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Numerical Simulation of Two Dimensional Coupled Burgers Equations by Rubin-Graves Type Linearization

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Abstract
In the present article, the numerical solution of the two-dimensional coupled Burgers equation has been sought by finite difference method based on Rubin-Graves type linearization. Three models with appropriate initial and boundary conditions are applied to the problem. In order to show the accuracy of the method, the error norms L_2, L_∞ are computed. The error norms L_2, L_∞ of the obtained numerical solutions are compared with the error norms of some of the numerical solutions in the literature.

Keywords: *Two-dimensional Burgers equation; Rubin-Graves type linearization; Finite difference method.*

AMS Subject Classification (2020): *Primary: 35Q51 ; Secondary: 74J35; 33F10.*

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1. Introduction

In nature, some of the pysical phenomena such as gas dynamics, traffic flow, Brusselator chemical reaction-diffusion and shock waves are modelled by nonlinear partial differential equation systems among others such as the two-dimensional coupled Burgers equation (2D-CBE). There are many theoretical and numerical studies about the 2D-CBE equation in the literature. Fletcher [1] has found its analytical solution by applying the two-dimensional Hopf-Cole transform to the two-dimensional coupled Burgers equation. 2D-CBE has been solved numerically by several scholars by means of various methods and techniques. Among others, Fletcher [2] have conducted a work for comparing finite difference and finite element methods. Goyon [3] applied multi level alternating direction implicit methods. Ali et al. [4] have used the collation method via the radial base functions. Jain and Holla [5] have implemented two algorithms using the cubic spline function technique. Bahadır [6] has dealt with the problem by a fully implicit finite difference method. Khater et al. [7] have found out the numerical solution of some Burgers type nonlinear partial differential equations by Chebyshev spectral collocation method.Mittal and Jiwari [8] have applied the differential quadrature method using the Chebyshev-Gauss-Lobatto nodal points. Liao [9] obtained the numerical solution of the two-dimensional coupled Burgers equation by solving the two-dimensional linear heat equation obtained by applying the two-dimensional Hopf-Cole transformation to the

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two-dimensional coupled Burgers equation using the fourth-dimensional finite difference method. Zhu et al. [10] applied the discrete Adomian decomposition method. Srivastava et al. have applied [11] Crank-Nicolson finite difference method, Tamsir and Srivastava [12] have used semi-implicit finite difference method, Srivastava and Tamsir [13] have utilized Crank-Nicolson semi-implicit finite difference method, Thakar and Wani [14] have used linear finite difference method, Srivastava et al. [15] have applied implicit logarithmic finite difference method, Srivastava et al. [16] have used implicit exponential finite difference method, Srivastava and Singh [17] have used explicit-implicit finite difference method, Zhang et al. [18] have used full finite difference and non-standard finite difference methods, Mittal and Tripathi [19] have applied modified bi-cubic B-spline collocation method, Tamsir et al. [20] have used exponential modified cubic-B-spline differential quadrature method, Zhanlav et al. [21] have applied high order explicit finite difference method and Ngondiep [22] has utilized three-level explicit time-split MacCormack algorithm. Saqib et al. [23] have dealt with numerical solutions of 2-dimensional time dependent coupled non-linear systems. Wubs and Goede [24], in their article, considered the fully explicit method resulting from the truncation in the solution process and chosen one of the test problems as the 2-dimensional coupled Burgers' equation. Chai and Ouyang [25] have used proper stabilized Galerkin methods.

The rest of this article is organized as follows: In the first section , the method based on Rubin-Graves type linearization together with finite difference method and used for the numerical solution of two dimensional coupled Burgers equation is presented . Then to see the performance accuracy of the method, the numerical solution of three test model problems has been made and presented in tables by calculating the pointwise values and the error norms L_2 and L_∞ of the model problems of which the analytical solution are known. In addition, comparisons have been made with the error norms of the numerical solutions obtained by various methods available in the literature. In the last section, a brief conclusion is given.

2. Application of the Method

In this article, we consider the the two-dimensional coupled Burgers equation of the general form given as

$$u_t + uu_x + vu_y = \varepsilon(u_{xx} + u_{yy}), \quad (x, y) \epsilon\, \Omega, t > 0 \tag{2.1}$$

$$v_t + uv_x + vv_y = \varepsilon(v_{xx} + v_{yy}), \quad (x, y) \epsilon\, \Omega, t > 0 \tag{2.2}$$

together with the initial

$$\begin{aligned} u(x, y, 0) &= \psi_1(x, y); \quad (x, y) \epsilon\, \Omega \\ v(x, y, 0) &= \psi_2(x, y); \quad (x, y) \epsilon\, \Omega \end{aligned}$$

and the boundary conditions

$$\begin{aligned} u(x, y, t) &= \xi(x, y, t); \quad (x, y) \epsilon\, \partial\Omega \\ v(x, y, t) &= \zeta(x, y, t); \quad (x, y) \epsilon\, \partial\Omega \end{aligned}$$

where $u\,(x, y, t)$ and $v\,(x, y, t)$ denote velocity components. Over the solution domain $\Omega = \{(x, y) : a \leqslant x \leqslant b,\, c \leqslant y \leqslant d\}$ together with its boundary $\partial\Omega$. ψ_1, ψ_2, ξ and ζ are known smooth functions. Re denotes the Reynold number. As it is widely known, at the large values of the Reynold number, a shock wave having a cusp results in and numerical stability near this shock wave is nearly always difficult to obtain.

For the solution process, the domain of the problem in $x-$direction $[a, b]$ is divided into N_x parts having equal length h_x, and in $y-$direction $[c, d]$ is divided into N_y parts having equal length h_y, $x_i = a + ih_x$, $i = 0(1)N_x$; $y_j = c + jh_y$, $j = 0(1)N_y$; a smooth grid is created in the solution domain of the problem with the help of nodal points (x_i, y_j). The step length Δt is taken in the direction of the time variable for $t_n = n\Delta t$, $n = 0(1)N_t$. Then, all the numerical calculations to be made in each time step t_n are obtained at the nodes of this smooth grid. The numerical solution of $u\,(x, y, t)$ and $v\,(x, y, t)$ at any node (x_i, y_j, t_n) is shown by $U\,^n_{i,j}$ and $V\,^n_{i,j}$, respectively.

When the finite difference method based on Rubin-Graves type linearization technique is applied, a linear algebraic equation system results in since the related finite difference approaches are written in place of the derivatives in the equation. In the proposed method, the nonlinear partial differential equation is written in the appropriate form and after applying the finite difference method, an iterative relationship between the $(n+1)^{th}$ and

$(n)^{th}$ time level steps of the dependent variables is obtained. This newly obtained iterative relationship resulted in a linear algebraic equation system, which can be easily solved by a symbolic programming language such as MatLab.

Now, for 2D-CBE

$$\begin{aligned} u_t + uu_x + vu_y &= \varepsilon(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= \varepsilon(v_{xx} + v_{yy}) \end{aligned}$$

in place of non-linear terms uu_x, vu_y, uv_x and vv_y Rubin-Graves type [26] linearization technique are used.

In place of u_t an approximation as $u_t \cong (U^{n+1}_{i,j} - U^n_{i,j})/k$ and in place of v_t an approximation as $v_t \cong (V^{n+1}_{i,j} - V^n_{i,j})/k$ and in places of the terms uu_x, vu_y, uv_x and vv_y the following Rubin-Graves type approximations

$$uu_x \cong U^{n+1}_{i,j}\left[\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2h_x}\right] + U^n_{i,j}\left[\frac{U^{n+1}_{i+1,j} - U^{n+1}_{i-1,j}}{2h_x}\right] - U^n_{i,j}\left[\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2h_x}\right]$$

$$vu_y \cong V^{n+1}_{i,j}\left[\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2h_y}\right] + V^n_{i,j}\left[\frac{U^{n+1}_{i+1,j} - U^{n+1}_{i-1,j}}{2h_y}\right] - V^n_{i,j}\left[\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2h_y}\right]$$

$$uv_x \cong U^{n+1}_{i,j}\left[\frac{V^n_{i+1,j} - V^n_{i-1,j}}{2h_x}\right] + U^n_{i,j}\left[\frac{V^{n+1}_{i+1,j} - V^{n+1}_{i-1,j}}{2h_x}\right] - U^n_{i,j}\left[\frac{V^n_{i+1,j} - V^n_{i-1,j}}{2h_x}\right]$$

$$vv_y \cong V^{n+1}_{i,j}\left[\frac{V^n_{i+1,j} - V^n_{i-1,j}}{2h_y}\right] + V^n_{i,j}\left[\frac{V^{n+1}_{i+1,j} - V^{n+1}_{i-1,j}}{2h_y}\right] - V^n_{i,j}\left[\frac{V^n_{i+1,j} - V^n_{i-1,j}}{2h_y}\right]$$

are written. Then in place of the derivatives u_{xx}, u_{yy}, v_{xx} and v_{yy} their central finite difference approximations

$$u_{xx} \cong \frac{U^{n+1}_{i-1,j} - 2U^{n+1}_{i,j} + U^{n+1}_{i+1,j}}{h_x^2}$$

$$u_{yy} \cong \frac{U^{n+1}_{i,j-1} - 2U^{n+1}_{i,j} + U^{n+1}_{i,j+1}}{h_y^2}$$

$$v_{xx} \cong \frac{V^{n+1}_{i-1,j} - 2V^{n+1}_{i,j} + V^{n+1}_{i+1,j}}{h_x^2}$$

$$v_{yy} \cong \frac{V^{n+1}_{i,j-1} - 2V^{n+1}_{i,j} + V^{n+1}_{i,j+1}}{h_y^2}$$

are written. Finally, the terms on the $(n+1)^{th}$ time level are taken on the left hand side and $(n)^{th}$ time level terms are taken on the right hand side. After some simpliciation process, the following

$$\begin{aligned} &U^{n+1}_{i-1,j}(-\frac{k}{2h_x}U^n_{i,j} - \frac{\varepsilon k}{h_x^2}) + U^{n+1}_{i,j}(1 + k(\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2h_x}) + 4\frac{\varepsilon k}{h_x^2}) \\ &+ U^{n+1}_{i+1,j}(\frac{k}{2h_x}U^n_{i,j} - \frac{\varepsilon k}{h_x^2}) + U^{n+1}_{i,j-1}(-\frac{k}{2h_y}V^n_{i,j} - \frac{\varepsilon k}{h_y^2}) \\ &+ U^{n+1}_{i,j+1}(\frac{k}{2h_y}V^n_{i,j} - \frac{\varepsilon k}{h_y^2}) + V^{n+1}_{i,j}(\frac{k(U^n_{i,j+1} - U^n_{i,j-1})}{2h_y}) \\ &= U^n_{i,j}\left[1 + k(\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2h_x})\right] + V^n_{i,j}\left[k(\frac{U^n_{i,j+1} - U^n_{i,j-1}}{2h_y})\right] \end{aligned}$$

and

$$V_{i-1,j}^{n+1}(-\frac{k}{2h_x}U_{i,j}^n-\frac{\varepsilon k}{h_x^2})+V_{i,j}^{n+1}(1+k(\frac{V_{i,j+1}^n-V_{i,j-1}^n}{2h_y})+4\frac{\varepsilon k}{h_x^2})$$
$$+V_{i+1,j}^{n+1}(\frac{k}{2h_x}U_{i,j}^n-\frac{\varepsilon k}{h_x^2})-V_{i,j-1}^{n+1}(\frac{k}{2h_y}V_{i,j}^n+\frac{\varepsilon k}{h_y^2})$$
$$+V_{i,j+1}^{n+1}(\frac{k}{2h_y}V_{i,j}^n-\frac{\varepsilon k}{h_y^2})+U_{i,j}^{n+1}(\frac{k(V_{i+1,j}^n-U_{i-1,j}^n)}{2h_x})$$
$$=V_{i,j}^n\left[1+k(\frac{V_{i+1,j}^n-V_{i-1,j}^n}{2h_x})\right]+U_{i,j}^n\left[k(\frac{V_{i,j+1}^n-V_{i,j-1}^n}{2h_y})\right]$$

linearized schemes are obtained, where $i,j=1(1)M-1$. In these schemes $h_x=h_y$, $\varepsilon k/h_x^2=\varepsilon k/h_y^2=a$, $k/2h_x=k/2h_y=b$ and $\varepsilon=1/\mathrm{Re}$ are taken as some simplifications are carried out. Finally, the following

$$-U_{i-1,j}^{n+1}\left[bU_{i,j}^n+a\right]+U_{i,j}^{n+1}\left[1+4a+b(U_{i+1,j}^n-U_{i-1,j}^n)\right]+U_{i+1,j}^{n+1}\left[bU_{i,j}^n-a\right]$$
$$-U_{i,j-1}^{n+1}\left[bV_{i,j}^n+a)\right]+U_{i,j+1}^{n+1}\left[bV_{i,j}^n-a\right]+V_{i,j}^{n+1}\left[b(U_{i,j+1}^n-U_{i,j-1}^n)\right]$$
$$=U_{i,j}^n\left[1+b(U_{i+1,j}^n-U_{i-1,j}^n)\right]+V_{i,j}^n\left[b(U_{i,j+1}^n-U_{i,j-1}^n)\right]$$

and

$$-V_{i-1,j}^{n+1}\left[bU_{i,j}^n+a\right]+V_{i,j}^{n+1}\left[1+4a+b(V_{i,j+1}^n-V_{i,j-1}^n)\right]+V_{i+1,j}^{n+1}\left[bU_{i,j}^n-a\right]$$
$$-V_{i,j-1}^{n+1}\left[bV_{i,j}^n+a)\right]+V_{i,j+1}^{n+1}\left[bV_{i,j}^n-a\right]+U_{i,j}^{n+1}\left[b(V_{i+1,j}^n-U_{i-1,j}^n)\right]$$
$$=V_{i,j}^n\left[1+b(V_{i,j+1}^n-V_{i,j-1}^n)\right]+U_{i,j}^n\left[b(V_{i+1,j}^n-V_{i-1,j}^n)\right]$$

schemes are obtained. Using the known U^n and V^n values in the finite difference diagrams obtained as a result of this linearization, the unknown values of U^{n+1} and V^{n+1} at the desired time t are obtained for all three model problems.

3. Numerical Results

In this section, the numerical solution of the two-dimensional coupled Burgers equation given by the equations (2.1)-(2.2), for three problems with appropriate initial and boundary conditions using the finite difference method based on Rubin-Graves type linearization has been obtained. In order to show the accuracy of the obtained numerical solutions, the following error norms L_2 and L_∞ are calculated

$$L_2=\sqrt{\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\left|U_{ij}-(u_{exact})_{ij}\right|^2}$$

and

$$L_\infty=\max_{i,j}\left|U_{i,j}-(u_{exact})_{i,j}\right|$$

where u_{ij}^n are analytical solutions and U_{ij}^n are approximate solutions at the nodal points (x_i,y_j,t_n) [27].

Problem I: As the first problem, finite difference method has been applied to 2D-CBE having the following exact solution over the region $\Omega=[0,1]\times[0,1]$ [6]

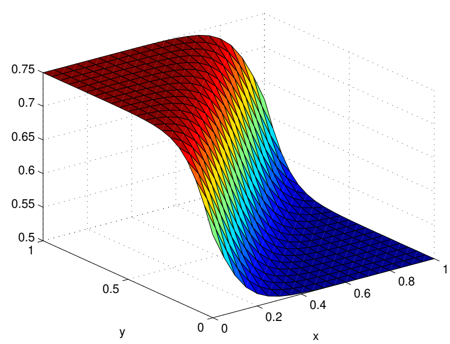
$$u(x,y,t)=\frac{3}{4}-\frac{1}{4\left[1+\exp((-4x+4y-t)\mathrm{Re}/32\right]} \tag{3.1}$$

$$v(x,y,t)=\frac{3}{4}+\frac{1}{4\left[1+\exp((-4x+4y-t)\mathrm{Re}/32\right]}. \tag{3.2}$$

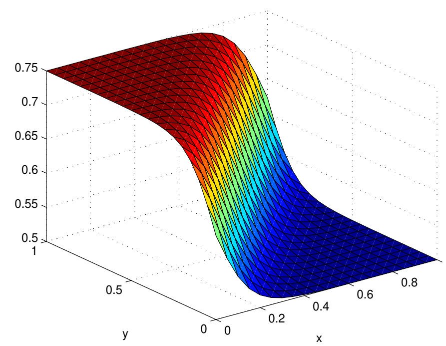
The initial and boundary conditions required for the application of the method are obtained from the analytical solution given by the equations (3.1)-(3.2). Table (1) presents the numerical solutions of Problem I for u for values of $h_x=h_y=0.05$, Re= 10, $\Delta t=10^{-4}$ at times $t=0.01$, 0.5 and 1.0. From the table it is clearly seen that both the

Table 1. Numerical solutions of Problem I for u for values of $h_x = h_y = 0.05$, Re= 10, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.624805	0.624805	0.615254	0.615254	0.605626	0.605626
$(0.5, 0.1)$	0.594202	0.594202	0.585396	0.585396	0.576840	0.576840
$(0.9, 0.1)$	0.567082	0.567082	0.559837	0.559837	0.553017	0.553017
$(0.3, 0.3)$	0.624805	0.624805	0.615255	0.615254	0.605627	0.605626
$(0.7, 0.3)$	0.594202	0.594202	0.585396	0.585396	0.576840	0.576840
$(0.1, 0.5)$	0.655431	0.655431	0.646276	0.646275	0.636685	0.636685
$(0.5, 0.5)$	0.624805	0.624805	0.615256	0.615254	0.605628	0.605626
$(0.9, 0.5)$	0.594202	0.594202	0.585396	0.585396	0.576840	0.576840
$(0.3, 0.7)$	0.655431	0.655431	0.646277	0.646275	0.636687	0.636685
$(0.7, 0.7)$	0.624805	0.624805	0.615256	0.615254	0.605629	0.605626
$(0.1, 0.9)$	0.682611	0.682611	0.674814	0.674814	0.666353	0.666353
$(0.5, 0.9)$	0.655431	0.655431	0.646277	0.646275	0.636687	0.636685
$(0.9, 0.9)$	0.624805	0.624805	0.615255	0.615254	0.605627	0.605626
L_2	8.419211×10^{-8}		2.169158×10^{-6}		2.354379×10^{-6}	
L_∞	6.693449×10^{-8}		2.451640×10^{-6}		2.804863×10^{-6}	



(a)



(b)

Figure 1. (a) Exact and (b) numerical solutions for u of Problem 1 for values of $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.5$.

numerical and analytical solutions at selected points for each time level are very close to each other. Moreover, it is also seen that the computed error norms L_2 and L_∞ are small enough to be acceptable. Table (2) presents the numerical solutions of Problem I for v for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 10$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0. Again from the table it can be observed that the numerical results are very close to their exact counterparts and computed error norms are small enough. Tables (3-4) show also pointwise values and the error norms L_2 and L_∞ of u and v but now for a larger value of Reynold number $\mathrm{Re} = 100$, respectively. As it is seen from the tables, both of the error norms increase as the Reynold number increases. Figures (1-2) show first exact and then numerical solutions for u and v of Problem 1 for values of $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.0$, respectively.

Problem II: Rubin-Graves type linearization finite difference method has been applied to 2D-CBE on the solution domain $\Omega = [0, 0.5] \times [0, 0.5]$ with the following initial

$$u(x, y, 0) = \sin \pi x + \cos \pi y,\ v(x, y, 0) = x + y \tag{3.3}$$

and boundary conditions

$$\left.\begin{array}{ll} u(0, y, t) = \cos(\pi y), & u(0.5, y, t) = 1 + \cos(\pi y) \\ v(0, y, t) = y, & v(0.5, y, t) = 0.5 + y \end{array}\right\} 0 \leq y \leq 0.5, t \geq 0 \tag{3.4}$$

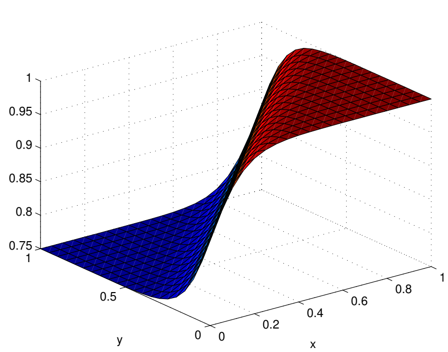
$$\left.\begin{array}{ll} u(x, 0, t) = 1 + \sin(\pi x) & u(x, 0.5, t) = \sin(\pi x) \\ v(x, 0, t) = x & v(x, 0.5, t) = x + 0.5 \end{array}\right\} 0 \leq x \leq 0.5, t \geq 0 \tag{3.5}$$

Table 2. Numerical solutions of Problem I for v for values of $h_x = h_y = 0.05$, Re$= 10$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

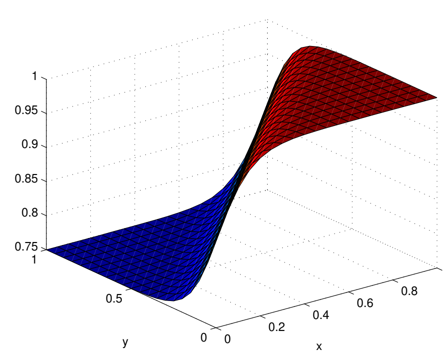
(x,y)	$t=0.01$		$t=0.5$		$t=1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.875195	0.875195	0.884746	0.884746	0.894374	0.894374
$(0.5, 0.1)$	0.905798	0.905798	0.914604	0.914604	0.923160	0.923160
$(0.9, 0.1)$	0.932918	0.932918	0.940163	0.940163	0.946983	0.946983
$(0.3, 0.3)$	0.875195	0.875195	0.884745	0.884746	0.894373	0.894374
$(0.7, 0.3)$	0.905798	0.905798	0.914604	0.914604	0.923160	0.923160
$(0.1, 0.5)$	0.844569	0.844569	0.853724	0.853725	0.863315	0.863315
$(0.5, 0.5)$	0.875195	0.875195	0.884744	0.884746	0.894372	0.894374
$(0.9, 0.5)$	0.905798	0.905798	0.914604	0.914604	0.923160	0.923160
$(0.3, 0.7)$	0.844569	0.844569	0.853723	0.853725	0.863313	0.863315
$(0.7, 0.7)$	0.875195	0.875195	0.884744	0.884746	0.894371	0.894374
$(0.1, 0.9)$	0.817389	0.817389	0.825186	0.825186	0.833647	0.833647
$(0.5, 0.9)$	0.844569	0.844569	0.853723	0.853725	0.863313	0.863315
$(0.9, 0.9)$	0.875195	0.875195	0.884145	0.884146	0.894373	0.894374
L_2	6.013832×10^{-8}		1.511454×10^{-6}		1.599711×10^{-6}	
L_∞	6.693447×10^{-8}		2.451640×10^{-6}		2.804862×10^{-6}	

Table 3. Numerical solutions of Problem I for u for values of $h_x = h_y = 0.05$, Re$= 100$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x,y)	$t=0.01$		$t=0.5$		$t=2.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.623106	0.623047	0.543002	0.543322	0.500470	0.500482
$(0.5, 0.1)$	0.501617	0.501622	0.500341	0.500353	0.500003	0.500003
$(0.9, 0.1)$	0.500011	0.500011	0.500002	0.500002	0.500000	0.500000
$(0.3, 0.3)$	0.623106	0.623047	0.642692	0.543322	0.500441	0.500482
$(0.7, 0.3)$	0.501617	0.501622	0.500317	0.500353	0.500003	0.500003
$(0.1, 0.5)$	0.748272	0.748274	0.742150	0.742214	0.555153	0.555675
$(0.5, 0.5)$	0.623106	0.623047	0.542509	0.543322	0.500414	0.500482
$(0.9, 0.5)$	0.501617	0.501622	0.500304	0.500353	0.500003	0.500003
$(0.3, 0.7)$	0.748272	0.748274	0.742114	0.742214	0.554816	0.555675
$(0.7, 0.7)$	0.623106	0.623047	0.542463	0.543322	0.500384	0.500482
$(0.1, 0.9)$	0.749988	0.749988	0.749945	0.749946	0.744196	0.744256
$(0.5, 0.9)$	0.748272	0.748274	0.742103	0.742214	0.554504	0.555675
$(0.9, 0.9)$	0.623106	0.623047	0.542282	0.543322	0.500525	0.500482
L_2	3.811712×10^{-5}		1.070747×10^{-3}		1.097702×10^{-3}	
L_∞	6.071263×10^{-5}		2.031654×10^{-3}		2.240898×10^{-3}	



(a)



(b)

Figure 2. (a) Exact and (b) numerical solutions for v of Problem 1 for values of $h_x = h_y = 0.05$, Re$= 100$, $\Delta t = 10^{-4}$ at $t = 0.5$.

Table 4. Numerical solutions of Problem I for v for values of $h_x = h_y = 0.05$, Re$= 100$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.876894	0.876953	0.956998	0.956678	0.999530	0.999518
$(0.5, 0.1)$	0.998383	0.998378	0.999659	0.999647	0.999997	0.999997
$(0.9, 0.1)$	0.999989	0.999989	0.999998	0.999998	1.000000	1.000000
$(0.3, 0.3)$	0.876894	0.876953	0.957308	0.956678	0.999559	0.999518
$(0.7, 0.3)$	0.998383	0.998378	0.999683	0.999647	0.999997	0.999997
$(0.1, 0.5)$	0.751728	0.751726	0.757850	0.757786	0.944847	0.944325
$(0.5, 0.5)$	0.876894	0.876953	0.957491	0.956678	0.999586	0.999518
$(0.9, 0.5)$	0.998383	0.998378	0.999696	0.999647	0.999997	0.999997
$(0.3, 0.7)$	0.751728	0.751726	0.757886	0.757786	0.945184	0.944325
$(0.7, 0.7)$	0.876894	0.876953	0957537	0.956678	0.999616	0.999518
$(0.1, 0.9)$	0.750012	0.750012	0.750055	0.750054	0.755804	0.755744
$(0.5, 0.9)$	0.751728	0.751726	0.757897	0.757786	0.945496	0.944325
$(0.9, 0.9)$	0.876894	0.876953	0.957718	0.956678	0.999475	0.999518
L_2	2.736786×10^{-5}		7.126002×10^{-4}		6.043011×10^{-4}	
L_∞	6.071263×10^{-5}		2.031654×10^{-3}		2.240898×10^{-3}	

Table 5. A comparison of numerical solutions for u of Problem 2 for values of $h_x = h_y = 0.025$, Re$= 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$, $N = 40$ with those in Refs. [5, 6, 12].

(x, y)	u				
	Present	[5]	[5] N=40	[6]	[12]
$(0.15, 0.1)$	0.96870	0.95691	0.96066	0.96650	0.96870
$(0.3, 0.1)$	1.03204	0.95616	0.96852	1.02970	1.03200
$(0.1, 0.2)$	0.84618	0.84257	0.84104	0.84449	0.86178
$(0.2, 0.2)$	0.87813	0.86399	0.86866	0.87631	0.87813
$(0.1, 0.3)$	0.67920	0.67667	0.67792	0.67809	0.67920
$(0.3, 0.3)$	0.79944	0.76876	0.77254	0.79792	0.79945
$(0.15, 0.4)$	0.54675	0.54408	0.54543	0.54601	0.66039
$(0.2, 0.4)$	0.58958	0.58778	0.58564	0.58874	0.58958

[12]. There is no analytical solution to this problem. Since Problem II has not analytical solution in Table (5), a comparison of numerical solutions for u of Problem 2 for values of $h_x = h_y = 0.025$, $\mathrm{Re} = 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$, $N = 40$ with those in Refs. [5, 6, 12] is presented. Again, due to the same reason, Table (6) presents a comparison of numerical solutions for v of Problem 2 for values of $h_x = h_y = 0.025$, $\mathrm{Re} = 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [5, 6, 12]. Tables (7-8) show also pointwise values of u and v but now for a smaller value of Reynold number $\mathrm{Re} = 50$, respectively. Figures (3) shows numerical solutions of u and v of Problem 2 for values of $h_x = h_y = 0.025$, Re$= 50$, $\Delta t = 10^{-4}$ at time $t = 0.625$, respectively.

Problem III: The solution domain of the third problem is $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and its analytical solution is of the form [12]

$$
\begin{aligned}
u(x, y, t) &= -\frac{4\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \cos(2\pi x) \sin(\pi y)}{\mathrm{Re}(2 + e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \sin(2\pi x) \sin(\pi y)} \\
v(x, y, t) &= -\frac{2\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \sin(2\pi x) \cos(\pi y)}{\mathrm{Re}(2 + e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \sin(2\pi x) \sin(\pi y)}
\end{aligned}
$$

Table (9) presents numerical solutions of u of Problem 3 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01$, 0.5 and 1.0. From the table one can easily see that the approximate and exact solutions are very close to each other and calculated error norms L_2 and L_∞ are small enough. In a similar manner, Table (10) presents numerical solutions of v of Problem 3 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01$, 0.5 and 1.0. Again, one can see from this table that both of the approximate and exact pointwise values are in good agreement. Th error norms L_2 and L_∞ show the general consistency between the approximate and exact solutions

Table 6. A comparison of numerical solutions for v of Problem 2 for values of $h_x = h_y = 0.025$, Re= 500, $\Delta t = 10^{-4}$ at time $t = 0.625$, $N = 40$ with those in Refs. [5, 6, 12].

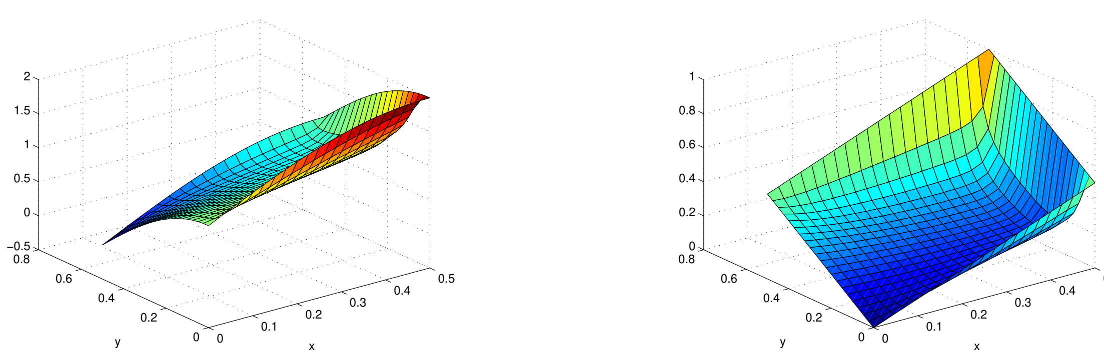
(x, y)	v				
	Present	[5]	[5] N=40	[6]	[12]
$(0.15, 0.1)$	0.09044	0.10177	0.08612	0.09020	0.09043
$(0.3, 0.1)$	0.10730	0.13287	0.07712	0.10690	0.10728
$(0.1, 0.2)$	0.18010	0.18503	0.17828	0.17972	0.17295
$(0.2, 0.2)$	0.16816	0.18169	0.16202	0.16777	0.16816
$(0.1, 0.3)$	0.26268	0.26560	0.26094	0.26222	0.26268
$(0.3, 0.3)$	0.23550	0.25142	0.21542	0.23497	0.23550
$(0.15, 0.4)$	0.31799	0.32084	0.31360	0.31753	0.29022
$(0.2, 0.4)$	0.30418	0.30927	0.29776	0.30371	0.30418

Table 7. A comparison of numerical solutions for u of Problem 2 for values of $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [5, 6, 12].

(x, y)	u			
	Present	[5]	[6]	[12]
$(0.1, 0.1)$	0.97146	0.97258	0.96688	0.97146
$(0.3, 0.1)$	1.15282	1.16214	1.14827	1.15280
$(0.2, 0.2)$	0.86308	0.86281	0.85911	0.86308
$(0.4, 0.2)$	0.97984	0.96483	0.97637	0.97984
$(0.1, 0.3)$	0.66316	0.66318	0.66019	0.66316
$(0.3, 0.3)$	0.77232	0.77030	0.76932	0.77232
$(0.2, 0.4)$	0.58181	0.58070	0.57966	0.58181
$(0.4, 0.4)$	0.75861	0.74435	0.75678	0.75860

Table 8. A comparison of numerical solutions for v of Problem 2 for values of $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [5, 6, 12].

(x, y)	v			
	Present	[5]	[6]	[12]
$(0.1, 0.1)$	0.09869	0.09773	0.09824	0.09869
$(0.3, 0.1)$	0.14158	0.14039	0.14112	0.14158
$(0.2, 0.2)$	0.16754	0.16660	0.16681	0.16754
$(0.4, 0.2)$	0.17110	0.17397	0.17065	0.17110
$(0.1, 0.3)$	0.26378	0.26294	0.26261	0.26378
$(0.3, 0.3)$	0.22654	0.22463	0.22576	0.22655
$(0.2, 0.4)$	0.32851	0.32402	0.32745	0.32851
$(0.4, 0.4)$	0.32500	0.31822	0.32441	0.32501



(a) **(b)**

Figure 3. Numerical solutions of (a) u and (b) v of Problem 2 for values of $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at time $t = 0.625$.

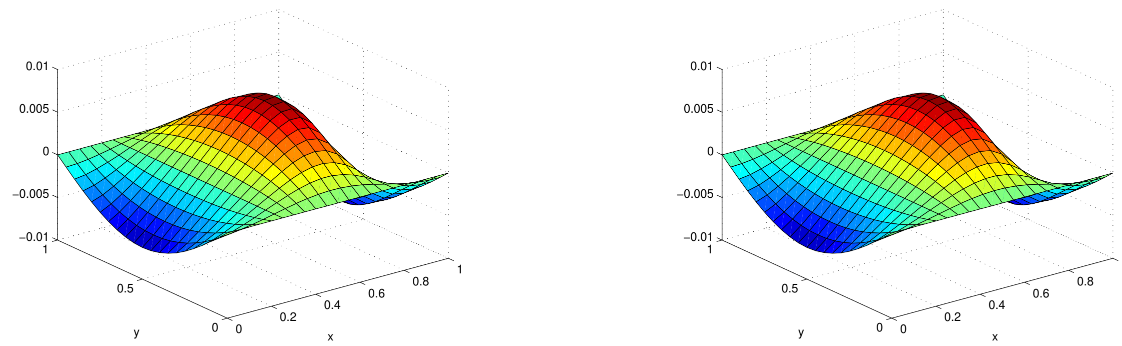
Table 9. Numerical solutions of u of Problem 3 for values of $h_x = h_y = 0.05$, Re$= 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	-0.001439	-0.001439	-0.001408	-0.001408	-0.001376	-0.001376
$(0.5, 0.1)$	0.001941	0.001941	0.001895	0.001894	0.001849	0.001848
$(0.9, 0.1)$	-0.001727	-0.001727	-0.001682	-0.001682	-0.001638	-0.001637
$(0.3, 0.3)$	0.001134	0.001134	0.001114	0.001114	0.001094	0.001094
$(0.7, 0.3)$	0.002551	0.002551	0.002458	0.002453	0.002368	0.002359
$(0.1, 0.5)$	-0.003927	-0.003927	-0.003854	-0.003854	-0.003780	-0.003781
$(0.5, 0.5)$	0.006280	0.006280	0.006130	0.006130	0.005981	0.005981
$(0.9, 0.5)$	-0.007194	-0.007194	-0.006960	-0.006953	-0.006731	-0.006718
$(0.3, 0.7)$	0.001134	0.001134	0.001114	0.001114	0.001094	0.001094
$(0.7, 0.7)$	0.002551	0.002551	0.002458	0.002453	0.002368	0.002359
$(0.1, 0.9)$	-0.001439	-0.001439	-0.001408	-0.001408	-0.001376	-0.001376
$(0.5, 0.9)$	0.001941	0.001941	0.001895	0.001894	0.001849	0.001848
$(0.9, 0.9)$	-0.001727	-0.001727	-0.001682	-0.001682	-0.001638	-0.001637
L_2	2.2105×10^{-5}		1.0312×10^{-3}		1.9287×10^{-3}	
L_∞	2.8241×10^{-7}		1.2663×10^{-5}		2.2938×10^{-5}	

throughout the solution domain. Figures (4-5) show first exact and then numerical solutions for u and v of Example 3 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at $t = 0.01$, respectively.

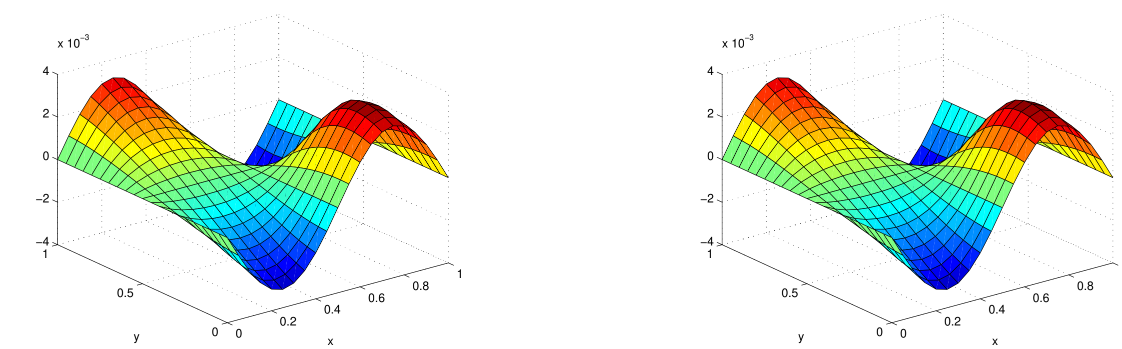
Table 10. Numerical solutions of v of Problem 3 for values of $h_x = h_y = 0.05$, Re$= 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Ecaxt
$(0.1, 0.1)$	-0.001609	-0.001609	-0.001574	-0.001574	-0.001539	-0.001539
$(0.5, 0.1)$	-0.000000	-0.000000	-0.000000	-0.000000	-0.000001	-0.000000
$(0.9, 0.1)$	0.001931	0.001931	0.001880	0.001880	0.001830	0.001830
$(0.3, 0.3)$	-0.001268	-0.001268	-0.001246	-0.001246	-0.001223	-0.001224
$(0.7, 0.3)$	0.002852	0.002852	0.002746	0.002743	0.002643	0.002637
$(0.1, 0.5)$	-0.000000	-0.000000	-0.000000	-0.000000	-0.000000	-0.000000
$(0.5, 0.5)$	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000
$(0.9, 0.5)$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$(0.3, 0.7)$	0.001268	0.001268	0.001246	0.001246	0.001223	0.001224
$(0.7, 0.7)$	-0.002852	-0.002852	-0.002746	-0.002743	-0.002643	-0.002637
$(0.1, 0.9)$	0.001609	0.001609	0.001574	0.001574	0.001539	0.001539
$(0.5, 0.9)$	0.000000	0.000000	0.000000	0.000000	0.000001	0.000000
$(0.9, 0.9)$	-0.001931	-0.001931	-0.001880	-0.001880	-0.001830	-0.001830
L_2	1.2846×10^{-5}		6.0214×10^{-4}		1.1320×10^{-3}	
L_∞	9.3390×10^{-8}		4.1431×10^{-6}		7.3722×10^{-6}	



(a) **(b)**

Figure 4. (a) Exact and (b) numerical solutions of u of Problem 3 for values $h_x = h_y = 0.05$, Re$= 1000$, $\Delta t = 10^{-3}$ at time $t = 0.01$.



(a) **(b)**

Figure 5. (a) Exact and (b) numerical solutions of v of Problem 3 for values $h_x = h_y = 0.05$, Re$= 1000$, $\Delta t = 10^{-3}$ at time $t = 0.01$.

4. Conclusion

In this study, numerical solutions of two dimensional coupled Burgers equation has been obtained by using finite difference method based on a Rubin-Graves type linearization. To demonstrate the accuracy and efficiency of the method, this method has been applied to three test problems with known analytical solutions and to one test problem with unknown analytical solution. The error norms L_2 and L_∞ have been calculated. From these calculations, it is seen that the proposed method yield good enough results, and it is simple and easy to apply. In conclusion, numerical solution of two dimensional coupled nonlinear partial differential equations arises in physical sciences can be achieved easily and effectively by the proposed method. The algebraic systems found out by using the proposed schemes can be easily stored and solved by the software systems of nowadays. As a conclusion, the proposed method can be easily and successfully applied to this type of problems arising in applied mathematics, mathematical physics and engineering science.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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