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Research Article

Commutative graded-n-coherent and graded valuation rings

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Abstract

Let $R = \oplus_{\alpha \in G} R_\alpha$ be a commutative ring with unity graded by an arbitrary grading commutative monoid G. For each positive integer, the notions of a graded-n-coherent module and a graded-n-coherent ring are introduced. In this paper many results are generalized from n-coherent rings to graded-n-coherent rings. In the last section, we provide necessary and sufficient conditions for the graded trivial extension ring to be a graded-valuation ring.

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1. Introduction

We devote this section to some conventions and a recall of some standard terminology. All rings are commutative with unity, and all modules are unital. G will denote a grading commutative monoid (that is, a commutative monoid, written additively, with an identity element denoted by 0), and all the graded rings and modules are graded by G.

If n is a nonnegative integer, we say that an R-module M is n-presented if there is an exact sequence $F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ of R-modules in which each F_i, is finitely generated and free. (Our usage follows [8]; in [15], such M is said to "have a finite n-presentation"). In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Following [5] we let $\lambda(M) = \lambda_R(M) = \sup\{n \mid M$ is an n-presented R-module$\}$, so that $0 \leq \lambda(M) \leq \infty$; the properties of the function λ are recalled in Lemma 2.3. Classically, the "n-presented" concept allows both ideal-theoretic and module-theoretic approaches to coherent rings. Indeed (cf. [5], p. 63, Exercise 12), a ring R is said to be coherent if each finitely generated ideal is finitely presented ; equivalently if each finitely presented R-module is 2-presented.

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Let n be a positive integer. Recall from [9] that R is n-coherent (as a ring) if each $(n-1)$-presented ideal of R is n-presented; and that R is a strong n-coherent ring if each n-presented R-module is $(n+1)$-presented. Thus, the 1-coherent rings are just the coherent rings. In general, any strong n-coherent ring is n-coherent (by, for instance, the version of Schanuel's Lemma in [15] p. 89). The converse holds if $n = 1$ by the result [5, p. 63, Exercise 12]. Noted that each Bezout (for instance, valuation) domain R is n-coherent for each $n \geq 1$; indeed, each $(n-1)$-presented ideal of R is principal and hence infinitely-presented (in the obvious sense). Moreover, each Noetherian ring is n-coherent for any $n \geq 1$. An excellent summary of work done on n-coherence can be found in [9]. And for background on coherence, we refer the reader to [11].

The concept of coherence has many graded generalizations, see e.g.,([2] and [3]). Among these generalizations, we have the graded-n-coherence. Accordingly, like it was done in [9], we use the λ-function to introduce both ideal and module theoretic approaches to "graded-n-coherence" for any positive integer n.

Section 2 begins, more generally, by defining graded-n-coherent modules for each integer $n \geq 1$. As one might expect, the graded-1-coherent modules are just the "graded-coherent modules" in [2]. Among other things, we show that, if R is a graded ring and $0 \to P \overset{u}{\to} N \overset{v}{\to} M \to 0$ an exact sequence of graded R-modules. Then if $\lambda(P) \geq n-1, N$ is a graded-n-coherent module and v has a cancellable degree then M is a graded-n-coherent module and if $\lambda(M) \geq n$ and N is a graded-n-coherent module, then P is a graded-n-coherent module. We also show that if $m \geq n$ is a positive integer and $M_0 \overset{u_1}{\to} M_1 \overset{u_2}{\to} M_2 \to \cdots \overset{u_m}{\to} M_m$ an exact sequence of graded-n-coherent R-modules such that the degree of every u_i is cancellable. Then $\operatorname{Im}(u_i)$, $\operatorname{Ker}(u_i)$ and $\operatorname{Coker}(u_i)$ are graded-n-coherent R-modules for each $i = 1, 2, \dots, m$. We also show that if $n \geq 1$, the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n$, and M is a graded R-module. Then M is graded-n-coherent as a graded R/I-module if and only if M is graded-n-coherent as a graded R-module. We also show that, if $R \to S$ is a graded ring homomorphism making S a faithfully flat R-module, M a graded R-module and $M \otimes S$ a graded-n-coherent S-module, then M is a graded-n-coherent R-module.

In Section 3, we introduce and study the notion of graded-n-coherent rings. Among other things, we show that, if R is a graded-n-coherent ring and I an $(n-1)$-presented homogeneous ideal of R. Then R/I is a graded-n-coherent ring. We also show that, if $R \to S$ is a graded ring homomorphism making S a faithfully flat R-module and S is a graded-n-coherent ring, then R is a graded-n-coherent ring. We also show that, if $(R_i)_{i=1,2,\dots,m}$ is a family of graded rings. Then $\prod_{i=1}^{m} R_i$ is a graded-n-coherent ring if and only if R_i is a graded-n-coherent ring, for each $i = 1, \dots, m$.

In section 4, we introduce and characterise the notion of graded-valuation rings and then, as a main results of this section, we characterise the gr-valuation property in the graded trivial extension ring, more precisely, we show that, in the case where the grading monoid is a torsionfree abelian group, if A is graded ring and E an nonzero graded A-module and $R := A \propto E$ the graded trivial extension ring of A by E. If E is a non-torsion graded A-module, then R is a gr-valuation ring if and only if A is a gr-valuation domain and E is isomorphic to A_H, the homogeneous quotient field of fractions of A. We also show that if A is a graded ring and E a nonzero graded A-module. Then $R := A \propto E$ the graded trivial extension ring is a gr-valuation ring if ond only if A is a gr-valuation domain, E is a gr-divisible and gr-uniserial A-module.

Now we will recall some definitions and basic properties on graded rings and modules, see for instance[6, II, §11, pp. 163-176]. Let G be a grading commutative monoid written additively with an identity element denoted by 0. By a graded ring R, we mean a ring graded by G, that is, a direct sum of subgroups R_α of R such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every α, $\beta \in G$. The set $h(R) = \cup_{\alpha \in G} R_\alpha$ is the set of homogeneous elements of R. A nonzero element $x \in R$ is called homogeneous if it belongs to one of the R_α, homogeneous of degree α if $x \in R_\alpha$. Every $z \in R$ may be written uniquely as a sum $z = z_{\alpha_1} + \cdots + z_{\alpha_n}$ of homogeneous elements $z_{\alpha_i} \in R_{\alpha_i}$ where $\alpha_1, \cdots, \alpha_n$ are distincts; z_{α_i} is called the homogeneous component of degree α_i of z. If G is cancellative, then R_0 is a subring of R (intuitionally $1 \in R_0$) and every R_α is an R_0-module.

By a graded R-module M, we mean an R-module graded by G, that is, a direct sum of subgroups M_α of M such that $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha, \beta \in G$. A graded R-module M is called a graded-free R-module (gr-free) if there exists a basis $(m_i)_{i\in I}$ of M consisting of homogeneous elements. Note that, any graded-free R-module is a free R-module; the converse is false[13, p. 21]. When G is cancellative, the M_α are R_0-modules. Obviously, R is a graded R-module.

Let R and R' be two graded rings, a ring homomorphism $f : R \to R'$ is called graded if $f(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in G$. A graded ring isomorphism is a bijective graded ring homomorphism. Let M and M' be two graded R-modules and let $v : M \to M'$ be an R-module homomorphism and $\beta \in G$; v is called graded of degree β if $v(M_\alpha) \subseteq M'_{\alpha+\beta}$ for all $\alpha \in G$. An R-module homomorphism $v : M \to M'$ is called graded if there exists $\beta \in G$ such that v is graded of degree β. A graded R-module isomorphism is a bijective graded R-module homomorphism of degree 0. If $v \neq 0$ and G is cancellative, the degree of v is, then determined uniquely. An exact sequence of graded R-modules is an exact sequence, where the R-modules and the R-module homomorphisms in question are graded.

A submodule N of M is called homogeneous if $N = \oplus_{\alpha\in G}(N \cap M_\alpha)$. It is well known that the following are equivalent for a submodule N of M: (1) N is homogeneous; (2) the homogeneous components of every element of N belong to N; (3) N is generated by homogeneous elements. A homogeneous submodule of R is called a homogeneous ideal of R. If N is a homogeneous submodule of a graded R-module M, then M/N is a graded R-module, where $(M/N)_\alpha := (M_\alpha + N)/N$. If I is a homogeneous ideal of a graded ring R, then R/I is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$. A homogeneous ideal M of R is called a maximal homogeneous ideal (gr-maximal) if it is maximal among proper homogeneous ideals; equivalently, if every nonzero homogeneous element of R/M is invertible. A graded ring is said to be graded-local (gr-local) if it has a unique gr-maximal ideal and a graded ring R is called a graded-field (gr-field) if every homogeneous element of R is invertible. Obviously, a field which is also a graded ring is a gr-field, while gr-field need not be a field[13, page 46].

Let R_1 and R_2 be two graded rings. Then $R = R_1 \times R_2$ is a graded ring with homogeneous elements $h(R) = \cup_{\alpha\in G} R_\alpha$, where $R_\alpha = (R_1)_\alpha \times (R_2)_\alpha$ for all $\alpha \in G$. It is well known that an ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$ for some ideals I_1 of R_1 and I_2 of R_2. Also it is easily seen that $I_1 \times I_2$ is a homogeneous ideal of $R_1 \times R_2$ if and only if I_1, I_2 are homogeneous ideals of R_1 and R_2, respectively.

Let R be a graded ring and let M and M' be graded R-modules. Define $(M \otimes_R M')_\alpha$ as the additive group of $M \otimes_R M'$ generated by the $m_\mu \otimes m'_\nu$, where $m_\mu \in M_\mu$, $m'_\nu \in M'_\nu$ and $\mu + \nu = \alpha$. Then $((M \otimes_R M')_\alpha)_{\alpha\in G}$ is a graduation of $M \otimes_R M'$ and $M \otimes_R M'$ is a graded R-module.

Assume that the grading monoid is a cancellative torsion-free monoid. Let R be a graded ring. R is called a graded-Noetherian ring (gr-Noetherian ring) if it satisfies the ascending chain condition (a.c.c.) on homogeneous ideals; equivalently, if each homogeneous prime ideal of R is finitely generated[14, Lemma 2.3]. Obviously, a Noetherian ring is a gr-Noetherian ring, while gr-Noetherian rings need not be Noetherian. It is known that the monoid ring $A[X; G]$ over a ring A is a Noetherian ring (resp. gr-Noetherian ring) if and only if R is a Noetherian ring and G (resp. each ideal of G) is finitely generated ([10, Theorem 7.7, p. 75] (resp. [14], Theorem 2.4). Hence, if $\mathbb{Q}$ is the additive group of rational numbers and D is a Noetherian ring, the group ring; $A = D[X; \mathbb{Q}]$ is a gr-Noetherian ring but not a Noetherian ring.

Finally, let R be a graded ring. R is called graded-coherent (gr-coherent) if every finitely generated homogeneous ideal is finitely presented. Obviously, every coherent graded ring is a graded-coherent ring while the converse is false in general[2, Example 3.2].

2. Graded-n-coherent modules

Definition 2.1. Let R be a graded ring and let n be a positive integer, we say that a graded R-module M is a graded-n-coherent module if M is n-presented and each $(n-1)$-presented homogeneous submodule of M is n-presented.

It follows from [2] that the graded-1-coherent modules are just the "graded-coherent modules".

Remark 2.2. Let R be a graded ring and let n be a positive integer. Then following assertions hold:

(1) Every $(n-1)$-presented homogeneous submodule of a graded-n-coherent R-module is a graded-n-coherent R-module.
(2) Any n-coherent graded R-module is a graded-n-coherent R-module.

For reference purposes, it will be helpful to recall the following elementry result [1, p. 61, Exercice 6] which summarize some behavior of λ.

Lemma 2.3. *Let R be a ring and let $0 \to P \to N \to M \to 0$ be an exact sequence of R-modules. Then:*

(1) $\lambda(N) \geq \inf\{\lambda(P), \lambda(M)\}$
(2) $\lambda(M) \geq \inf\{\lambda(N), \lambda(P)+1\}$
(3) $\lambda(P) \geq inf\, \{\lambda(N), \lambda(M)-1\}$
(4) *If $N = P \oplus M$ then $\lambda(N) = inf\{\lambda(M), \lambda(P)\}$.*

Theorem 2.4. *Let R be a graded ring and let $0 \to P \xrightarrow{u} N \xrightarrow{v} M \to 0$ be an exact sequence of graded R-modules.*

(1) *If $\lambda(P) \geq n-1$, N is a graded-n-coherent module and v has a cancellable degree then M is a graded-n-coherent module.*
(2) *If $\lambda(M) \geq n$ and N is a graded-n-coherent module, then P is a graded-n-coherent module.*

Proof. (1) P is $(n-1)$-presented and N is n-presented; therefore, M is n-presented by Lemma 2.3. Let M_1 be an $(n-1)$-presented homogeneous submodule of M. Since v has a cancellable degree the submodule $v^{-1}(M_1)$ of N is homogeneous. Then the exact sequence : $0 \to P \xrightarrow{u} v^{-1}(M_1) \xrightarrow{v} M_1 \to 0$ shows that $\lambda\left(v^{-1}(M_1)\right) \geq \inf\{\lambda(P), \lambda(M_1)\} \geq n-1$ (Lemma 2.3 (1)); therefore, $\lambda\left(v^{-1}(M_1)\right) \geq n$ since $v^{-1}(M_1) \subseteq N$ and N is graded-n-coherent. We conclude, by Lemma 2.3(2), that $\lambda(M_1) \geq \inf\left\{\lambda\left(v^{-1}(M_1)\right), \lambda(P)+1\right\} \geq n$.

(2) M and N are both n-presented; therefore, P is $(n-1)$-presented by Lemma 2.3(3). Every $(n-1)$-presented homogeneous submodule of a graded-n-coherent module is a graded-n-coherent module by Remark 2.2(1); hence, P is a graded-n-coherent R-module.

□

Theorem 2.5. *Let $m \geq n$ be positive integer and let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \cdots \xrightarrow{u_m} M_m$ be an exact sequence of graded-n-coherent R-modules such that the degree of every u_i is cancellable. Then $\operatorname{Im}(u_i), \operatorname{Ker}(u_i)$ and $\operatorname{Coker}(u_i)$ are graded-n-coherent R-modules for each $i = 1, 2, \ldots, m$.*

Proof. It suffices to prove the assertion for $m = n$. Let $M_0 \xrightarrow{u_1} M_1 \xrightarrow{u_2} M_2 \to \ldots \xrightarrow{u_n} M_n$ be an exact sequence of graded-n-coherent R-modules. We then have the exact sequences of

graded R-modules:
$0 \to K\,\mathrm{er}\,(u_1) \to M_0 \to \mathrm{Im}\,(u_1) \to 0$
$0 \to \mathrm{Im}\,(u_i) = \mathrm{Ker}\,(u_{i+1}) \to M_i \to \mathrm{Im}\,(u_{i+1}) \to 0$, for each $i = 1, \ldots, n-1$, and
$0 \to \mathrm{Im}\,(u_n) \to M_n \to \mathrm{Coker}\,(u_n) \to 0$
Since the degree of u_1 is cancellable, $Im\,(u_1)$ is a finitely generated homogeneous submodule of M_1 since M_0 is finitely generated (for M_0 is graded-n-coherent) therefore, $\mathrm{Im}\,(u_2)$ is 1-presented; and by induction, we conclude that $\mathrm{Im}\,(u_n)$ is $(n-1)$-presented. Thus $\mathrm{Im}\,(u_n)$ is a graded-n-coherent module by Remark 2.2(1) since $\mathrm{Im}\,(u_n)$ is a homogeneous submodule of the graded-n-coherent module M_n. Therefore $Im\,(u_i)$ and $\mathrm{Ker}\,(u_i)$ are graded-n-coherent modules by applying Theorem 2.4 to the above exact sequences of graded R-modules. Finally, Theorem 2.4 and the exact sequences of graded R-modules of degree 0, $0 \to Im\,(u_i) \to M_i \to \mathrm{Coker}\,(u_i) \to 0$ show that $\mathrm{Coker}\,(u_i)$ is graded-n-coherent module for each $i = 1, \ldots, m$. □

Theorem 2.6. *Let M be a graded R-module and I be a homogeneous ideal of R such that $IM = 0$. Let $n \geq 1$ and let the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n$. Then M is graded-n-coherent as a graded R/I-module if and only if M is graded-n-coherent as a graded R-module.*

Before establishing this theorem, we first prove the following three Lemmas.

Lemma 2.7. *Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) > n$ and let M be an n-presented graded S-module. Then M is an n-presented graded R-module*

Proof. We proceed by induction on n. Case $n = 0$: If M is a finitely generated graded S-module and S a finitely generated graded R-module, it is clear that M is a finitely generated graded R-module. Assume that the result is true for n. Let M be an $(n+1)$-presented graded S-module and let $\lambda_R(S) \geq n+1$. We have to show that $\lambda_R(M) \geq n+1$. Let $F_{n+1} \overset{u_{n+1}}{\to} F_n \overset{u_n}{\to} \ldots \to F_1 \overset{u_1}{\to} F_0 \overset{u_0}{\to} M \to 0$ be a finite $(n+1)$-presentation of M as a graded S-module. The exact sequence of S-modules $0 \to Ker\,(u_0) \to F_0 \to M \to 0$ shows that $\lambda_S\,(\mathrm{Ker}\,(u_0)) \geq n$; so by induction we have $\lambda_R\,(\mathrm{Ker}\,(u_0)) \geq n$ since $\lambda_R(S) \geq n+1 \geq n$. Moreover $\lambda_R\,(F_0) \geq n+1$ since $\lambda_R(S) \geq n+1$ and F_0 is a finitely generated free graded S-module. Hence $\lambda_R(M) \geq \inf\,\{\lambda_R\,(F_0)\,, \lambda_R\,(\mathrm{Ker}\,(u_0)) + 1\} \geq n+1$ by Lemma 2.3(2) as desired. □

Lemma 2.8. *Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module. If M is n-presented as a graded R-module, then it is n-presented as a graded S-module.*

Proof. We proceed by induction on n. Case $n = 0$: If M is a finitely generated graded R-module, then M is also a finitely generated graded S-module.

We conclude the proof by induction on n. Let M be a graded S-module such that $\lambda_R(M) \geq n+1$ and $\lambda_R(S) \geq n$. We have to show that $\lambda_S(M) \geq n+1$. By induction, we have $\lambda_S(M) \geq n$. The exact sequence of S-modules $0 \to K \to F_0 \to M \to 0$ (in which F_0 is a finitely generated free S-module), considered as an exact seguence of R-modules, shows that $\lambda_R(K) \geq \inf\,\{\lambda_R\,(F_0)\,; \lambda_R(M) - 1\} \geq n$ (Lemma 2.3(3)). Moreover, we have $\lambda_R(S) \geq n \geq n-1$; then by induction we have $\lambda_{\mathrm{S}}(K) \geq n$; hence, $\lambda_S(M) \geq n+1$ by Lemma 2.3(2) as desired. □

Lemma 2.9. *Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be an S-module. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as a graded S-module.*

Proof. Let $R \to S$ be a graded ring homomorphism such that $\lambda_R(S) \geq n-1$ and let M be a graded S-module such that M is graded-n-coherent as a graded R-module. Lemma 2.8 shows that $\lambda_S(M) \geq n$ since $\lambda_R(M) \geq n$ and $\lambda_R(S) \geq n-1$. Let N be a homogeneous submodule of the graded S-module M such that $\lambda_S(N) \geq n-1$. Then by Lemma 2.7 we have $\lambda_R(N) \geq n-1$. Thus, $\lambda_R(N) \geq n$ since M is a graded-n-coherent R-module; therefore, $\lambda_S(N) \geq n$ by Lemma 2.8 as desired. □

(Proof of Theorem 2.6): Let $R \to R/I$ be the canonical graded ring homomorphism such that $\lambda_R(R/I) \geq n$ and let M be a graded R-module such that $IM = 0$. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as a graded R/I-module by Lemma 2.9 since $\lambda_R(R/I) \geq n \geq n-1$. Conversely, let M be a graded-n-coherent R/I-module. By Lemma 2.7 we have $\lambda_R(M) \geq n$ since $\lambda_R(R/I) \geq n$. Let N be a homogeneous submodule of the graded R-module M such that $\lambda_R(N) \geq n-1$. By Lemma 2.8 we have $\lambda_{R/I}(N) \geq n-1$ since $\lambda_R(R/I) \geq n$. Thus $\lambda_{R/I}(N) \geq n$ since M is a graded-n-coherent R/I-module and N is a homogeneous submodule of M as a graded R/I-module. Hence, $\lambda_R(N) \geq n$ by Lemma 2.7 ($\lambda_R(R/I) \geq n$) and this completes the proof of Theorem 2.6. □

Remark 2.10. Let the canonical graded ring homomorphism $R \to R/I$ satisfy $\lambda_R(R/I) \geq n-1$, and let M be a graded R-module such that $IM = 0$, where I is a homogeneous ideal of R. If M is graded-n-coherent as a graded R-module, then it is graded-n-coherent as an R/I-module by Lemma 2.9.

Theorem 2.11. *Let $R \to S$ be a graded ring homomorphism making S a faithfully flat R-module and let M be a graded R-module. If $M \otimes S$ is a graded-n-coherent S-module, then M is a graded-n-coherent R-module.*

Proof. We have $\lambda_S(M \otimes S) \geq n$ since $M \otimes S$ is a graded-n-coherent S-module; therefore, $\lambda_R(M) \geq n$ since S is a faithfully flat R-module. Let N be an $(n-1)$-presented homogeneous submodule of M. Since S is a flat R-module, $\lambda_S(N \otimes S) \geq n-1$ and we may assume that $N \otimes S \subseteq M \otimes S$. Thus, $\lambda_S(N \otimes S) \geq n$ (since $N \otimes S$ is homogeneous and $M \otimes S$ is a graded-n-coherent S-module); therefore, $\lambda_R(N) \geq n$ since S is a faithfully flat R-module. □

3. Graded-n-coherent rings

Definition 3.1. A graded ring R is called graded-n-coherent if it is graded-n-coherent as a graded R-module, that is, if each $(n-1)$-presented homogeneous ideal of R is n-presented.

Remark 3.2. Obviously, every n-coherent graded ring is a graded-n-coherent ring. The converse is not true in general, example 3.2 in [2] gives an example of graded-1-coherent ring which is not 1-coherent.

The next result shows that we have already many examples of graded-n-coherent rings.

Example 3.3.
(1) Every graded-valuation domain is a graded-n-coherent ring for each $n \geq 1$, see [1].
(2) Every graded-Noetherian ring is a graded-n-coherent for each $n \geq 1$, see [7].

Proposition 3.4. *Let R be a graded-n-coherent ring and let I be an $(n-1)$-presented homogeneous ideal of R. Then R/I is a graded-n-coherent ring.*

Proof. Since R is a graded-n-coherent R-module, it follows from Theorem 2.4(1) that R/I is a graded-n-coherent R-module; therefore, by Theorem 2.6, R/I is a graded-n-coherent ring. □

Remark 3.5. The case $n = 1$ recovers the known fact that if I is a finitely generated homogeneous ideal of a graded-1-coherent ring R, then R/I is a graded-1-coherent ring [2, Theorem 3.7(1)].

Theorem 3.6. *Let $R \to S$ be a graded ring homomorphism making S a faithfully flat R-module. If S is a graded-n-coherent ring, then R is a graded-n-coherent ring.*

Proof. This is straightforward by taking $M = R$ in Theorem 2.11. □

Theorem 3.7. *Let $(R_i)_{i=1,2,\ldots,m}$ be a family of graded rings. Then $\prod_{i=1}^{m} R_i$ is a graded-n-coherent ring if and only if R_i is a graded-n-coherent ring, for each $i = 1, \ldots, m$.*

To establish this Theorem, we need to prove the following Lemma.

Lemma 3.8. *Let R_1 and R_2 be two graded rings. Then R_i is an infinitely presented homogeneous ideal of $R_1 \times R_2$, for $i = 1, 2$.*

Proof. The graded rings R_1 and R_2, more precisely $R_1 \times 0$ and $0 \times R_2$, are two finitely generated homogeneous ideals of $R_1 \times R_2$ since $0 \to R_1 \to R_1 \times R_2 \to R_2 \to 0$ and $0 \to R_2 \to R_1 \times R_2 \to R_1 \to 0$ are two exact sequences of graded rings. We finish the proof of this Lemma by induction on the degrees of presentation of the R_i using the above two exact sequences of graded rings. □

(Proof of Theorem 3.7): We proceed by induction on m, it suffices to prove the assertion for $m = 2$. Let R_1 and R_2 be two graded rings such that $R_1 \times R_2$ is a graded-n-coherent ring. Since $R_1 \cong (R_1 \times R_2)/R_2$, $R_2 \cong (R_1 \times R_2)/R_1$ are two graded ring isomorphism, and the R_i are infinitely presented homogeneous ideals of $R_1 \times R_2$ by Lemma 3.8, then Proposition 3.4 shows that R_i $(i = 1, 2)$ are graded-n-coherent rings. Conversely, let R_1 and R_2 be two graded-n-coherent rings and let $I = I_1 \times I_2$ be an $(n-1)$-presented homogeneous ideal of $R_1 \times R_2$, where I_i is a homogeneous ideal of R_i; then for each $i = 1, 2$: $\lambda_{R_1 \times R_2}(I_i) \geq \inf\{\lambda_{R_1 \times R_2}(I_1), \lambda_{R_1 \times R_2}(I_2)\} = \lambda_{R_1 \times R_2}(I) \geq n-1$ (Lemma 2.3(4)). By Lemma 2.8, we have $\lambda_{R_i}(I_i) \geq n-1$ ($\lambda_{R_1 \times R_2}(R_i) = \infty$ (Lemma 3.8)). Thus, $\lambda_{R_i}(I_i) \geq n$ since R_i is a graded-n-coherent ring and by Lemma 2.7, we have $\lambda_{R_1 \times R_2}(I_i) \geq n$ since $\lambda_{R_1 \times R_2}(R_i) = \infty$ (Lemma 3.8). Hence : $\lambda_{R_1 \times R_2}(I) = \lambda_{R_1 \times R_2}(I_1 \times I_2) = \inf\{\lambda_{R_1 \times R_2}(I_1), \lambda_{R_1 \times R_2}(I_2)\} \geq n$ and this completes the proof of Theorem 3.7. □

4. Graded-valuation property in graded trivial extension

Assume that the grading monoid G is torsionless, that is a commutative, cancellative monoid and the quotient group of G is a torionfree abelian group. Let A be a graded ring, and let $Q(A)$ denote the total ring of quotients of A and H the saturated multiplicative set of regular homogeneous elements of A. Then, by extending some definitions to the case where rings are with zero divisors, A_H, called the homogeneous total ring of quotients of A, is a ring graded by $\langle G \rangle$, where $A_H = \oplus_{\alpha \in \langle G \rangle} (A_H)_\alpha$ with $(A_H)_\alpha = \{\frac{a}{b} \mid a \in A_\beta, b$ a regular element of A_γ and $\beta - \gamma = \alpha\}$. If A is a graded integral domain (An integral domain graded by G), then A_H is called the homogeneous quotient field of A. Clearly, every nonzero homogeneous element of A_H is invertible and $(A_H)_0$ is a field. We say that A is a graded-valuation ring (gr-valuation ring for short) if either $x \in A$ or $x^{-1} \in A$ for every nonzero homogeneous element $x \in A_H$. Recall that if A is a graded ring and E is a graded A-module, then $A \propto E$ is a graded ring where $A \propto E = \oplus_{\alpha \in G}(A \propto E)_\alpha = \oplus_{\alpha \in G}(A_\alpha \oplus E_\alpha)$. This section gives a result of the transfer of gr-valuation property to graded trivial extension ring.

We begin with the following result extending Theorem 1.2 in [1] to the case where rings are with zero divisors and which characterize gr-valuation rings.

Theorem 4.1. *Let $A = \oplus_{\alpha \in G} A_\alpha$ be a graded ring. The following statements are equivalent:*

(1) *A is a gr-valuation ring.*
(2) *Either $a \mid b$ or $b \mid a$ for every nonzero homogeneous elements $a, b \in A$, one at least of which is regular.*
(3) *Every pair of homogeneous (fractional) ideals of A, one at least of which is regular, are totally ordered under inclusion.*

(4) *Every pair of principal homogeneous ideals of A, one at least of which is regular, are totally ordered under inclusion.*

Proof. (1) implies (2) by definition.
(2) implies (3): Let I, J be two homogeneous ideals of A, one at least of which is regular, suppoose that $I \nsubseteq J$ and $J \nsubseteq I$. Let x be an homogeneous element of $I \setminus J$ and y be an homogeneous element of $J \setminus I$. Since $x = (\frac{x}{y})y \notin J$ we have $\frac{x}{y} \notin A$ and since $y = (\frac{y}{x})x \notin I$, we have $\frac{y}{x} \notin A$; therefore A is not a gr-valuation ring, a contradiction.
(3) implies (4) is trivial.
(4) implies (1): Let $x = \frac{a}{b} \in A_H$, where $a, b \in h(A)$. If $x \notin A$, then $Aa \nsubseteq Ab$, therefore $Ab \subseteq Aa$, hence $x^{-1} = \frac{b}{a} \in A$. Hence A is a gr-valuation ring. □

Let A be a graded ring, where G is a commutative monoid. Following [4], a proper homogeneous ideal P of A is said to be a homogeneous 2-prime ideal if whenever $ab \in P$ for some $a, b \in h(A)$, then either $a^2 \in P$ or $b^2 \in P$. Many characterizations of gr-valuation domains are given in [1]. Now, we give a new characterization of gr-valuation domains in terms of homogeneous 2-prime ideals.

Theorem 4.2. *Let $A = \bigoplus_{\alpha \in G} A_\alpha$ be a graded integral domain, where the grading monoid G is torsionless. The following statements are equivalent.*
(i) A is a gr-valuation domain.
(ii) Every proper homogeneous ideal is a homogeneous 2-prime ideal.
(iii) Every proper principal homogeneous ideal is a homogeneous 2-prime ideal.

Proof. $(i) \Rightarrow (ii)$: Suppose that A is a gr-valuation domain and P be a proper homogeneous ideal of A. Choose $a, b \in h(A)$ such that $ab \in P$. Assume that a, b are nonzero homogeneous elements of A. Since A is a gr-valuation domain, by Theorem 4.1, $a|b$ or $b|a$. This implies that $ab|a^2$ or $ab|b^2$. Without loss of generality, we may assume that $ab|a^2$. In this case, $a^2 \in (ab) \subseteq P$ which completes the proof.
$(ii) \Rightarrow (iii)$: It is clear.
$(iii) \Rightarrow (i)$: Suppose that every proper principal homogeneous ideal is homogeneous 2-prime. Let $a, b \in h(A)$ be nonzero homogeneous elements of A. Assume that a, b are nonunits. Then $P = (ab)$ is a proper homogeneous ideal of A. By assumption, P is homogeneous 2-prime. Since $ab \in P$, we have $a^2 \in P$ or $b^2 \in P$. If $a^2 \in P$, then we have $a^2 = xab$ for some $x \in A$. Since A is graded, we may assume that $x \in h(A)$. As A is graded integral domain, we conclude that $a = xb$, that is, $b|a$. In other case, one can prove that $a|b$. Then by Theorem 4.1, A is a gr-valuation domain. □

Definition 4.3. Let A be a graded ring, a graded A-module E is said to be graded-uniserial (gr-uniserial for short) if the set of its homogeneous submodules is totally ordered by inclusion.

We next give a characterization for the graded trivial extension ring to be a gr-valuation ring. Note that here the assumption "G is a torsionfree abelian group" is necessary since the grading monoid of A and E must be the same and the fact that is "torsionfree" is used by Lemma 4.5.

Theorem 4.4. *Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring and E a nonzero graded A-module. Let $R := A \propto E$ be the graded trivial extension ring of A by E. Assume that E is a non-torsion graded A-module. Then R is a gr-valuation ring if and only if A is a gr-valuation domain and E is isomorphic to A_H, the homogeneous quotient field of fractions of A.*

Before proving Theorem 4.4 we establish the following lemma.

Lemma 4.5. *Assume that the grading monoid is a torsionfree abelian group. Let A be a graded ring, E a nonzero graded A-module, and $R := A \propto E$ be the graded trivial extension ring of A by E. If R is a gr-valuation ring, then A is a gr-valuation domain and E is a gr-uniserial A-module.*

Proof. Assume that R is a gr-valuation ring. First we wish to show that A is a gr-valuation ring and E is a gr-uniserial A-module. Let $a, b \in h(A)$, one at least of which is regular, if $(a, 0)$ divides $(b, 0)$ (resp., $(b, 0)$ divides $(a, 0)$), then a divides b (resp., b divides a). Hence A is a gr-valuation ring. On the other hand, let $x, y \in h(E)$. If $(0, x)$ divides $(0, y)$ (resp., $(0, y)$ divides $(0, x)$) then there exists $(c, z) \in R$ such that $(0, y) = (c, z)(0, x)$ (resp., $(0, x) = (c, z)(0, y)$) and so $y \in Ax$(resp., $x \in Ay$). Therefore, E is a gr-uniserial A-module.

We prove that A is an integral domain. Deny. Let $a, b \in h(A)$ such that $ab = 0, a \neq 0$ and $b \neq 0$. For each $x \in h(E)$, $(b, 0)$ divides $(0, x)$ (since R is a gr-valuation ring and $(0, x)$ does not divide $(b, 0)$ (since $b \neq 0$)), and so there exists $y \in E$ such that $by = x$, thus $ax = 0$ and so $a \in (0 : E)$. Also, for each $x \in h(E)$, $(a, 0)$ divides $(0, x)$ and so $x \in aE = 0$, a contradiction since $E \neq 0$. Therefore since the grading monoid is a torsionfree abelian group. Thus A is an integral domain. □

(Proof of Theorem 4.4): Assume that A is a gr-valuation domain and let $R := A \propto A_H$, where A_H is the homogeneous quotient field of A. Our aim is to show that R is a gr-valuation ring. Let (a, x), $(b, y) \in h(R) - \{(0, 0)\}$. Two cases are then possible.

Case 1. $a = b = 0$. There exists then $c \in A$ such that $x = cy$ (resp., $y = cx$) since A_H is the homogeneous quotient field of fractions of A and A is a gr-valuation domain. Hence, $(0, x) = (c, 0)(0, y)$ (resp., $(0, y) = (c, 0)(0, x)$) as desired.

Case 2. $a \neq 0$ or $b \neq 0$. We may assume that $a \neq 0$ and $b \in Aa$. Let $c \in A$ such that $ac = b$, and let $z \in A_H$ such that $az + cx = y$. Hence, $(a, x)(c, z) = (b, y)$ as desired.

Conversely, assume that E is a non-torsion graded A-module, and $R = A \propto E$ is a gr-valuation ring. By Lemma 4.5, A is a gr-valuation domain. It remains to show that $E \simeq A_H$. Let $u \in h(E)$ such that $(0 : u) = 0$, and let $f : A_H \otimes Au \to A_H \otimes E$ be the homomorphism of A-module induced by the inclusion map $Au \hookrightarrow E$. Since the homogeneous quotient field of A is a flat A-module, hence f is injective. Let $(\lambda, x) \in h(A_H \times E)$, by Lemma 4.5 we get that $x \in Au$ or $u \in Ax$. If $x = au$ for some $a \in A$, then $f(\lambda \otimes au) = \lambda \otimes x$. If $u \in Ax$, then there exists $a \in A$ such that $u = ax$. Thus

$$f\left(\frac{\lambda}{a} \otimes u\right) = \frac{\lambda}{a} \otimes u = \frac{\lambda}{a} \otimes ax = \lambda \otimes x$$

Since f is an homomorphism of A-module, then for every element $(\lambda, x) \in A_H \times E$, there exists an element $y \in A_H \otimes Au$, such that $f(y) = (\lambda, x)$. Consequently, f is an isomorphism of A-module.

Now, consider the homomorphism of A-module $g : E \to A_H \otimes E$ defined by $g(x) = 1 \otimes x$. If $g(x) = 1 \otimes x = 0$, for some homogeneous element $x \in E$ then there exists $0 \neq a \in A$ such that $ax = 0$. By Lemma 4.5 $x \in Au$ or $u \in Ax$. But $u \notin Ax$ since $ax = 0$, $a \neq 0$ and $(0 : u) = 0$. Hence, $x = bu$ for some $b \in A$. Then $abu = 0$, hence $ab = 0$ since $(0 : u) = 0$ and so $b = 0$ (since A is a gr-valuation domain and $a \neq 0$); thus $x = 0$. Now if $g(x) = 1 \otimes x = 0$, for some $x = \sum_{\alpha \in G} x_\alpha \in E$. It follows that $1 \otimes x_\alpha = 0$ for all $\alpha \in G$ and then, by the above sentence, we have $x_\alpha = 0$ for all $\alpha \in G$, then $x = 0$. Therefore g is injective. Let $(\lambda, x) \in h(A_H \times E)$. If $\lambda \in A$, then $\lambda \otimes x = 1 \otimes \lambda x = g(\lambda x)$. Now if $\lambda^{-1} \in A$, then there exists $y \in E$ such that $\lambda^{-1} y = x$, since $\left(\lambda^{-1}, 0\right)$ divides $(0, x)$. Hence

$$\lambda \otimes x = \lambda \otimes \left(\lambda^{-1} y\right) = 1 \otimes y = g(y)$$

Since g is an homomorphism of A-module, then for every element $(\lambda, x) \in A_H \times E$, there exists an element $y \in E$, such that $g(y) = (\lambda, x)$. Consequently, g is an isomorphism of

A-module. We deduce that

$$E \simeq_g A_H \otimes_A E \simeq_f A_H \otimes_A Au \simeq A_H \otimes_A A$$

Finally, since for all multiplicatively closed subset S of A, the $S^{-1}A$-modules $S^{-1}E$ and $S^{-1}A \otimes_A E$ are isomorphic; more precisely, the map $\varphi : S^{-1}E \to S^{-1}A \otimes_A E$, where $\varphi\left(\frac{x}{s}\right) = \frac{1}{s} \otimes x$ is isomorphism. we have $A_H \otimes_A A \simeq A_H$. Hence $E \simeq A_H$. □

Theorem 4.4 enriches the literature with new examples of gr-valuation rings.

Example 4.6. Let K be a graded-field which is graded by an arbitrary torsionfree group. Let $K_H = K$ be its homogeneous quotient field of fractions. The trivial ring extension of K by K_H, $K \propto K_H$ is a gr-valuation ring.

Example 4.7. Let k be a field. Let $A = k[[x]]$ the ring of formal power series with coefficients in k graded by $\mathbb{Z}$ and A_H its homogeneous quotient field of fractions. The trivial ring extension of A by A_H, $A \propto A_H$ is a gr-valuation ring.

The next theorem characterize the gr-valuation property in the graded trivial extension ring in a general case. Recall from [12, page 179] that a graded A-module is said to be gr-divisible if $ax = b$ with $a \in h(A)$, $b \in h(E)$ has a solution in E.

Theorem 4.8. *Let A be a graded ring and E a nonzero graded A-module. Then $R := A \propto E$ is a gr-valuation ring if and only if A is a gr-valuation domain, E a gr-divisible and gr-uniserial A-module.*

Proof. Assume that R is a gr-valuation ring, then by Lemma 4.5, A is a gr-valuation domain and E is gr-uniserial A-module. It remains to show that E is gr-divisible, let $x \in h(E)$ and $a \in h(A) \setminus \{0\}$, $(0, x)$ and $(a, 0)$ are two homogeneous elements, since R is a gr-valuation ring, two cases are then possible:

Case 1: $(0, x)$ divides $(a, 0)$, then there exist (b, f) suth that $(a, 0) = (0, x)(b, f)$ and so $a = 0$, a contradiction.

Case 2: $(a, 0)$ divides $(0, x)$, then there exist (c, z) suth that $(0, x) = (a, 0)(c, z)$ implies that $x = az$, then E is gr-divisible.

Conversely, let (a, c) and (b, d) be two elements in $h(R)$. Our aim is to show that R is a gr-valuation ring. Two cases are then possible:

Case 1: $a = b = 0$. Since E is gr-divisible, then there exists $x \in h(E)$ such that $dx = c$ (resp., $d = cx$). Hence, $(0, c) = (x, 0)(0, d)$ (resp., $(0, d) = (x, 0)(0, c)$) as desired.

Case 2: $a \neq 0$ or $b \neq 0$. Since A is a gr-valuation domain, we may assume that $a \neq 0$ and b divides a. Then there exists $x \in h(A)$ such that $ax = b$, then since E is gr-divisible and since $d - xc$ is a homogeneous element (we can check this easily since $deg(x) = deg(b) - deg(a) = deg(d) - deg(c)$), there exists $y \in h(E)$ such that $ay = d - xc$. Hence $(a, c)(x, y) = (b, d)$ as desired. □

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