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On New Inequalities Involving AB-fractional Integrals for Some Convexity Classes

Ali Karaoğlan [1], **Barış Çelik** [1], **Erhan Set** [1]*, **Ahmet Ocak Akdemir** [2]
[1] Ordu University, Faculty of Arts and Sciences, Department of Mathematics
Ordu, Türkiye, alikaraoglan@odu.edu.tr, bariscelik15@hotmail.com
[2] Ağrı İbrahim Çeçen University, Faculty of Arts and Sciences, Department of Mathematics
Ağrı, Türkiye, aocakakdemir@gmail.com

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Abstract: During the past two decades or so, fractional integral operators have been one of the most important tools in the development of inequality theory. By this means, a lot of generalized integral inequalities involving various fractional integral operators have been presented in the literature. Very recently, Atangana-Baleanu fractional integral operators has been introduced by Atangana and Baleanu and with the help of these operators some new integral inequalities are obtained. The main aim of the paper is to establish some new integral inequalities for quasi-convex function and P-function via Atangana-Baleanu integral operators by using identity which was produced by Set *et al.* in [16].

Keywords: Differentiable convex functions, Hölder inequality, Young inequality, power mean inequality, Atangana-Baleanu fractional integral.

1. Introduction

Convex functions, which are of great importance for the inequality theory, have been the focus of many researchers from the past to the present. Let's start by remembering this useful function class.

We use an interval $I \subseteq \mathbb{R}$ with the nonempty interior.

Definition 1.1 *A function $\Upsilon : I \to \mathbb{R}$ is said to be convex if the inequality*

$$\Upsilon(\lambda x + (1-\lambda)y) \leq \lambda\Upsilon(x) + (1-\lambda)\Upsilon(y) \tag{1}$$

holds for all $x, y \in I$ and all $\lambda \in [0,1]$. We say that Υ is concave if $-\Upsilon$ is convex.

There are many types of convexity in the literature. The two types of convexity that will be used in this article are as follows.

*Correspondence: erhanset@yahoo.com
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Definition 1.2 [10] *The function* $\Upsilon : I \to \mathbb{R}$ *is called a quasi-convex if the following inequality*

$$\Upsilon\left(\lambda x + (1-\lambda)y\right) \leq \max\{\Upsilon(x), \Upsilon(y)\}$$

holds for all $x, y \in I$ *and all* $\lambda \in [0,1]$.

Definition 1.3 [14] *A function* $\Upsilon : I \to \mathbb{R}$ *is* P*-function or that* Υ *belongs to the class of* $P(I)$, *if it is nonnegative andsatisfies the inequality*

$$\Upsilon\big(\lambda x + (1-\lambda)y\big) \leq \Upsilon(x) + \Upsilon(y) \tag{2}$$

for all $x, y \in I$ *and all* $\lambda \in [0,1]$.

After recalling the definitions of convex function, quasi-convex function, and P-function, let us now give other definitions that we will use in this article.

$B(\alpha)$ is normalization function with $B(0) = B(1) = 1$.

Definition 1.4 [7] *Let* $\Upsilon \in H^1(a,b)$, $b > a$ $\alpha \in [0,1]$. *Then, the definition of the new fractional derivative is given:*

$${}_{a}^{ABC}D_{\kappa}^{\alpha}\left[\Upsilon(\kappa)\right] = \frac{B(\alpha)}{1-\alpha}\int_{a}^{\kappa} \Upsilon'(x) E_{\alpha}\left[-\alpha\frac{(\kappa-x)^{\alpha}}{(1-\alpha)}\right] dx. \tag{3}$$

Definition 1.5 [7] *Let* $\Upsilon \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$. *Then, the definition of the new fractional derivative is given:*

$${}_{a}^{ABR}D_{\kappa}^{\alpha}\left[\Upsilon(\kappa)\right] = \frac{B(\alpha)}{1-\alpha}\frac{d}{d\kappa}\int_{a}^{\kappa} \Upsilon(x) E_{\alpha}\left[-\alpha\frac{(\kappa-x)^{\alpha}}{(1-\alpha)}\right] dx. \tag{4}$$

Equations (3) and (4) have a non-local kernel. Also, in Equation (4) when the function is constant we get zero.

Associated integral operator for Atangana-Baleanu fractional derivative has been defined as follows.

Definition 1.6 [7] *The fractional integral associate to the new fractional derivative with non-local kernel of a function* $\Upsilon \in H^1(a,b)$ *as defined:*

$${}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} = \frac{1-\alpha}{B(\alpha)}\Upsilon(\kappa) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{a}^{\kappa} \Upsilon(y)(\kappa-y)^{\alpha-1} dy$$

where $b > a, \alpha \in [0,1]$.

In [1], the authors have given the right hand side of integral operator as following;

$$^{AB}I_b^\alpha\left\{\Upsilon(\kappa)\right\} = \frac{1-\alpha}{B(\alpha)}\Upsilon(\kappa) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_\kappa^b \Upsilon(y)(y-\kappa)^{\alpha-1}dy.$$

For some recent results connected with fractional integral, fractional derivatives and some different convex classes, we suggest to the interested readers the papers [2–6, 8, 9, 11–13, 15, 17–19].

The main aim of this paper is to prove new integral inequalities using Atangana-Baleanu integral operators for quasi-convex and P-function. In this paper, integration techniques, quasi-convex function and P-function definition were used and new integral inequalities that produce different boundaries were proved.

2. Main Results

In the main findings part, first of all, an integral identity is reminded. Then, new integral inequalities are obtained by using this integral identity, some convex function types and some basic inequality derivation methods. Since the results contain Atangana-Baleanu fractional integral operators, new integral inequalities, whose proofs are given, introduce new approaches and add a new dimension to the literature.

In [16], Set *et al.* proved identity for Atangana-Baleanu fractional integral operators as following. We use a closed interval $[a,b]\subset\mathbb{R}$ with $a<b$.

Lemma 2.1 *$\Upsilon:[a,b]\to\mathbb{R}$ be differentiable function on (a,b). Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&{}^{AB}_{\ a}I\alpha\left\{\Upsilon(\kappa)\right\} +^{AB} I_b^\alpha\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^\alpha\Upsilon(a)+(b-\kappa)^\alpha\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\\
= \quad & \frac{(\kappa-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\Upsilon'(a) + \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha+1}\Upsilon''\left(k\kappa+(1-\kappa)a\right)dk\\
&-\frac{(b-\kappa)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\Upsilon'(b) + \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\Upsilon''\left(kb+(1-k)\kappa\right)dk
\end{aligned}$$

where $\alpha\in[0,1],\kappa\in[a,b]$ and $\Gamma(.)$ is Gamma function.

In this part, we obtained some fractional integral inequalities for quasi-convex function and P-function by using Atangana-Baleanu fractional integral operators as follows.

Theorem 2.2 *$\Upsilon:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) and $\Upsilon''\in L_1[a,b]$. If $|\Upsilon''|$ is a quasi-convex function, we have the following inequality for Atangana-Baleanu fractional integral* *operators*

$$\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.$$
$$\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|$$
$$\leq \frac{(\kappa-a)^{\alpha+2}\max\left\{|\Upsilon''(\kappa)|,|\Upsilon''(a)|\right\}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} + \frac{(b-\kappa)^{\alpha+2}\max\left\{|\Upsilon''(b)|,|\Upsilon''(\kappa)|\right\}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}$$

where $\kappa \in [a,b]$, $\alpha \in [0,1]$, $\Gamma(.)$ *is Gamma function and* $B(\alpha)$ *is normalization function.*

Proof By using the identity that is given in Lemma 2.1, we can write

$$\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.$$
$$\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|$$
$$= \left| \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1}\Upsilon''\left(k\kappa+(1-k)a\right)dk \right.$$
$$\left. + \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1}\Upsilon''\left(kb+(1-k)\kappa\right)dk \right|$$
$$\leq \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|dk$$
$$+ \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|dk.$$

By using quasi-convexity of $|\Upsilon''|$, we get

$$\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} \right.$$
$$\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|$$
$$\leq \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1}\max\{|\Upsilon''(\kappa)|,|\Upsilon''(a)|\}dk$$
$$+ \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1}\max\{|\Upsilon''(b)|,|\Upsilon''(\kappa)|\}dk$$
$$= \frac{(\kappa-a)^{\alpha+2}\max\left\{|\Upsilon''(\kappa)|,|\Upsilon''(a)|\right\}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} + \frac{(b-\kappa)^{\alpha+2}\max\left\{|\Upsilon''(b)|,|\Upsilon''(\kappa)|\right\}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}$$

and the proof is completed. □

Corollary 2.3 *In Theorem 2.2, if we choose $\kappa = \frac{a+b}{2}$, we obtain*

$$\begin{aligned}&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big]\right.\\ &\left. -\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|\\ \leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\max\left\{\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|,\left|\Upsilon''(a)\right|\right\} + \max\left\{\left|\Upsilon''(b)\right|,\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|\right\}\right].\end{aligned}$$

Corollary 2.4 *In Theorem 2.2, if we choose $\alpha = 1$, we obtain*

$$\begin{aligned}&\left|\int_a^b \Upsilon(x)dx - [(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)] - \frac{(\kappa-a)^2\Upsilon'(a)-(b-\kappa)^2 f'(b)}{2}\right|\\ \leq\ & \frac{1}{6}\left[(\kappa-a)^3\max\{|\Upsilon''(\kappa)|,|\Upsilon''(a)|\} + (b-\kappa)^2\max\{|\Upsilon''(b)|,|\Upsilon''(\kappa)|\}\right].\end{aligned}$$

Theorem 2.5 *$\Upsilon : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) and $\Upsilon'' \in L_1[a,b]$. If $|\Upsilon''|^q$ is a quasi-convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\left| {}_{a}^{AB}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I_{b}^{\alpha}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)}\right.\\ &\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\ \leq\ & \frac{(\kappa-a)^{\alpha+2}\left(\max\{|\Upsilon''(\kappa)|^q,|\Upsilon''(a)|^q\}\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\\ &+\frac{(b-\kappa)^{\alpha+2}\left(\max\{|\Upsilon''(b)|^q,|\Upsilon''(\kappa)|^q\}\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $\kappa\in[a,b]$, $\alpha\in[0,1]$, $q>1$, $\Gamma(.)$ is Gamma function and $B(\alpha)$ is normalization function.

Proof By using Lemma 2.1, we have

$$\begin{aligned}
&\Bigg| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|dk \\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|dk.
\end{aligned}$$

By applying Hölder inequality, we have

$$\begin{aligned}
&\Bigg| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|^q dk\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|^q dk\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By using quasi-convexity of $|\Upsilon''|^q$, we obtain

$$\begin{aligned}
&\Bigg| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \max\{|\Upsilon''(\kappa)|^q,|\Upsilon''(a)|^q\}dk\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \max\{|\Upsilon''(b)|^q,|\Upsilon''(\kappa)|^q\}dk\right)^{\frac{1}{q}}\right] \\
=\ & \frac{(\kappa-a)^{\alpha+2}\left(\max\{|\Upsilon''(\kappa)|^q,|\Upsilon''(a)|^q\}\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}} \\
&+\frac{(b-\kappa)^{\alpha+2}\left(\max\{|\Upsilon''(b)|^q,|\Upsilon''(\kappa)|^q\}\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}.
\end{aligned}$$

So, the proof is completed. □

Corollary 2.6 *In Theorem 2.5, if we choose* $\kappa = \frac{a+b}{2}$, *we obtain*

$$\begin{aligned}
&\Bigg| {}^{AB}_{a}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \\
&-\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left[\left(\max\left\{\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^{q}, |\Upsilon''(a)|^{q}\right\}\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\max\left\{|\Upsilon''(b)|^{q}, \left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2.7 *In Theorem 2.5, if we choose* $\alpha = 1$, *we obtain*

$$\begin{aligned}
&\left|\int_a^b \Upsilon(x)dx - [(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)] - \frac{(\kappa-a)^2\Upsilon'(a)-(b-\kappa)^2\Upsilon'(b)}{2}\right| \\
\leq\ & \frac{1}{2}\left[(\kappa-a)^3\left(\max\{|\Upsilon''(\kappa)|^q, |\Upsilon''(a)|^q\}\right)^{\frac{1}{q}} + (b-\kappa)^3\left(\max\{|\Upsilon''(b)|^q, |\Upsilon''(\kappa)|^q\}\right)^{\frac{1}{q}}\right]\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}.
\end{aligned}$$

Theorem 2.8 $\Upsilon : [a,b] \to \mathbb{R}$ *be differentiable function on* (a,b) *and* $\Upsilon'' \in L_1[a,b]$. *If* $|\Upsilon''|^q$ *is a quasi-convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\Bigg| {}^{AB}_{a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\max\{|\Upsilon''(\kappa)|^q, |\Upsilon''(a)|^q\}}{\alpha+2}\right)^{\frac{1}{q}} \\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\max\{|\Upsilon''(b)|^q, |\Upsilon''(\kappa)|^q\}}{\alpha+2}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\kappa \in [a,b]$, $\alpha \in [0,1]$, $q \geq 1$, $B(\alpha)$ *is normalization function.*

Proof By using Lemma 2.1, we get

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)}\right.\\
&\left. - \frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''(k\kappa+(1-k)a)\right|dk\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\left|\Upsilon''(kb+(1-k)\kappa)\right|dk.
\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)}\right.\\
&\left. - \frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''(k\kappa+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha+1}\left|\Upsilon''(kb+(1-k)\kappa)\right|^q dk\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By using quasi-convexity of $|\Upsilon''|^q$, we obtain

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\right.\\
&\left. - \frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-k)^{\alpha+1}\max\{\left|\Upsilon''(\kappa)\right|^q,\left|\Upsilon''(a)\right|^q\}dk\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha+1}\max\{\left|\Upsilon''(b)\right|^q,\left|\Upsilon''(t)\right|^q\}dk\right)^{\frac{1}{q}}\right]\\
=\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\max\{|\Upsilon''(\kappa)|^q,|\Upsilon''(a)|^q\}}{\alpha+2}\right)^{\frac{1}{q}}\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\max\{|\Upsilon''(b)|^q,|\Upsilon''(\kappa)|^q\}}{\alpha+2}\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is completed. □

Corollary 2.9 *In Theorem 2.8, if we choose* $\kappa = \frac{a+b}{2}$, *we obtain*

$$\begin{aligned}&\Bigg| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \\ &-\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)}\Bigg| \\ \leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{\max\left\{\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^{q}, |\Upsilon''(a)|^{q}\right\}}{\alpha+2}\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\frac{\max\left\{|\Upsilon''(b)|^{q}, \left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^{q}\right\}}{\alpha+2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 2.10 *In Theorem 2.8, if we choose* $\alpha = 1$, *we obtain*

$$\begin{aligned}&\left|\int_{a}^{b}\Upsilon(x)dx - [(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)] - \frac{(\kappa-a)^{2}\Upsilon'(a)-(b-\kappa)^{2}f'(b)}{2}\right| \\ \leq\ & \left(\frac{1}{3}\right)^{1-\frac{1}{q}}\frac{(\kappa-a)^{3}}{2}\left(\frac{\max\{|\Upsilon''(\kappa)|^{q}, |\Upsilon''(a)|^{q}\}}{3}\right)^{\frac{1}{q}} + \left(\frac{1}{3}\right)^{1-\frac{1}{q}}\frac{(b-\kappa)^{3}}{2}\left(\frac{\max\{|\Upsilon''(b)|^{q}, |\Upsilon''(\kappa)|^{q}\}}{3}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.11 $\Upsilon : [a,b] \to \mathbb{R}$ *be differentiable function on* (a,b) *and* $\Upsilon'' \in L_1[a,b]$. *If* $|\Upsilon''|^q$ *is a quasi-convex function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\Bigg| {}_{a}^{AB}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I_{b}^{\alpha}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\ &-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\Bigg| \\ \leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)} + \frac{\max\{|\Upsilon''(\kappa)|^{q}, |\Upsilon''(a)|^{q}\}}{q}\right) \\ &+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)} + \frac{\max\{|\Upsilon''(b)|^{q}, |\Upsilon''(\kappa)|^{q}\}}{q}\right)\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $\kappa \in [a,b]$, $\alpha \in [0,1]$, $q > 1$, $\Gamma(.)$ *is Gamma function and* $B(\alpha)$ *is normalization function.*

Proof By using the identity that is given in Lemma 2.1, we get

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.\\
&\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|dk\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|dk.
\end{aligned}$$

By using the Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$,

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}f(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.\\
&\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 (1-k)^{(\alpha+1)p}dk + \frac{1}{q}\int_0^1 \left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|^q dk\right]\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{(\alpha+1)p}dk + \frac{1}{q}\int_0^1 \left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|^q dk\right].
\end{aligned}$$

By using quasi-convexity of $|\Upsilon''|^q$ and by a simple computation, we have the desired result. □

Corollary 2.12 *In Theorem 2.11, if we choose $\kappa = \frac{a+b}{2}$, we obtain*

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \right.\\
&\left. -\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)} \right|\\
\leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)B(\alpha)\Gamma(\alpha)}\\
&\left(\frac{2}{p(\alpha p+p+1)} + \frac{\max\left\{\left|\Upsilon''(\frac{a+b}{2})\right|^q, \left|\Upsilon''(a)\right|^q\right\} + \max\left\{\left|\Upsilon''(b)\right|^q, \left|\Upsilon''(\frac{a+b}{2})\right|^q\right\}}{q}\right).
\end{aligned}$$

Corollary 2.13 *In Theorem 2.11, if we choose* $\alpha = 1$, *we obtain*

$$\begin{aligned}
&\left| \int_a^b \Upsilon(x)dx - [(\kappa - a)\Upsilon(a) + (b-\kappa)\Upsilon(b)] - \frac{(\kappa - a)^2\Upsilon'(a) - (b-\kappa)^2 f'(b)}{2} \right| \\
\leq\ & \frac{(\kappa - a)^3}{2}\left(\frac{1}{p(2p+1)} + \frac{\max\{|\Upsilon''(\kappa)|^q, |\Upsilon''(a)|^q\}}{q}\right) \\
&+ \frac{(b-\kappa)^3}{2}\left(\frac{1}{p(2p+1)} + \frac{\max\{|\Upsilon''(b)|^q, |\Upsilon''(\kappa)|^q\}}{q}\right).
\end{aligned}$$

Theorem 2.14 $\Upsilon : [a,b] \to \mathbb{R}$ *be differentiable function on* (a,b) *and* $\Upsilon'' \in L_1[a,b]$. *If* $|\Upsilon''|$ *is a* P*-function, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa - a)^{\alpha}\Upsilon(a) + (b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. - \frac{(\kappa - a)^{\alpha+1}\Upsilon'(a) - (b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right| \\
\leq\ & \frac{(\kappa - a)^{\alpha+2}\left(|\Upsilon''(\kappa)| + |\Upsilon''(a)|\right)}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} + \frac{(b-\kappa)^{\alpha+2}\left(|\Upsilon''(b)| + |\Upsilon''(\kappa)|\right)}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}
\end{aligned}$$

where $\kappa \in [a,b]$, $\alpha \in [0,1]$, $\Gamma(.)$ *is Gamma function and* $B(\alpha)$ *is normalization function.*

Proof By using the identity that is given in Lemma 2.1, we can write

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa - a)^{\alpha}\Upsilon(a) + (b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. - \frac{(\kappa - a)^{\alpha+1}\Upsilon'(a) - (b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right| \\
\leq\ & \frac{(\kappa - a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa + (1-k)a\right)\right|dk \\
&+ \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1}\left|\Upsilon''\left(kb + (1-k)\kappa\right)\right|dk.
\end{aligned}$$

Since $|\Upsilon''|$ is a P-function, we obtain

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1}\left(|\Upsilon''(\kappa)|+|\Upsilon''(a)|\right)dk \\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1}\left(|\Upsilon''(b)|+|\Upsilon''(\kappa)|\right)dk \\
=\ & \frac{(\kappa-a)^{\alpha+2}\left(|\Upsilon''(\kappa)|+|\Upsilon''(a)|\right)}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} + \frac{(b-\kappa)^{\alpha+2}\left(|\Upsilon''(b)|+|\Upsilon''(\kappa)|\right)}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}
\end{aligned}$$

and the proof is completed. □

Corollary 2.15 *In Theorem 2.14, if we choose* $\kappa = \frac{a+b}{2}$, *we obtain*

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \right. \\
&\left. -\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)} \right| \\
\leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[2\left|\Upsilon''(\frac{a+b}{2})\right| + |\Upsilon''(a)| + |\Upsilon''(b)|\right].
\end{aligned}$$

Corollary 2.16 *In Theorem 2.14, if we choose* $\alpha = 1$, *we obtain*

$$\begin{aligned}
&\left| \int_a^b \Upsilon(x)dx - \left[(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)\right] - \frac{(\kappa-a)^2\Upsilon'(a)-(b-\kappa)^2\Upsilon'(b)}{2} \right| \\
\leq\ & \frac{1}{6}\left[(\kappa-a)^3\left(|\Upsilon''(\kappa)|+|\Upsilon''(a)|\right)+(b-\kappa)^3\left(|\Upsilon''(b)|+|\Upsilon''(\kappa)|\right)\right].
\end{aligned}$$

Theorem 2.17 $\Upsilon : [a,b] \to \mathbb{R}$ *be differentiable function on* (a,b) *and* $\Upsilon'' \in L_1[a,b]$. *If* $|\Upsilon''|^q$ *is a* P*-function, then we have the following inequality for Atangana-Baleanu fractional integral* *operators*

$$\begin{aligned}&\left|{}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\}+{}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\}-\frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)}\right.\\&\left.-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\\leq\ &\frac{(\kappa-a)^{\alpha+2}\left(|\Upsilon''(\kappa)|^{q}+|\Upsilon''(a)|^{q}\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\\&+\frac{(b-\kappa)^{\alpha+2}\left(|\Upsilon''(b)|^{q}+|\Upsilon''(\kappa)|^{q}\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\end{aligned}$$

where $p^{-1}+q^{-1}=1,\ \kappa\in[a,b],\ \alpha\in[0,1],\ q>1,\ B(\alpha)$ *is normalization function.*

Proof By using Lemma 2.1, we get

$$\begin{aligned}&\left|{}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\}+{}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\}-\frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)}\right.\\&\left.-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\\leq\ &\frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_{0}^{1}(1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|dk\\&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_{0}^{1}k^{\alpha+1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|dk.\end{aligned}$$

By applying Hölder inequality, we get

$$\begin{aligned}&\left|{}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\}+{}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\}-\frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)}\right.\\&\left.-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\right|\\\leq\ &\frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_{0}^{1}(1-k)^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|^{q}dk\right)^{\frac{1}{q}}\right]\\&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_{0}^{1}k^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|^{q}dk\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $|\Upsilon''|^q$ is a P-function, we obtain

$$\begin{aligned}
&\Bigg| {}^{AB}_{\ a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&-\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left(|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q\right)dk\right)^{\frac{1}{q}}\right] \\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left(|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q\right)dk\right)^{\frac{1}{q}}\right] \\
=\ & \frac{(\kappa-a)^{\alpha+2}\left(|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}} \\
&+\frac{(b-\kappa)^{\alpha+2}\left(|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q\right)^{\frac{1}{q}}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}.
\end{aligned}$$

So, the proof is completed. □

Corollary 2.18 *In Theorem 2.17, if we choose* $\kappa=\frac{a+b}{2}$, *we obtain*

$$\begin{aligned}
&\Bigg| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \\
&-\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left[\left(\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^q+|\Upsilon''(a)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(|\Upsilon''(b)|^q+\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2.19 *In Theorem 2.17, if we choose* $\alpha=1$, *we obtain*

$$\begin{aligned}
&\left|\int_a^b \Upsilon(x)dx - [(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)] - \frac{(\kappa-a)^2\Upsilon'(a)-(b-\kappa)^2\Upsilon'(b)}{2}\right| \\
\leq\ & \frac{1}{2}\left[(\kappa-a)^3\left(|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q\right)^{\frac{1}{q}}+(b-\kappa)^3\left(|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q\right)^{\frac{1}{q}}\right]\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}.
\end{aligned}$$

Theorem 2.20 *$\Upsilon : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) and $\Upsilon'' \in L_1[a,b]$. If $|\Upsilon''|^q$ is a P-function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.\\
&\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q}{\alpha+2}\right)^{\frac{1}{q}}\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q}{\alpha+2}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\kappa \in [a,b]$, $\alpha \in [0,1]$, $q \geq 1$, $B(\alpha)$ is normalization function.

Proof By using Lemma 2.1, we get

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.\\
&\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|dk\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|dk.
\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}
&\left| {}_{a}^{AB}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I_{b}^{\alpha}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.\\
&\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|\\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''\left(k\kappa+(1-k)a\right)\right|^q dk\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha+1}\left|\Upsilon''\left(kb+(1-k)\kappa\right)\right|^q dk\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|\Upsilon''|^q$ is a P-function, we have

$$\left| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon(\kappa)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon(\kappa)\right\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \right.$$
$$\left. -\frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \right|$$
$$\leq \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-k)^{\alpha+1}\left(|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q\right)dk\right)^{\frac{1}{q}}\right]$$
$$+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha+1}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha+1}\left(|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q\right)dk\right)^{\frac{1}{q}}\right]$$
$$= \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q}{\alpha+2}\right)^{\frac{1}{q}}$$
$$+\frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q}{\alpha+2}\right)^{\frac{1}{q}}.$$

So, the proof is completed. □

Corollary 2.21 *In Theorem 2.20, if we choose* $\kappa=\frac{a+b}{2}$, *we obtain*

$$\left| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \right.$$
$$\left. -\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right|$$
$$\leq \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^q+|\Upsilon''(a)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|\Upsilon''(b)|^q+\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].$$

Corollary 2.22 *In Theorem 2.20, if we choose* $\alpha=1$, *we obtain*

$$\left|\int_a^b \Upsilon(x)dx - [(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)] - \frac{(\kappa-a)^2\Upsilon'(a)-(b-\kappa)^2\Upsilon'(b)}{2}\right|$$
$$\leq \left(\frac{1}{3}\right)^{1-\frac{1}{q}}\frac{(\kappa-a)^3}{2}\left(\frac{|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q}{3}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\frac{(b-\kappa)^3}{2}\left(\frac{|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q}{3}\right)^{\frac{1}{q}}.$$

Theorem 2.23 *$\Upsilon : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) and $\Upsilon'' \in L_1[a,b]$. If $|\Upsilon''|^q$ is a P-function, then we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\Bigg| {}^{AB}_{a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&- \frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)} + \frac{|\Upsilon''(\kappa)|^q+|\Upsilon''(a)|^q}{q}\right) \\
&+ \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)} + \frac{|\Upsilon''(b)|^q+|\Upsilon''(\kappa)|^q}{q}\right)
\end{aligned}$$

where $p^{-1}+q^{-1}=1$, $\kappa \in [a,b]$, $\alpha \in [0,1]$, $q>1$, $\Gamma(.)$ is Gamma function and $B(\alpha)$ is normalization function.

Proof By using the identity that is given in Lemma 2.1, we have

$$\begin{aligned}
&\Bigg| {}^{AB}_{a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&- \frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha+1}\left|\Upsilon''(k\kappa+(1-k)a)\right|dk \\
&+ \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\left|\Upsilon''(kb+(1-k)\kappa)\right|dk.
\end{aligned}$$

By using the Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$,

$$\begin{aligned}
&\Bigg| {}^{AB}_{a}I^{\alpha}\{\Upsilon(\kappa)\} + {}^{AB}I^{\alpha}_{b}\{\Upsilon(\kappa)\} - \frac{(\kappa-a)^{\alpha}\Upsilon(a)+(b-\kappa)^{\alpha}\Upsilon(b)}{B(\alpha)\Gamma(\alpha)} \\
&- \frac{(\kappa-a)^{\alpha+1}\Upsilon'(a)-(b-\kappa)^{\alpha+1}\Upsilon'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)\Upsilon(\kappa)}{B(\alpha)} \Bigg| \\
\leq\ & \frac{(\kappa-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 (1-k)^{(\alpha+1)p}dk + \frac{1}{q}\int_0^1 \left|\Upsilon''(k\kappa+(1-k)a)\right|^q dk\right] \\
&+ \frac{(b-\kappa)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{(\alpha+1)p}dk + \frac{1}{q}\int_0^1 \left|\Upsilon''(kb+(1-k)\kappa)\right|^q dk\right].
\end{aligned}$$

Since $|\Upsilon''|^q$ is a P-function and by a simple computation, we have the desired result. □

Corollary 2.24 *In Theorem 2.23, if we choose* $\kappa = \frac{a+b}{2}$, *we obtain*

$$\begin{aligned}
&\Bigg| {}^{AB}_{\ a}I^{\alpha}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} + {}^{AB}I^{\alpha}_{b}\left\{\Upsilon\left(\frac{a+b}{2}\right)\right\} - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}\Big[\Upsilon(a)+\Upsilon(b)\Big] \\
&- \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)B(\alpha)\Gamma(\alpha)}\Big[\Upsilon'(a)-\Upsilon'(b)\Big] - \frac{2(1-\alpha)\Upsilon\left(\frac{a+b}{2}\right)}{B(\alpha)}\Bigg| \\
\leq\ & \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{2}{p(\alpha p+p+1)} + \frac{2\left|\Upsilon''\left(\frac{a+b}{2}\right)\right|^{q} + |\Upsilon''(a)|^{q} + |\Upsilon''(b)|^{q}}{q}\right).
\end{aligned}$$

Corollary 2.25 *In Theorem 2.23, if we choose* $\alpha = 1$, *we obtain*

$$\begin{aligned}
&\left|\int_a^b \Upsilon(x)dx - [(\kappa-a)\Upsilon(a)+(b-\kappa)\Upsilon(b)] - \frac{(\kappa-a)^2\Upsilon'(a)-(b-\kappa)^2\Upsilon'(b)}{2}\right| \\
\leq\ & \frac{(\kappa-a)^3}{2}\left(\frac{1}{p(2p+1)} + \frac{|\Upsilon''(\kappa)|^{q}+|\Upsilon''(a)|^{q}}{q}\right) \\
&+ \frac{(b-\kappa)^3}{2}\left(\frac{1}{p(2p+1)} + \frac{|\Upsilon''(b)|^{q}+|\Upsilon''(\kappa)|^{q}}{q}\right).
\end{aligned}$$

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