskaja type. Also, we refer some recent works based on shape parameter $\lambda \in [-1, 1]$, see details: [5, 6, 8, 19–24, 26–28].

Motivated by all above-mentioned papers, we construct the following $\lambda-$Szász-Mirakjan-Durrmeyer operators as:

$$D_{m,\lambda}(\mu;y) = m\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty} s_{m,j}(t)\mu(t)\,dt, \quad y\in[0,\infty), \tag{6}$$

where $\widetilde{s}_{m,j}(\lambda;y)$ $(j = 0, 1, ..\infty)$ given in (5) and $\lambda \in [-1, 1]$.

This work is organized as follows: In Sect. 2, we compute some preliminaries results such as moments and central moments. Then, in Sect. 3, we obtain the order of convergence in respect of the usual modulus of continuity, for the functions belong to Lipschitz class and Peetre's K-functional, respectively. Next, In Sect. 4, we prove a Korovkin type convergence theorem on weighted spaces also in Sect. 5, we establish a Voronovskaya type asymptotic theorem. Finally, with the aid of Maple software, we present the comparison of the convergence of operators (6) to certain functions with some graphics and error of approximation table.

2. Preliminaries

Lemma 1. *[25]. For the $\lambda-$Szász-Mirakjan operators $S_{m,\lambda}(\mu;y)$, following results are satisfied:*

$$\begin{aligned}
S_{m,\lambda}(1;y) &= 1;\\
S_{m,\lambda}(t;y) &= y+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda;\\
S_{m,\lambda}(t^2;y) &= y^2+\frac{y}{m}+\left[\frac{2y+e^{-(m+1)y}-1-4(m+1)y^2}{m^2(m-1)}\right]\lambda;\\
S_{m,\lambda}(t^3;y) &= y^3+\frac{3y^2}{m}+\frac{y}{m^2}\\
&\quad+\left[\frac{1-e^{-(m+1)y}-2y+3(m-3)(m+1)y^2-6(m+1)y^3}{m^3(m-1)}\right]\lambda;\\
S_{m,\lambda}(t^4;y) &= y^4+\frac{6y^3}{m}+\frac{7y^2}{m^2}+\frac{y}{m^3}\\
&\quad+\left[\frac{e^{-(m+1)y}-1+2my+2(3m-11)(m+1)y^2}{m^4(m-1)}\right.\\
&\quad\left.+\frac{4(m-8)(m+1)^2y^3-8(m+1)^3y^4}{m^4(m-1)}\right]\lambda.
\end{aligned}$$

Lemma 2. *For the operators defined by (6), we obtain the following moments*

$$D_{m,\lambda}(1;y)=1; \tag{7}$$

$$D_{m,\lambda}(t;y)=y+\frac{1}{m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda; \tag{8}$$

$$D_{m,\lambda}(t^2;y)=y^2+\frac{4y}{m}+\frac{2}{m^2}+\left[\frac{1-e^{-(m+1)y}-2y-2(m+1)y^2}{m^2(m-1)}\right]2\lambda; \tag{9}$$

$$\begin{aligned}
D_{m,\lambda}(t^3;y) &= y^3+\frac{9y^2}{m}+\frac{18y}{m^2}+\frac{6}{m^3}\\
&\quad+\left[\frac{2-2e^{-(m+1)y}-4y+(m-11)(m+1)y^2-2(m+1)y^3}{m^3(m-1)}\right]3\lambda;
\end{aligned} \tag{10}$$

$$D_{m,\lambda}(t^4;y) = y^4 + \frac{16y^3}{m} + \frac{72y^2}{m^2} + \frac{96y}{m^3} + \frac{24}{m^4} + \left[\frac{24 - 24e^{-(m+1)y} + y(m-25) + 18(m-7)(m+1)y^2}{m^4(m-1)} - \frac{2(m^2-7m-23)(m+1)y^3 + 4(m+1)^3y^4}{m^4(m-1)}\right]2\lambda. \tag{11}$$

Proof. In view of the following relation

$$\int_0^\infty s_{m,j}(t)t^u dt = m^{-(u+1)}\frac{\Gamma(j+u+1)}{\Gamma(j+1)},$$

it is easy to get $\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;y) = 1$, hence we find (7).

Now, with the help of Lemma 1, we will compute the expressions (8) and (9).

$$\begin{aligned}
D_{m,\lambda}(t;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^\infty e^{-mt}\frac{(mt)^j}{j!}tdt \\
&= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{1}{m^2}\frac{\Gamma(j+2)}{\Gamma(j+1)} \\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j+1}{m} \\
&= S_{m,\lambda}(t;y) + \frac{1}{m}S_{m,\lambda}(1;y) \\
&= y + \frac{1}{m} + \left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda.
\end{aligned}$$

$$\begin{aligned}
D_{m,\lambda}(t^2;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^\infty e^{-mt}\frac{(mt)^j}{j!}tdt \\
&= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{1}{m^3}\frac{\Gamma(j+3)}{\Gamma(j+1)} \\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{(j+2)(j+1)}{m^2} \\
&= S_{m,\lambda}(t^2;y) + \frac{3}{m}S_{m,\lambda}(t;y) + \frac{2}{m^2}S_{m,\lambda}(1;y) \\
&= y^2 + \frac{4y}{m} + \frac{2}{m^2} + \left[\frac{1-e^{-(m+1)y}-2y-2(m+1)y^2}{m^2(m-1)}\right]2\lambda.
\end{aligned}$$

Similarly, from Lemma 1, we can get expressions (10) and (11) by simple computation, thus we have omitted details. □

Corollary 1. *As a consequence of Lemma 2, we arrive the following relations:*

$$
\begin{aligned}
(i)\ D_{m,\lambda}(t-y;y) &= \frac{1}{m} + \left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda \\
&\leq \frac{m+e^{-(m+1)y}+2y}{m(m-1)} := \alpha_m(y); \\
(ii)\ D_{m,\lambda}((t-y)^2;y) &= \frac{2y}{m} + \frac{2}{m^2} \\
&+ \left[\frac{1+(m-1)ye^{-(m+1)y}+(m+2)y+(2m-2(m+1)^2)y^2}{m^2(m-1)}\right]2\lambda \\
&\leq \frac{2y}{m} + \frac{2}{m^2} \\
&+ \frac{2+2(m-1)ye^{-(m+1)y}+2(m+2)y+2(2m-2(m+1)^2)y^2}{m^2(m-1)} \\
&:= \beta_m(y); \\
(iii)\ D_{m,\lambda}((t-y)^4;y) &= \frac{12y^2}{m^2} + \frac{48y}{m^3} + \frac{24}{m^4} \\
&+ \left(\frac{24-24e^{-(m+1)y}+y(m-25)+18(m-7)(m+1)y^2}{m^4(m-1)}\right. \\
&- \frac{2(m^2-7m-23)(m+1)y^3+4(m+1)^3y^4}{m^4(m-1)} \\
&+ \frac{12ye^{-(m+1)y}-12y+24y^2-6(m-11)(m+1)y^3+12(m+1)y^4}{m^3(m-1)} \\
&+ \frac{6y^2(1-e^{-(m+1)y})-12y^3-12(m+1)^2y^4}{m^2(m-1)} \\
&+ \left.\frac{2y^3(1-e^{-(m+1)y})+4y^4}{m(m-1)}\right)2\lambda.
\end{aligned}
$$

3. Direct Theorems of $D_{m,\lambda}$ Operators

In this section, we discuss the order of convergence in connection with the usual modulus of continuity, for the function belong to Lipschitz type class and Peetre's K-functional, respectively. Let the space $C_B[0,\infty)$ denotes the all continuous and bounded functions on $[0,\infty)$ and it has the sup-norm for a function μ as below:

$$\|\mu\|_{[0,\infty)} = \sup_{y\in[0,\infty)} |\mu(y)|.$$

The Peetre's K-functional is defined as

$$K_2(\mu,\eta)=\inf_{\nu\in C^2[0,\infty)}\left\{\|\mu-\nu\|+\eta\left\|\nu''\right\|\right\},$$

where $\eta>0$ and $C_B^2[0,\infty)=\left\{\nu\in C_B[0,\infty):\nu',\nu''\in C_B[0,\infty)\right\}$.

From [9], there exists an absolute constant $C>0$ such that

$$K_2(\mu;\eta)\leq C\omega_2(\mu;\sqrt{\eta}),\qquad \eta>0, \tag{12}$$

where

$$\omega_2(\mu;\eta)=\sup_{0<z\leq\eta}\sup_{y\in[0,\infty)}\left|\mu(y+2z)-2\mu(y+z)+\mu(y)\right|,$$

is the second order modulus of smoothness of the function $\mu\in C_B[0,\infty)$. Also, we define the usual modulus of continuity of $\mu\in C_B[0,\infty)$ as follows

$$\omega(\mu;\eta):=\sup_{0<\alpha\leq\eta}\sup_{y\in[0,\infty)}\left|\mu(y+z)-\mu(y)\right|.$$

Since $\eta>0$, $\omega(\mu;\eta)$ has some useful properties see details in [4].

Further, we give an elements of Lipschitz type continuous function with $Lip_L(\zeta)$, where $L>0$ and $0<\zeta\leq1$. If the following expression

$$\left|\mu(t)-\mu(y)\right|\leq L\left|t-y\right|^{\zeta},\qquad (t,y\in\mathbb{R}),$$

holds, then one can say a function μ belongs to $Lip_L(\zeta)$.

Theorem 1. *Let $\mu\in C_B[0,\infty)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, the following inequality is satisfied:*

$$\left|D_{m,\lambda}(\mu;y)-\mu(y)\right|\leq 2\omega(\mu;\sqrt{\beta_m(y)}),$$

where $\beta_m(y)$ given as in Corollary 1.

Proof. Using the well-known property of modulus of continuity $\left|\mu(t)-\mu(y)\right|\leq\left(1+\frac{|t-y|}{\delta}\right)\omega(\mu;\delta)$ and operating $D_{m,\lambda}(.;y)$, we arrive

$$\left|D_{m,\lambda}(\mu;y)-\mu(y)\right|\leq\left(1+\frac{1}{\delta}D_{m,\lambda}(\left|t-y\right|;y)\right)\omega(\mu;\delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality and by Corollary 1, we get

$$\begin{aligned}\left|D_{m,\lambda}(\mu;y)-\mu(y)\right|&\leq\left(1+\frac{1}{\delta}\sqrt{D_{m,\lambda}((t-y)^2;y)}\right)\omega(\mu;\delta)\\&\leq\left(1+\frac{1}{\delta}\sqrt{\beta_m(y)}\right)\omega(\mu;\delta).\end{aligned}$$

Choosing $\delta=\sqrt{\beta_m(y)}$, thus we have the proof of this theorem. □

Theorem 2. *Let $\mu\in Lip_L(\zeta)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, we obtain*

$$\left|D_{m,\lambda}(\mu;y)-\mu(y)\right|\leq L(\beta_m(y))^{\frac{\zeta}{2}}.$$

Proof. Taking into consideration the linearity and monotonicity properties of the operators (6), it gives

$$\begin{aligned} |D_{m,\lambda}(\mu;y)-\mu(y)| &\leq D_{m,\lambda}(|\mu(t)-\mu(y)|\,;y) \\ &\leq m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}e^{-mt}\frac{(mt)^j}{j!}\,|\mu\,(t)-\mu(y)|\,dt \\ &\leq L\left(m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}e^{-mt}\frac{(mt)^j}{j!}\,|t-y|^{\zeta}\,dt\right). \end{aligned}$$

Utilizing the Hölder's inequality with $p_1=\frac{2}{\zeta}$ and $p_2=\frac{2}{2-\zeta}$, from Corollary 1 and Lemma 2, we arrive

$$\begin{aligned} |D_{m,\lambda}(\mu;y)-\mu(y)| &\leq L\left\{m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}e^{-mt}\frac{(mt)^j}{j!}\,(t-y)^2\,dt\right\}^{\frac{\zeta}{2}} \\ &\quad\cdot\left\{\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\right\}^{\frac{2-\zeta}{2}} \\ &= L\left\{D_{m,\lambda}((t-y)^2;y)\right\}^{\frac{\zeta}{2}}\left\{D_{m,\lambda}(1;y)\right\}^{\frac{2-\zeta}{2}} \\ &\leq L(\beta_m(y))^{\frac{\zeta}{2}}. \end{aligned}$$

Hence, we obtain the required sequel. □

Theorem 3. *For all $\mu\in C_B[0,\infty)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$, the following inequality holds:*

$$|D_{m,\lambda}(\mu;y)-\mu(y)|\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y)+(\alpha_m(y))^2})+\omega(\mu;|\alpha_m(y)|),$$

where $C>0$ is a constant, $\alpha_m(y)$, $\beta_m(y)$ defined as in Corollary 1.

Proof. We denote $\gamma_{m,\lambda}(y):=y+\frac{1}{m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda$, it is obvious that $\gamma_{m,\lambda}(y)\in[0,\infty)$ for sufficently large m. We define the following auxiliary operators:

$$\widehat{D}_{m,\lambda}(\mu;y)=D_{m,\lambda}(\mu;y)-\mu(\gamma_{m,\lambda}(y))+\mu(y). \tag{13}$$

From (7) and (8), we find

$$\widehat{D}_{m,\lambda}(t-y;y)=0.$$

Using Taylor's formula, one has

$$\xi(t)=\xi(y)+(t-y)\xi'(y)+\int_y^t(t-u)\xi''(u)du,\quad(\xi\in C_B^2[0,\infty)). \tag{14}$$

Operating $\widehat{D}_{m,\lambda}(.;y)$ to (14), it gives

$$\begin{aligned}
\widehat{D}_{m,\lambda}(\xi;y)-\xi(y) &= \widehat{D}_{m,\lambda}((t-y)\xi'(y);y)+\widehat{D}_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y) \\
&= \xi'(y)\widehat{D}_{m,\lambda}(t-y;y)+D_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y) \\
&- \int_y^{\gamma_{m,\lambda}(y)} (\gamma_{m,\lambda}(y)-u)\xi''(u)du \\
&= D_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)-\int_y^{\gamma_{m,\lambda}(y)} (\gamma_{m,\lambda}(y)-u)\xi''(u)du.
\end{aligned}$$

Taking Lemma 2 and (13) into the account, we get

$$\begin{aligned}
\left|\widehat{D}_{m,\lambda}(\xi;y)-\xi(y)\right| &\leq \left|D_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)\right| \\
&+\left|\int_y^{\gamma_{m,\lambda}(y)} (\gamma_{m,\lambda}(y)-u)\xi''(u)du\right| \\
&\leq D_{m,\lambda}(\int_y^t (t-u)\left|\xi''(u)\right|du;y) \\
&+\int_y^{\gamma_{m,\lambda}(y)} \left(\gamma_{m,\lambda}(y)-u\right)\left|\xi''(u)\right|du \\
&\leq \left\|\xi''\right\|\left\{D_{m,\lambda}((t-y)^2;y)+\left(\gamma_{m,\lambda}(y)-y\right)^2\right\} \\
&\leq \left\{\beta_m(y)+(\alpha_m(y))^2\right\}\left\|\xi''\right\|.
\end{aligned}$$

From (7), (8) and (13), it deduces the following

$$\left|\widehat{D}_{m,\lambda}(\mu;y)\right| \leq |D_{m,\lambda}(\mu;y)|+2\|\mu\| \leq \|\mu\| D_{m,\lambda}(1;y)+2\|\mu\| \leq 3\|\mu\|.$$

Also by (14) and using above relation, we get

$$|D_{m,\lambda}(\mu;y)-\mu(y)| \leq \left|\widehat{D}_{m,\lambda}(\mu-\xi;y)-(\mu-\xi)(y)\right|$$

$$+\left|\widehat{D}_{m,\lambda}(\xi;y)-\xi(y)\right|+|\mu(y)-\mu(\alpha_{m,\lambda}(y))|$$
$$\leq 4\|\mu-\xi\|+\left\{\beta_m(y)+(\gamma_{m,\lambda}(y))^2\right\}\left\|\xi''\right\|+\omega\left(\mu;|\alpha_m(y)|\right).$$

Hence, if we take the infimum on the right hand side over all $\xi \in C_B^2[0,\infty)$ and by (12), we arrive

$$|D_{m,\lambda}(\mu;y)-\mu(y)| \leq 4K_2(\mu;\frac{\left\{\beta_m(y)+(\alpha_m(y))^2\right\}}{4})+\omega(\mu;|\alpha_m(y)|)$$
$$\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y)+(\alpha_m(y))^2})+\omega(\mu;|\alpha_m(y)|).$$

Thus, the proof is completed. □

4. Weighted Approximation

In this section, we will establish the Korovkin type convergence theorem on weighted spaces. Let $B_{y^2}[0,\infty)$ be the space of all functions κ verifying the condition $|\kappa(y)| \leq M_\kappa(1+y^2)$, $y \in [0,\infty)$ with constant M_κ, which depend only on κ. We denote with $C_{y^2}[0,\infty)$ the set of all continuous functions belonging to $B_{y^2}[0,\infty)$ and it is endowed with the norm $\|\kappa\|_{y^2} = \sup\limits_{y\in[0,\infty)} \frac{|\kappa(y)|}{1+y^2}$ and also we define $C^*_{y^2}[0,\infty) := \{\kappa :\ \kappa \in C_{y^2}[0,\infty),\ \lim\limits_{y\to\infty} \frac{|\kappa(y)|}{1+y^2} < \infty\}$.

Theorem 4. *For all $\mu \in C^*_{y^2}[0,\infty)$, we arrive*

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(\mu;y)-\mu(y)|}{1+y^2}=0.$$

Proof. Considering to the Korovkin type convergence theorem presented by Gadzhiev [10], we want to show that operators (3) verifies the following condition:

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(t^s;y)-y^s|}{1+y^2}=0, \qquad s=0,1,2. \tag{15}$$

By (7), the first condition in (15) is clear for $s=0$.

For $s=1$, using (8), we have

$$\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(t;y)-y|}{1+y^2} \leq \left|\frac{m-1+\lambda}{m(m-1)}\right|\sup_{y\in[0,\infty)}\frac{1}{1+y^2}+\left|\frac{3\lambda}{m(m-1)}\right|\sup_{y\in[0,\infty)}\frac{y}{1+y^2}.$$

Hence,

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(t;y)-y|}{1+y^2}=0.$$

Similarly for $s=2$, using (9), we get

$$\sup_{y\in[0,\infty)} \frac{\left|D_{m,\lambda}(t^2;y)-y^2\right|}{1+y^2} \le \left|\frac{2((m-1)+\lambda)}{m^2(m-1)}\right| \sup_{y\in[0,\infty)} \frac{1}{1+y^2}$$
$$+\left|\frac{4m(m-1)-6\lambda}{m^2(m-1)}\right| \sup_{y\in[0,\infty)} \frac{y}{1+y^2} + \left|\frac{4(m+1)\lambda}{m^2(m-1)}\right| \sup_{y\in[0,\infty)} \frac{y^2}{1+y^2}.$$

It follows

$$\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{\left|D_{m,\lambda}(t^2;y)-y^2\right|}{1+y^2} = 0.$$

This gives the proof of this theorem. □

5. Voronovskaya Type Asymptotic Theorem

In this section, we will prove Voronovskaya type asymptotic theorem. Firstly we consider the following lemma, which we will use in the proof of our main theorem.

Lemma 3. *Let $y \in [0,\infty)$ and $\lambda \in [-1,1]$. Then, the following results are satisfied:*

$$(i) \lim_{m\to\infty} mD_{m,\lambda}(t-y;y) = 1,$$
$$(ii) \lim_{m\to\infty} mD_{m,\lambda}((t-y)^2;y) = 2y(1-2y),$$
$$(iii) \lim_{m\to\infty} m^2 D_{m,\lambda}((t-y)^4;y) = 4y^2(1-y)(2y+3).$$

Theorem 5. *Let $\mu \in C_{y^2}[0,\infty)$ such that $\mu', \mu'' \in C_{y^2}[0,\infty)$ and $\lambda \in [-1,1]$, then we have for any $y \in [0,\infty)$ that*

$$\lim_{m\to\infty} m\left[D_{m,\lambda}(\mu;y) - \mu(y)\right] = \mu'(y) + y(1-2y)\mu''(y).$$

Proof. Suppose that $\mu, \mu', \mu'' \in C_{y^2}[0,\infty)$ and $y \in [0,\infty)$. Using Taylor's expansion formula, we find

$$\mu(t) = \mu(y) + (t-y)\mu'(y) + \frac{1}{2}(t-y)^2\mu''(y) + (t-y)^2\Delta(t;y). \tag{16}$$

In (16), $\Delta(t;y)$ is a Peano of the remainder term and by the fact that $\Delta(.;y) \in C^*_{y^2}[0,\infty)$, we arrive $\lim\limits_{t\to y}\Delta(t;y) = 0$.

After operating $D_{m,\lambda}(.;y)$ to (16), then

$$D_{m,\lambda}(\mu;y) - \mu(y) = D_{m,\lambda}((t-y);y)\mu'(y) + \frac{1}{2}D_{m,\lambda}((t-y)^2;y)\mu''(y)$$
$$+ D_{m,\lambda}((t-y)^2\Delta(t;y);y).$$

If we take the limit of the both sides of above expression as $m \to \infty$, hence

$$\lim_{m\to\infty} m(D_{m,\lambda}(\mu;y)-\mu(y))$$
$$= \lim_{m\to\infty} m\left(D_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}D_{m,\lambda}((t-y)^2;y)\mu''(y)+D_{m,\lambda}((t-y)^2\Delta(t;y);y)\right). \tag{17}$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality to the last term on the right hand side of the above relation, it gives

$$\lim_{m\to\infty} mD_{m,\lambda}((t-y)^2\Delta(t;y);y) \leq \sqrt{\lim_{m\to\infty} D_{m,\lambda}(\Delta^2(t;y);y)}\sqrt{\lim_{m\to\infty} m^2 D_{m,\lambda}((t-y)^4;y)}.$$

Since $\Delta(t;y)\in C_{y^2}[0,\infty)$, then from Theorem 4, $\lim_{t\to y}\Delta(t;y)=0$. It becomes

$$\lim_{m\to\infty} D_{m,\lambda}(\Delta^2(t;y);y)=\Delta^2(y;y)=0. \tag{18}$$

Combining (17)-(18) and by Lemma 3 (iii), one has

$$\lim_{m\to\infty} mD_{m,\lambda}((t-y)^2\Delta(t;y);y)=0.$$

Hence, we obtain the following desired sequel

$$\lim_{m\to\infty} m\left[D_{m,\lambda}(\mu;y)-\mu(y)\right]=\mu'(y)+y(1-2y)\mu''(y).$$

□

6. Graphical and Numerical Analysis

In this section, with the aid of Maple software, we present some graphics and an error of approximation table to see the convergence of operators (6) to certain functions with the different values of m and λ parameters.

In Figure 1, we show the convergence of operators (6) to the function $\mu(y) = ysin(y)/2$ (black) for $\lambda=1$, $m=10$ (red), $m=30$ (green) and $m=75$ (blue). In Figure 2, we show the convergence of operators (6) to the function $\mu(y)=ysin(y)/2$ (black) for $\lambda=-1$, $m=10$ (red), $m=30$ (green) and $m=75$ (blue). It is obvious from Figure 1 and Figure 2 that, as the values of m increases than the convergence of operators (6) to the functions $\mu(y)$ becomes better. In Figure 3, we compare the convergence of operators (3) (green) and operators (6) (red) with the function $\mu(y)=1-\sin(\pi y)$ (black) for $\lambda=1$ and $m=10$. It is clear from Figure 3 that, operators (6) has better approximation than operators (3). Also, in Table 1, we present an error of approximation of operators (6) to function $\mu(y)=ysin(y)/2$ for the certain values of m and $\lambda\in[-1,1]$. We can check from Table 1 that, as the value of m increases than the error of approximation of operators (6) to $\mu(y)$ is decreases. One the other hand, for $\lambda>0$, the absolute difference between operators (6) and $\mu(y)$ is smaller than between operators (3) and $\mu(y)$.

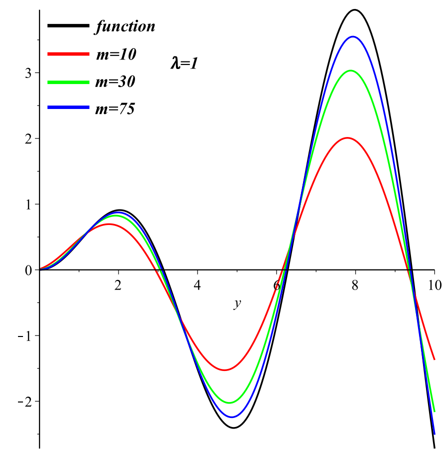

FIGURE 1. The convergence of operators $D_{m,\lambda}(\mu; y)$ to the function $\mu(y) = ysin(y)/2$ for $\lambda = 1$ and $m = 10, 30, 75$

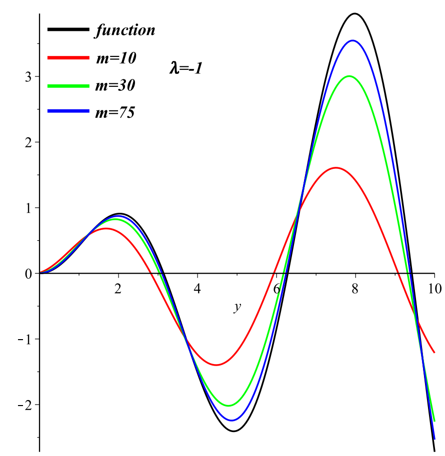

FIGURE 2. The convergence of operators $D_{m,\lambda}(\mu; y)$ to the function $\mu(y) = ysin(y)/2$ for $\lambda = -1$ and $m = 10, 30, 75$

Declaration of Competing Interests The author declares no competing interests.

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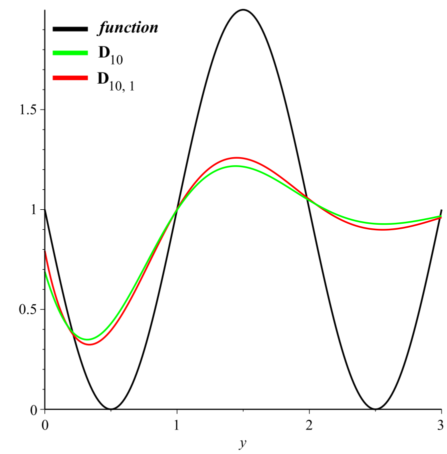

FIGURE 3. The convergence of operators $D_{m,\lambda}(\mu;y)$ and $D_m(\mu;y)$ to the function $\mu(y) = 1 - \sin(\pi y)$ for $\lambda = 1$ and $m = 10$

TABLE 1. Error of approximation $D_{m,\lambda}(\mu;y)$ operators to $\mu(y) = ysin(y)/2$ for $m = 10, 30, 75, 150$

λ	$\lvert\mu(y) - D_{m,\lambda}(\mu;y)\rvert$			
	$m = 10$	$m = 30$	$m = 75$	$m = 150$
-1	0.0779267654	0.0274289801	0.0110996263	0.0055687488
-0.75	0.0778118774	0.0274227761	0.0110991971	0.0055686940
0	0.0774672138	0.0274041639	0.0110979093	0.0055685292
0.75	0.0771225502	0.0273855517	0.0110966215	0.0055683644
1	0.0770076622	0.0273793477	0.0110961923	0.0055683096

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