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Temporal Proximity of 1-cycles in CW Spaces, Time-Varying Cell Complexes

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Abstract: This paper introduces approximate temporal proximities of 1-cycle cell complexes in a space-time view of a planar Whitehead CW space. Divergence of the vector field of a 1-cycle provides a natural basis for an approximate Temporal Proximity (aTP) of time-varying 1-cycles useful in the detection, characterization, analysis, and measurement of the closeness of changing geometric realizations of simplicial complexes in a J.H.C Whitehead CW topological space. A practical application of aTP is given in terms of the temporal closeness of 1-cycle shapes in sequences of video frames. A main result in this paper is that every pair of cell complexes with the same descriptions over the same temporal interval have two properties, namely, (i) persistence and (ii) approximate temporal closeness.

Keywords: CW space, 1-cycle, cell complex, divergence, temporal proximity.

1. Introduction

Time-varying cell complexes in a J.H.C Whitehead Closure-finite Weak (CW) space [35] provide a natural basis for the introduction of temporal proximities [4], which are an extension of traditional Čech [33] and Efremovič-Smirnov [28] spatial proximities as well as the more recent computational proximities [16], descriptive proximities [24, 26] and proximal relators [7, 17, 30–32]. For an overview of temporal proximities, see Appendix E.

Another thread leading to more recent views of proximity spaces stems from results obtained by members of the Sugar (Turkish Şeker) group [2, 3, 14, 25]. Approximate temporal proximity is the counterpart of approximate proximity introduced in [20] and temporal proximity introduced in [4, §3].

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In this paper, the temporal proximity of changing 1-cycles $\mathrm{cyc}E, \mathrm{cyc}E'$ is considered in terms of the divergence of each 1-cycle vector field that is a scalar field [9], *i.e.*,

$$\mathrm{div}\ \mathrm{cyc}E = \frac{\partial \mathrm{cyc}E}{\partial \mathrm{bdy}} + \frac{\partial \mathrm{cyc}E}{\partial \mathrm{int}} + \frac{\partial \mathrm{cyc}E}{\partial \mathrm{t}}.$$

$\mathrm{div}\ \mathrm{cyc}E$ quantifies the flux of a 1-cycle in terms of its boundary, interior, temporal variation as the 1-cycle moves through space. For a threshold $th > 0$, $\|\mathrm{div}\ E - \mathrm{div}\ E'\| < th$ implies cycles $\mathrm{cyc}E, \mathrm{cyc}E'$ are temporally near (denoted by $\mathrm{cyc}E\ \delta_{\Delta t}\ \mathrm{cyc}E'$). For an introduction to divergence and its counterpart (gradient $\nabla \mathrm{cyc}E$ over a vector field considered as a stream field), see [11, §1.1, pp. 16-19], [1, §13.2.4, pp. 651-652]. For our purposes, we associate a vector field $\overrightarrow{E}(g)$ (for a 1-cycle E) with a distinguished (representative) vertex g on the 1-cycle and obtain the divergence $\mathrm{div}\ \mathrm{cyc}E$.

Let 2^k be a collection of sub-complexes of a space tCW and let

$$\Phi_t(E) = \left\{\Phi_{t,n}(x) \in \mathbb{R}^n : x \in 2^E \in 2^K\right\}$$

where $\Phi_{t,n}(x)$ is a feature vector with n feature values that describe a subcomplex in space tCW that varies over time.

Example 1.1 *In the description of a time-varying 1-cycle $cycE$ at time t in a tCW space, let*

$$\Phi_{t,1}(cycE) = div\ cycE,$$

$$\Phi_{t,2}(cycE) = \overline{f\hbar}\ \textit{(average cycE vertex Planck energy)},$$

$$\Phi_{t,2}(cycE) = (\Phi_{t,1}(cycE), \Phi_{t,2}(cycE)).$$

For a pair of 1-cycles $cycE, cycE'$, we have

$$cycE\ \delta_{\|\Phi_t\|}\ cycE' \Leftrightarrow \|\Phi_{t,2}(cycE) - \Phi_{t,2}(cycE')\| < th,\ \textit{for } th > 0.$$

Let $E \underset{\Delta t}{\cap} E'$ denote a time-dependent form of the descriptive intersection [15] of nonempty subsets $E, E' \in 2^K$ at times t, t', defined by

$$E \underset{\|\Phi\|_t}{\cap} E' = \overbrace{\{x \in E \cup E' : \Phi_t(x) \in \Phi_t(E), \Phi_{t'}(x) \in \Phi_{t'}(E'), |t - t'| < th\}}^{\textbf{Descriptions } \Phi_t(E), \Phi_t(E') \textbf{ overlap, temporally}}.$$

These observations lead to the following elementary as well as useful result.

Lemma Let E, E' be cell complexes in a temporal proximity space E (denoted by tCW E). $E \underset{\|\Phi\|_t}{\cap} E' \neq \emptyset$ if and only if $E \delta_{\|\Phi_t\|} E'$.

Remark 1.2 *For the proof, see Lemma 2.6. This elementary lemma provides a useful result, since we now can observe that a pair of cell complexes are close to each other over some temporal interval without the requirement that we first determine their temporal intersection.*

The main results given in this paper are twofold.

(1) Every persistent pair of cell complexes are approximately temporally close (see Theorem 4.3).

(2) Every pair of cell complexes with the same divergence over the same interval of time have two properties, namely, (i) persistence and (ii) approximate temporal closeness (see Theorem 4.4).

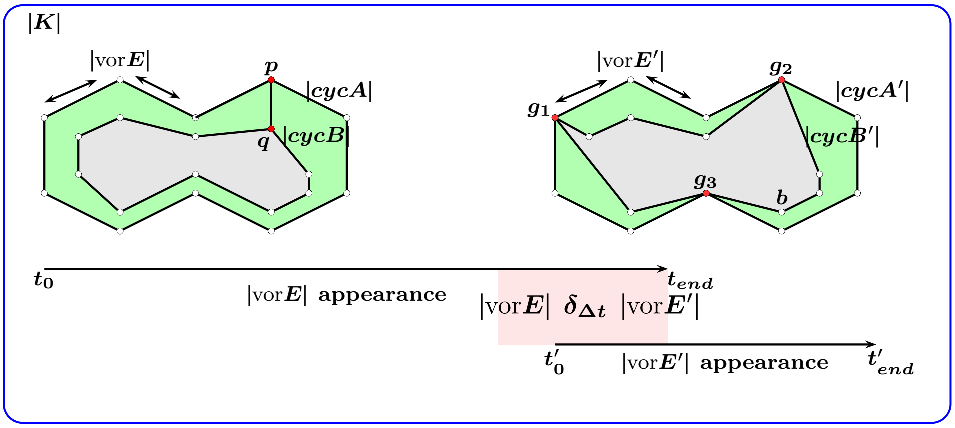


Figure 1: Temporally near vortexes

2. Preliminaries

In this section, we briefly present a framework for $\delta_{\|\Phi_t\|}$ (approximately temporally-near) cell complexes in tCW spaces. For the traditional view of a Whitehead CW space, see Appendix. A. Temporal closeness of cell complexes leads naturally to fixed points in temporal intervals (see, *e.g.*, for example, [22] and a recent study of time-constrained, self-similar surface shapes recorded and approximated geometrically in triangulated video frames [4]). Briefly, a planar **cell complex** is a collection of elementary cells called simplexes attached to each other, namely, (0-cells (vertexes), 1-cells (edges) and 2-cells (filled triangles). In this work, the focus is on vortexes that are collections of nested cell complexes called 1-cycles.

Definition 2.1 (1-Cycle) *A 1-cycle cycE in a CW space K is a collection of path-connected vertexes on 1-cells (edges) attached to each other with no end vertex and with nonvoid interior.*

Definition 2.2 (Temporal CW (tCW) Space) *A temporally changing Whiteheadean CW space (denoted by tCW) is a Whitehead CW space containing cell complexes that change over time, measured by the divergence of the vector field of each cell complex in terms of its boundary and interior at any instant in time.*

Definition 2.3 (Temporal Proximity) *A CW space K is a temporal proximity space (briefly, tCW) equipped with relation $\delta_{\Delta t}$, provided cell complexes appear, disappear and possibly reappear during temporal intervals. In a tCW space k, cell complexes E, E' are temporally near, provided cell complex E appears and persists (continues) during the same interval of time in which complex E' appears and persists. This temporal proximity (closeness) of complexes E, E' is denoted by $E \ \delta_{\Delta t} \ E'$.*

Example 2.4 *Vortexes $vorE, vorE'$ have temporal proximity in Figure 1. From Definition 2.3, we then write $vorE \ \delta_{\Delta t} \ vorE'$.*

In other words, cell complexes in a tCW space change over time, appearing momentarily and eventually disappearing over varying temporal intervals.

Let 2^k be a collection of subcomplexes of a space tCW and let

$$\Phi_t(E) = \left\{\Phi_{t,n}(x) \in \mathbb{R}^n : x \in 2^E \in 2^K\right\}$$

where $\Phi_{t,n}(x)$ is a feature vector with n feature values that describe a sub-complex in space tCW that varies over time.

Example 2.5 *In the description of a time-varying 1-cycle $cycE$ at time t in a tCW space, let*

$$\Phi_{t,1}(cycE) = div\ cycE,$$

$$\Phi_{t,2}(cycE) = \overline{f\hbar} \text{ (average } cycE \text{ vertex Planck energy)},$$

$$\Phi_{t,2}(cycE) = (\Phi_{t,1}(cycE), \Phi_{t,2}(cycE)).$$

For a pair of 1-cycles $cycE, cycE'$, we have

$$cycE \ \delta_{\|\Phi_t\|} \ cycE' \Leftrightarrow \|\Phi_{t,2}(cycE) = \Phi_{t,2}(cycE')\| < th, \text{ for } th > 0.$$

Let $E \underset{\Delta t}{\cap} E'$ denote a time-dependent form of the descriptive intersection [15] of nonempty subsets $E, E' \in 2^K$ at times t, t', defined by

$$E \underset{\|\Phi\|_t}{\cap} E' = \overbrace{\{x \in E \cup E' : \Phi_t(x) \in \Phi_t(E), \Phi_{t'}(x) \in \Phi_{t'}(E'), |t - t'| < th\}}^{\textbf{Descriptions } \Phi_t(E), \Phi_t(E') \textbf{ overlap, temporally}}.$$

Lemma 2.6 *Let E, E' be cell complexes in a tCW space. Then $E \underset{\|\Phi\|_t}{\cap} E' \neq \varnothing$ if and only if $E \ \delta_{\|\Phi_t\|} \ E'$.*

Proof

$\Rightarrow$: For cell complexes E, E', assume that $E \underset{\|\Phi\|_t}{\cap} E' \neq \emptyset$. Consequently, E, E' occur during the same temporal interval, *i.e.*, cell complex E appears at some time t_E during the appearance of cell complex E' at time $t_{E'}$. In other words, times $t_E, t_{E'}$ occur during a temporal interval

$$[t, \ldots, t_E, \ldots, t_{E'}, \ldots t'].$$

Hence $E\delta_{\|\Phi_t\|}E'$.

$\Leftarrow$: If $E\ \delta_{\|\Phi_t\|}\ E'$, then, from Definition 2.3, it must be the case that cell complexes E, E' appear during the same temporal interval. Hence $E \underset{\|\Phi\|_t}{\cap} E' \neq \emptyset$. $\square$

The focus here is on cell complexes that are nested 1-cycles that form vortexes in a tCW space. Briefly, a **vortex** in a tCW space is a collection of path-connected vertexes in nested 1-cycles (for the details, see Appendix C).

Remark 2.7 *In this paper, the notation $|K|$ (commonly used in algebraic topology to denote a subspace of an abstract space K [10, §1.2,p. 8]) denotes the geometric (graphical) realization of an abstract CW space K (see,* e.g., *$|K|$ in Figure 1). If K is a planar complex, then $|K|$ is a 2-dimensional polytope* [8, §II.1,p. 47]. *Similarly, the geometric realization (polytope) of vortexes $vorE, vorE'$ is denoted by $|vorE|, |vorE'|$ in Figure 1. For a 1-cycle $cycE$ cell complex in a space K, we write $|cycE|$ for its geometrical realization,* i.e., *$cycE$ is a cell complex on $|cycE|$ (see,* e.g., [29, §5.8,p. 69]*).*

Definition 2.8 [19]**(Planar Vortex)** *A* **planar vortex** *$vorE$ is a finite cell complex, which is a collection of path-connected vertices in nested, filled 1-cycles in a CW complex K. A* **1-cycle** *in $vorE$ (denoted by $cycA$) is a sequence of edges with no end vertex and with a nonempty interior. A geometric realization of cell complex $vorE$ in space K in the Euclidean plane is denoted by $|vorE|$ on $|K|$.*

Example 2.9 *A sample, time-constrained (clocked) vortex $vorE$ spiraling through space is represented graphically as $|vorE|$ in Figure 1.*

3. Approximate Temporal Proximities

A nonempty set X equipped with the relation $\delta_{\Delta t}$ is a temporal Čech proximity space (denoted by $(X, \delta_{\Delta t})$), provided temporal form of the Čech axioms in Appendix D are satisfied. For an overview of the temporal Čech axioms, see Appendix E. When we write, for example, vor$E\ \delta_{\Delta t}$ vorE' in tCW space K, we mean that vor$E$ persists over a temporal interval $\Delta t = [t_0, t_{end}]$ that overlaps with a temporal interval $\Delta t' = [t'_0, t'_{end}]$ over which a vortex vorE' persists.

Example 3.1 *In Figure 1, vorE $\delta_{\Delta t}$ vorE′ in the temporal CW space K, since vortex vorE persists over a temporal interval that overlaps the temporal interval over which vortex vorE′ persists.*

An interest in overlapping temporal intervals leads to the introduce of the temporal intersection between cell complexes such as vortexes vor$E$, vorE' that overlap in time (denoted by vor$E \underset{\Delta t}{\cap}$ vorE').

Definition 3.2 *Let vorE, vorE′ be vortexes in a temporal CW space K. Also, let $\underset{vorE}{\Delta} t$ be a temporal interval over which vorE appears and persists and let $\underset{vorE'}{\Delta} t'$ be a temporal interval over which vorE′ appears and persists. Then*

$$vorE \underset{\|\Phi\|_t}{\cap} vorE' = \left\{ instant\ t : t \in \underset{vorE}{\Delta} t \cap \underset{vorE'}{\Delta} t' \right\}.$$

Example 3.3 *In Figure 1, vorE $\underset{\|\Phi\|_t}{\cap}$ vorE′ $\neq \emptyset$. Because*

$$\underset{vorE}{\Delta} t \cap \underset{vorE'}{\Delta} t' \neq \emptyset$$

since the pair of vortexes persist over the interval $[t', t_{end}]$.

Approximate Temporal Čech Axioms

Let shapes A, B, C be in a time-varying space X. The space X is an approximate temporal proximity space aTP, provided the following axioms are satisfied:

(tP.0) All nonempty subsets in X are temporally far from the empty set, *i.e.*, $A \ \not\delta_{\|\Phi_t\|} \ \emptyset$ for all $A \subseteq X$.

(tP.1) $A \ \delta_{\|\Phi_t\|} \ B \Rightarrow B \ \delta_{\|\Phi_t\|} \ A$.

(tP.2) $A \underset{\|\Phi\|_t}{\cap} B \neq \emptyset \Leftrightarrow A \ \delta_{\|\Phi_t\|} \ B$.

(tP.3) $A \ \delta_{\|\Phi_t\|} \ (B \cup C) \Rightarrow A \ \delta_{\|\Phi_t\|} \ B$ or $A \ \delta_{\|\Phi_t\|} \ C$.

Lemma 3.4 *Let E, E' be cell complexes in a space tCW. $E \underset{\|\Phi\|_t}{\cap} E' \neq \emptyset$ if and only if $E \ \delta_{\|\Phi_t\|} \ E'$.*

Proof Immediate from Axiom **tP.2**. □

Theorem 3.5 $(K, \delta_{\|\Phi_t\|})$ *be a tCW proximity space* K *and let vortexes* $vorE, vorE' \in 2^K$, i.e., *vortexes* $vorE, vorE'$ *are subcomplexes in the collection of cell complexes in space* K. *Then* $vorE \underset{\|\Phi\|_t}{\cap} vorE' \neq \varnothing \Leftrightarrow vorE \ \delta_{\|\Phi_t\|} \ vorE'$.

Proof Immediate from Lemma 3.4. □

4. Main Results

The end-goal, here, is to arrive at a means of tracking 1-cycles either by themselves or vortexes (nested 1-cycles) that are short-lived, appearing over a temporal interval, disappearing, and then possibly reappearing, resembling a sequences of sunrises, appearing each morning, disappearing each evening and then reappearing the next day.

Lemma 4.1 *Every pair of subcomplexes with close rates-of-change in a tCW space* K *are approximately temporally proximal.*

Proof Let $E, E' \in 2^K$ with rates-of-change $\frac{\partial E}{\partial t}, \frac{\partial E'}{\partial t'}$ such that

$$\Phi_t(E) = \frac{\partial E}{\partial t}, \ \Phi_{t'}(E') = \frac{\partial E'}{\partial t'},$$

$$\|\Phi_t(E) - \Phi_{t'}(E')\| < th \text{ for } th > 0.$$

From Lemma 3.4, $E \ \delta_{\|\Phi_t\|} \ E'$. □

Definition 4.2 *Let* E, E' *be cell complexes in a space tCW.* E, E' *persist, provided* $E \ \delta_{\|\Phi_t\|} \ E'$, i.e., E *and* E' *appear with the same rates-of-change during the same interval of time.*

Cell complexes with the same rate-of-change of their boundary or of their interior during the same interval of time are also persistent, even if their rates-of-change are different. It is also the case that cell complexes with close divergence during the same interval of time are persistent, independent of differing rates-of-change as well as differing perimeters (boundaries) and interiors.

Theorem 4.3 *Every persistent pair of cell complexes in a space tCW is approximately temporally close.*

Proof Let E, E' be persistent cell complexes with the same rates-of-change over the same temporal interval in a space tCW. From Lemma 4.1, $E \ \delta_{\|\Phi_t\|} \ E'$. □

Theorem 4.4 *Every pair of cell complexes in a space tCW with the same divergence over the same interval of time are* *(i) persistent, and*

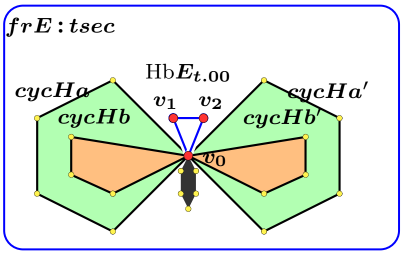
(ii) approximately temporally close.

Proof

(i) from Definition 4.2, cell complexes with close divergences over the sample time interval are persistent.

(ii) Cell complexes E, E in tCW that appear during the interval of time, are temporally close. Hence $E\ \delta_{\|\Phi_t\|}\ E$. □

Vigolo Hawaiian butterfly $\mathrm{Hb}\boldsymbol{E_{t.00}}$ in video frame $\boldsymbol{frE}$ at time $\boldsymbol{t}$ at the beginning of a temporal interval $[\boldsymbol{t}, \boldsymbol{t} + \mathbf{05}\boldsymbol{sec}]$, Betti no. $\boldsymbol{\beta}(\mathrm{Hb}\boldsymbol{E_{t.00}}) = \mathbf{3}$, $\mathrm{Hb}\boldsymbol{E_{t.00}}\ \ \boldsymbol{\delta_{\|\Phi_t\|}}\ \mathrm{Hb}\boldsymbol{E_{t.01}}$



Vigolo Hawaiian butterfly $\mathrm{Hb}\boldsymbol{E_{t.01}}$ in video frame $\boldsymbol{frE'}$ at time $\boldsymbol{t} + \mathbf{0.1}\boldsymbol{sec}$ in temporal interval $[\boldsymbol{t}, \boldsymbol{t} + \mathbf{05}\boldsymbol{sec}]$, Betti no. $\boldsymbol{\beta}(\mathrm{Hb}\boldsymbol{E_{t.01}}) = \mathbf{3}$, $\mathrm{Hb}\boldsymbol{E_{t.00}}\ \boldsymbol{\delta_{\|\Phi_t\|}}\ \mathrm{Hb}\boldsymbol{E_{t.01}}$

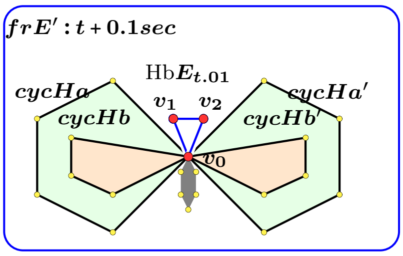


Figure 2: Persistent butterfly shapes in a pair of video frames that appear over a short temporal interval

5. Application

This section briefly introduces an application of temporally proximal 1-cycles in a topology of data approach to detecting persistent video frame cycles that appear, disappear and sometimes reappear within a temporal interval. The basic approach is to track the persistence of video frame shapes covered with filled cycles.

In the study of temporally proximal 1-cycles on video frames, 1-cycles have free group presentations with corresponding Betti numbers that simplify the comparison of frame 1-cycles. Briefly, a finite group G is free, provided every element $x \in G$ is a linear combination of its basis elements (called generators) [10, §1.4, p. 21]. For the details about free groups, see Appendix F.

Lemma 5.1 [23] *Every 1-cycle in a CW space has a free group presentation.*

For simplicity, assume a pair of 1-cycles attached to each other have a vertex in common. Recall that a Betti number is a count of the number generators in a free group [10, §4,p. 24]. Frame 1-cycles are, for example, descriptively close, provided the difference between the Betti numbers of their free group presentations are close. Determining the persistence of frame cycles then reduces to tracking the appearance, disappearance and possible reappearance of the cycles in terms of their recurring Betti numbers over temporal intervals. For more about this, see [21].

Theorem 5.2 *Every free group presentation of nested 1-cycles attached to each other has a Betti number.*

Proof Let shape $\mathrm{sh}E = \mathrm{cyc}E, \mathrm{cyc}E'$, *i.e.*, $\mathrm{sh}E$ consists of a pair of 1-cycles attached to each other such that $\mathrm{cyc}E \cap \mathrm{cyc}E' \neq \emptyset$ and the interior $\mathrm{int}(\mathrm{sh}E)$ is nonvoid. From Lemma 5.1, every cycle in $\mathrm{sh}E$ has a free group presentation. Without loss of generality, assume $\mathrm{cyc}E \cap \mathrm{cyc}E' = v$, a vertex common to both 1-cycles in $\mathrm{sh}E$. In that case, starting at v, the vertexes in $\mathrm{sh}E = \mathrm{cyc}E, \mathrm{cyc}E'$ are path-connected with no end vertex. Hence, $\mathrm{sh}E$ is also a 1-cycle. Again, from Lemma 5.1, $\mathrm{sh}E$ has a free group presentation . Since very vertex v' in $\mathrm{sh}E$ can be written as a linear combination of v, then $\mathcal{B}(\mathrm{sh}E) = 1$. What we observed for a 1-cycle with one generator easily extends to nested 1-cycles with more than one generator with a corresponding Betti number greater than 1. This gives us the desired result. □

Example 5.3 *A Hawaiian butterfly shape that persists in a sequence of video frames over a short temporal interval is shown in Figure 2. The wings of each butterfly shape are constructed from nested cycles, which have nonempty intersection with a common vertex v_0. Also notice that vertex*

v_0 and antenna vertices v_1, v_2 also constitute a 3rd 1-cycle (call it $cycE''$) so that we have

$$wing\ w1 = cycHa \cup cycHb\ (\textbf{wing } \boldsymbol{w1} \textbf{ has nested 1-cycles}),$$

$$wing\ w2 = cycHa' \cup cycHb'\ (\textbf{wing } \boldsymbol{w2} \textbf{ has nested 1-cycles}),$$

$$w1 \cap w2 = \{v_0\}\ (\textbf{wings have a common vertex}),$$

$$cycE'' = \{v_0 \to v_1 \to v_2 \to v_0\}\ (\textbf{antenna cycle}),$$

$$HbE_{t.00} = \{w1, w2, cycE''\}\ (\textbf{butterfly } \boldsymbol{HbE_{t.00}} \textbf{ parts}),$$

$$\mathcal{B}(HbE_{t.00}) = 3\ (\textbf{butterfly free group Betti number}).$$

For simplicity, we will ignore the 4th 1-cycle in Figure 2, namely, the butterfly body, also attached to v_0.

Recall that a 1-cycle is a sequence of path-connected edges with no end edge and with nonvoid interior. Hence butterfly $HbE_{t.00}$ in Figure 2 is a massive 1-cycle. From Theorem 5.2, $HbE_{t.00}$ has a free group presentation with a corresponding Betti number. Observe that the three vertexes in the antenna cycle $cycE''$ in Figure 2 serve as generators of the $HbE_{t.00}$ free group. Hence $\mathcal{B}(HbE_{t.00}) = 3$ Betti number for the butterfly free group, which persists over the pair of video frames shown in Figure 2.

So tracking a butterfly across a sequence of video frames reduces to tracking the appearance of the Betti number of a butterfly shape over a sequence of video frames in a particular temporal interval. This situation is represented in Figure 2 such that

$$HbE_{t.00}\ \delta_{\|\Phi_t\|}\ HbE_{t.01},$$

i.e., *butterfly $\mathcal{B}(HbE_{t.00})$ persists, reappearing in a second video frame with a slightly altered appearance* (e.g., *wing color in the first frame is less bright in the second frame) and with Betti number = 3. In effect, $\mathcal{B}(HbE_{t.00})$ persists for a 0.1 second. What we observed about temporal proximity in this example has been used effectively in a recent, similar study of highway traffic [4].*

Example 5.4 *Repeat the steps in Example 5.3 using the divergence of vertex v_0 instead of the Betti number of the butterfly 1-cycle. In other words, track the persistence of a frame shape cycle over time using the divergence of a butterfly vertex, shared by the wings and antenna of the butterfly. Then compare the results with those obtained in Example 5.3.*

Example 5.5 *Repeat the steps in Example 5.3 using both the divergence of vertex v_0 as well as the Betti number of the butterfly 1-cycle. In other words, track the persistence of a frame shape cycle over time using the butterfly Betti number and the divergence of a butterfly vertex, shared by the* *wings and antenna of the butterfly. Then compare the results with those obtained in Example 5.3 and in Example 5.4.*

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A. Cell Complexes

A planar Whitehead cell complex K [20] (usually called a CW complex) is a collection of n-dimensional minimal cells $e_\alpha^n, n \in \{0,1,2\}$, *i.e.*,

$$K = \left\{e_\alpha^n \subset \mathbb{R}^2 : n \in \{0,1,2\}\right\}.$$

in the Euclidean plane π.

Definition A.1 *A cell subcomplex* $shE := \{e_\alpha^n\} \in 2^K$ *(**shape complex**) is a **closed subcomplex**, provided the subcomplex includes both a nonempty interior (denoted by* $int(e_\alpha^n)$*) and its boundary (denoted by* $bdy(e_\alpha^n)$*). In effect,* shE *is closed, provided*

$$shE = int(shE) \cup bdy(shE) \ (\textbf{Closed subcomplex}).$$

Let 2^π be the collection of all subsets in the Euclidean plane π. In the plane, a Whitehead **Closure-finite Weak (CW)** cell complex $K \in 2^\pi$ has two properties, namely,

C: A cell complex K is **closure-finite**, provided each cell $e_\alpha^n \in K$ is contained in a finite subcomplex of K. In addition, each cell $e_\alpha^n \in K$ has a finite number of immediate faces. One cell e_α^n is an **immediate face** of another cell e_α^m, provided $e_\alpha^n \cap e_\alpha^m \neq \emptyset$ [29] (also called a **common face**).

W: The plane π has a **weak topology** induced by cell complex K, *i.e.*, a subset $S \in 2^\pi$ is closed, if and only if $S \cap e_\alpha^n$ is also closed in e_α^n for each n, α [29, §5.3, p. 65].

A collection $K \in 2^\pi$ is called a **CW complex**, provided it has the closure-finite property and π has the weak topology property induced by K.

Minimal cell planar complexes are given in Table 1.

Remark A.2 *Closure finite cell complexes with weak topology (briefly, CW complexes) were introduced by J.C.H. Whitehead [34], later formalized in [35]. In this work*[1]*, a* **cell complex**

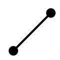
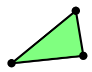
[1] Here, we use $cl(e^n)$ (closure of a cell) and $\mathrm{bdy}(e^n)$ (contour of a cell) used in this paper, instead of Whitehead's $\bar{e^n}$ and $\partial(e^n)$.

K (or complex) [35, §4, p. 221] is a Hausdorff space (union of disjoint open cells e, e^n, e^n_i) such that the closure of an n cell $e^n \in K$ (denoted by $cl(e^n)$ is the image of a map $f : \sigma^n \to cl(e^n)$, where σ^n is a fixed n-simplex and where the boundary $bdy(e^n)$ (otherwise known as the **contour** *of a complex) is defined by*

$$bdy(e^n) = \overbrace{f(bdy(e^n)) = cl(e^n) - int(e^n).}^{\textbf{Complex contour} \to \textbf{closure cl}(e^n)\ \textbf{minus}\ Int(e^n)\ \textbf{interior}}$$

Notice that a subcomplex $X \subset K$ has the weak topology, since X is the union of a finite number intersections $X \cap cl(e)$ for single cells $e \in K$ [35, §5, p. 223]. From a geometric perspective, a cell complex is a triangulation of the CW space K [34, p. 246].

Table 1: Minimal Planar Cell Complexes

Minimal Complex	Cell $e^n : n \in \{0,1,2\}$	Planar Geometry	Interior
	e^0	Vertex	nonempty
	e^1	Edge	line segment w/o end points
	e^2	Filled triangle	nonempty triangle interior w/o edges

B. Shape Complexes

A shape complex has two basic parts, namely, contour and interior, introduced in [18].

Each shape complex sh$E$ has a nonempty interior that excludes all points on the shape contour.

The fundamental parts of every shape complex are gathered together in the closure of a shape complex, definite using the Hausdorff distance [5] (see, also, [6, §23, p. 128]) between all points in a CW complex K and a shape sh$E$.

Definition B.1 *Closure of a planar shape shE (denoted by $cl(sh\boldsymbol{E})$) in a CW space K is defined by*

$$\overbrace{cl(sh\boldsymbol{E}) = \inf\{\boldsymbol{x} \in \boldsymbol{X} : \boldsymbol{D}(\boldsymbol{x}, sh\boldsymbol{E}) = 0\}.}^{\textbf{Hausdorff distance}\ \boldsymbol{D}(\boldsymbol{x}, \mathbf{sh}\boldsymbol{E}) = \inf\{\|\boldsymbol{x} - \boldsymbol{p}\| : \boldsymbol{p} \in \mathbf{sh}\boldsymbol{E}\}}$$

In other words, we have the closure of a planar shape sh$\boldsymbol{E}$ is a finite bounded region of the Euclidean plane such that

$$\overbrace{\mathrm{cl}(\mathrm{sh}\boldsymbol{E}) = \mathrm{bdy}(\mathrm{sh}\boldsymbol{E}) \cup \boldsymbol{Int}(\mathrm{sh}\boldsymbol{E}).}^{\textbf{cl(sh}E\textbf{) includes its contour \& its interior}}$$

C. Planar Vortexes

This section briefly looks at planar vortex structures in planar CW spaces. For simplicity, we consider only 2 cycle vortexes containing a pair of nested 1-cycles that intersect or attached to each other via at least one bridge edge.

Definition C.1 [21] **(Planar 2 Cycle Vortex)** *Let $cycA, cycB$ be a collection of path-connected vertexes on nested filled 1-cycles (with $cycB$ in the interior of $cycA$) defined on a finite, bounded, planar region in a CW space K. A* planar 2 cycle vortex *$vorE$ is defined by*

$$vorE = \overbrace{\{cl(cycA) : cl(cycB) \subset int(cl(cycA))\}}^{\textbf{cl(cyc}B\textbf{) is contained (nested) in the interior of cl(cyc}A\textbf{)}}.$$

A vortex containing adjacent non-intersecting cycles has a bridge edge attached to vertexes on the cycles.

Definition C.2 *A vortex* **bridge edge** *is an edge attached to vertexes on a pair of non-interecting, filled 1-cycles.*

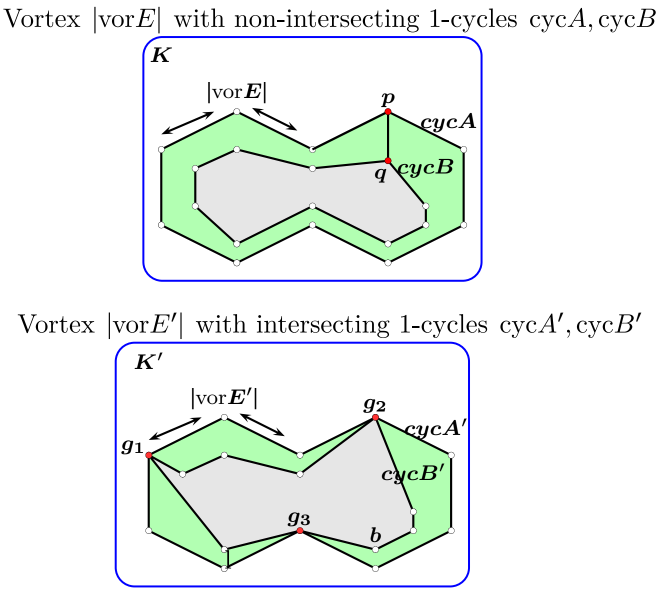


Figure 3: Sample planar 2-cycle vortexes

Remark C.3 *From Definition C.1, the cycles in a 2 cycle vortex can either have nonempty intersection (see,* e.g., $cycA' \cap cycB' \neq \emptyset$ *in* $|vorE'|$ *in Figure C) or there is a bridge edge between the cycles (see,* e.g., $\widehat{pq}$ $|vorE|$ *in Figure C). In effect, every pair of vertexes in a 2 cycle vortex is path-connected.*

Remark C.4 *The structure of a 2 cycle vortex extends to a vortex with* $k > 2$ *nested filled 1-cycles, provided adjacent pairs of cycles* $cycA, cycA'$ *in a* k*-cycle vortex either intersect or there is a bridge edge attached between* $cycA, cycA'$.

D. Čech Proximity Spaces

This section briefly introduces Čech proximity spaces, paving the way for temporal proximity spaces.

The simplest form of proximity relation (denoted by δ) on a nonempty set was introduced by E.Čech [33]. A nonempty set X equipped with the relation δ is a Čech proximity space (denoted by (X, δ)), provided the following axioms are satisfied:

Čech Axioms

(P.0) All nonempty subsets in X are far from the empty set, $A \not\delta \emptyset$ for all $A \subseteq X$.

(P.1) $A \ \delta \ B \Rightarrow B \ \delta \ A$.

(P.2) $A \cap B \neq \emptyset \Rightarrow A \ \delta \ B$.

(P.3) $A \ \delta \ (B \cup C) \Rightarrow A \ \delta \ B$ or $A \ \delta \ C$.

Given that a nonempty set E has $k \geq 1$ features such as Fermi energy E_{Fe}, cardinality E_{card}, a description $\Phi(E)$ of E is a feature vector, *i.e.*, $\Phi(E) = (E_{Fe}, E_{card})$. Nonempty sets A, B with overlapping descriptions are descriptively proximal (denoted by $A \ \delta_\Phi \ B$). The descriptive intersection of nonempty subsets in $A \cup B$ (denoted by $A \underset{\Phi}{\cap} B$) is defined by

$$A \underset{\Phi}{\cap} B = \overbrace{\{x \in A \cup B : \Phi(x) \in \Phi(A) \ \cap \ \Phi(B)\}}^{\textbf{Descriptions } \Phi(A) \ \& \ \Phi(B) \textbf{ overlap}}.$$

Let 2^X denote the collection of all subsets in a nonempty set X. A nonempty set X equipped with the relation δ_Φ with nonempty subsets $A, B, C \in 2^X$ is a Čech proximity space, provided the Čech axioms are satisfied.

Lemma D.1 *Let K be a CW space, $2^{Vor(K)}$ be a collection of planar vortexes equipped with the proximity δ and let $vorA, vorB \in Vor(K)$. Then $vorA \cap vorB \neq \emptyset$ implies $vorA\ \delta\ vorB$.*

Proof Immediate from Axiom **P.2**. □

Let (X,δ_1) and (Y,δ_2) be two Čech proximity spaces. Next, we consider proximally continuous maps informally introduced by Smirnov [28, p. 5]. Then a map $f:(X,\delta_1)\to(Y,\delta_2)$ is *proximally continuous*, provided $A\ \delta_1\ B$ implies $f(A)\ \delta_2\ f(B)$, *i.e.*, if A,B are close in δ_1 proximity space X, then $f(A),f(B)$ are close in the δ_2 proximity space Y [13, §1.4]. In general, a proximal continuous map preserves the nearness of pairs of sets in the pre-image space X for the corresponding pairs of set in the image space Y [12, §1.7, p. 16].

Theorem D.2 *Let K,K' be a pair of CW spaces, equipped with the proximities δ_1,δ_2, respectively and let $f:(K,\delta_1)\to(K',\delta_2)$ be a* proximally continuous *map. If vortexes $vorA, vorB \in 2^K$ are δ_1 close in space K, then $f(vorA), f(vorB)$ are close in the δ_2 proximity space K'*

Proof Given $\text{vor}A\ \delta_1\ \text{vor}B$ in space K, then by the definition of a proximally continuous map, $f(\text{vor}A)\ \delta_2\ f(\text{vor}B)$ in space K'. □

The notation 2^K denotes the collection of all subcomplexes of the CW space K.

Example D.3 *Let $vorA, vorB \in 2^K$ be vortexes, each with same number of cycles, which overlap. In that case, $vorA \cap vorB \neq \emptyset$. By Axiom* **P.2** *$vorA\ \delta_1\ vorB$ in space K. Let $f:(K,\delta_1)\to(K',\delta_2)$ be a* proximally continuous*, defined for any vortex $vorA$ by the identify map.*

$$f(vorA) = vorA.$$

In that case, let $vorE, vorE' \in 2^K$. Then

$$f(vorE) = vorE,$$

$$f(vorE') = vorE',$$

$$vorE\ \delta_1\ vorE' \Rightarrow f(vorE)\ \delta_2\ f(vorE').$$

In other words, if vortexes $vorE, vorE' \in 2^K$ are close, then $f(vorE), f(vorE')$ are close.

E. Temporal Proximity Space

This section briefly introduces a temporal proximity space, introduced in [4, §3].

A nonempty set X equipped with the relation $\delta_{\Delta t}$ is a temporal Čech proximity space (denoted by $(X,\delta_{\Delta t})$), provided temporal form of the Čech axioms in Appendix D are satisfied.

When write, for example, $\text{vor}E\ \delta_{\Delta t}\ \text{vor}E'$ in tCW space K, we mean that $\text{vor}E$ persists over a temporal interval $\Delta t = [t_0, t_{end}]$ that overlaps with a temporal interval $\Delta t' = [t'_0, t'_{end}]$ over which a vortex $\text{vor}E'$ persists.

Example E.1 *In Figure 1, $vorE\ \delta_{\Delta t}\ vorE'$ in the temporal CW space K, since vortex $vorE$ persists over a temporal interval that overlaps the temporal interval over which vortex $vorE'$ persists.*

An interest in overlapping temporal intervals leads to the introduce of the temporal intersection between cell complexes such as vortexes $\text{vor}E, \text{vor}E'$ that overlap in time (denoted by $\text{vor}E \underset{\Delta t}{\cap} \text{vor}E'$).

Definition E.2 *Let $vorE, vorE'$ be vortexes in a temporal CW space K. Also let $\underset{vorE}{\Delta} t$ be a temporal interval over which $vorE$ appears and persists and let $\underset{vorE'}{\Delta} t'$ be a temporal interval over which $vorE'$ appears and persists. Then*

$$vorE \underset{\Delta t}{\cap} vorE' = \left\{ instant\ t : t \in \underset{vorE}{\Delta} t \cap \underset{vorE'}{\Delta} t' \right\}.$$

Example E.3 *In Fig. 1, $vorE \underset{\Delta t}{\cap} vorE' \neq \emptyset$. Because*

$$\underset{vorE}{\Delta} t \cap \underset{vorE'}{\Delta} t' \neq \emptyset$$

since the pair of vortexes persist over the interval $[t', t_{end}]$.

Temporal Čech Axioms

(tP.0) All nonempty subsets in X are temporally far from the empty set, *i.e.*, $A\ \underline{\delta}\!\!\!/_{\Delta t}\ \emptyset$ for all $A \subseteq X$.

(tP.1) $A\ \delta_{\Delta t}\ B \Rightarrow B\ \delta_{\Delta t}\ A$.

(tP.2) $A \underset{\Delta t}{\cap} B \neq \emptyset \Leftrightarrow A\ \delta_{\Delta t}\ B$.

(tP.3) $A\ \delta_{\Delta t}\ (B \cup C) \Rightarrow A\ \delta_{\Delta t}\ B$ or $A\ \delta_{\Delta t}\ C$.

The Temporal Čech Axioms includes a time-constrained form of the Čech proximity Axiom **P.2**, with an important property introduced in Lemma E.4.

Lemma E.4 *The temporal proximity Axiom* **tP.2** *is an equivalence between overlapping temporal intervals and the $\delta_{\Delta t}$ proximity of subsets in the space $(X, \delta_{\Delta t})$.*

Proof $\Rightarrow$: Let the temporal intervals for $A, B \in 2^X$ overlap (*i.e.*, let $A \underset{\Delta t}{\cap} B \neq \emptyset$). Then, by Definition 2.3, $A\ \delta_{\Delta t}\ B$.

$\Leftarrow$: $A\ \delta_{\Delta t}\ B$ means that A and B persist over the same temporal interval $\underset{\substack{A\in 2^X\\ B\in 2^X}}{\Delta} t$. Hence the converse of Axiom **tP.2** holds, namely,

$$A\ \delta_{\Delta t}\ B \Rightarrow A \underset{\Delta t}{\cap} B \neq \emptyset.$$

□

Theorem E.5 $(K, \delta_{\Delta t})$ *be a tCW proximity space and let vortexes* $vorE, vorE' \in 2^K$, i.e., *vortexes* $vorE, vorE'$ *are subcomplexes in the collection of cell complexes in space* K. *Then* $vorE \underset{\Delta t}{\cap} vorE' \neq \emptyset \Leftrightarrow vorE\ \delta_{\Delta t}\ vorE'$.

Proof Immediate from Lemma E.4. □

Vortex $|\text{vor}E'|$ with intersecting 1-cycles $\text{cyc}A', \text{cyc}B'$

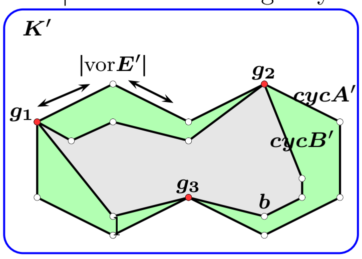


Figure 4: Nested, intersecting planar 1-cycles in a vortex

F. Free Group Presentation of a Vortex

A finite group G is free, provided every element $x \in G$ is a linear combination of its basis elements (called generators) [10, §1.4, p. 21]. We write $\mathcal{B}$ to denote a nonempty basis set of generators $\{g_1, \ldots, g_{|\mathcal{B}|}\}$ and $G(\mathcal{B}, +)$ to denote the free group with binary operation "+".

Example F.1 *The basis* $\{g_1, g_2, g_3\}$ *generates a group* G *whose geometric realization is* $|vorE'|$ *in Fig. E. The* "+" *operation on* G *corresponds to a move from a generator to a neighbouring* *vertex. For example:*

$$b = \overbrace{3g_1 + 3g_2}^{\textbf{traversing 3 cyc}A'\ \&\ \textbf{3 cyc}B'\ \textbf{edges to reach}\ b\ \textbf{via}\ g_1, g_2}$$

$$b = \overbrace{4g_1 + 3g_2}^{\textbf{traversing 7 cyc}B'\ \textbf{edges to reach}\ b\ \textbf{via}\ g_1, g_2}$$

$$b = \overbrace{0g_1 + 1g_3}^{\textbf{traversing 1 cyc}B'\ \textbf{edge to reach}\ b\ \textbf{via}\ g_3}$$

The identity element 0 in G is represented by a zero move from a generator g to another vertex (denoted by $0g$) and an inverse in G is represented by a reverse move $-g$.

Definition F.2 [27, p.239] **(Rotman Presentation)** *Let $X = \{g_i : i = 1, 2, \dots\}$, $\triangle = \{v = \sum kg_i : v \in groupG, g_i \in X\}$ be a set of generators of members of a nonempty set X and set of relations between members of G and the generators in X. A mapping of the form $\{X, \triangle\} \to G$, a free group, is called a presentation of G.*

We write $G(V, +)$ to denote a group G on a nonvoid set V with a binary operation "+". For a group $G(V, +)$ presentable as a collection of linear combinations of members of a basis set $\mathcal{B} \subseteq V$, we write $G(\mathcal{B}, +)$.

Definition F.3 (Free Group Presentation of a Cell Complex) *Let 2^K be the collection of cell complexes in a CW space K, $E \in 2^K$ containing n vertexes, $G(E, +)$ a group on nonvoid set E with binary operation "+", $\triangle = \{v = \sum kg_i : v \in E, g_i \in E\}$ be a set of generators of members in E, set of relations between members of E and the generators $\mathcal{B} \subset E$, $g_i \in \mathcal{B}, v = h_{imod\ n}(0) \in K$, k_i the i^{th} integer coeficient $mod n$ in a linear combination $\sum\limits_{i,j} k_i g_j$ of generating elements $g_j = h_j(0) \in \mathcal{B}$. A free group presentation of G is a continuous map $f : 2^K \times \triangle \to 2^K$ defined by*

$$f(\mathcal{B}, \triangle) = \left\{ v := \sum_{i,j} k_i g_j \in \triangle : v \in E, g_j \in \mathcal{B}, k_i \in \mathbb{Z} \right\}.$$

Lemma F.4 [22, §4, p. 10] *Every 1-cycle in a CW space K has a free group presentation.*

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