On Involute-Evolute Curve Pair in Semi-Euclidean Space

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ABSTRACT.   In this study, a kind of generalized involute and evolute curve pair is considered in 4 dimensional semi Euclidean space with 2 index. The curvatures and the associated Frenet Frame of this kind of generalized involute-evolute curve pair are presented.

1. INTRODUCTION

Different curve couples are defined by examining the situations of the Frenet vectors of the curves relative to each other at their corresponding points. Involute-evolute curve pair is one of them. The involute-evolute curve pair were discovered by Huggens while trying to build a more accurate clock [2]. Fuchs defined the involute of a given curve as a curve in such the tangent vector of the given curve at each point corresponds its normal vector [5]. The involute-evolute curve pair is well known in 3-dimensional Euclidean space [6, 11]. The relations Frenet vectors of involute-evolute curve pair were given in E^3 [7]. Turgut and Erdoğan examined this curve pair in E^n [16]. Özyılmaz and Yılmaz determined that an evolute Frenet apparatus can be formed by an involute apparatus in four dimensional Euclidean space, so, in this way, another orthonormal frame in the same space can be obtained [14].

On the other hand, this curve pair has been studied by many researchers in Minkowski space [10, 12]. Bükçü and Karacan presented the involute-evolute curves of the spacelike curve with a spacelike principal normal in Minkowski 3-Space [3]. Hanif et al. presented a special kind of generalized involute and evolute curve pair in 4-dimensional Minkowski space by considering this curve pair in a different way. [9].

In this article, we consider $(0, 2)$-involute curve of a given timelike curve with respect to the causal characteristics of the $(0, 2)$-plane that is spanned by the tangent and first binormal of the vector fields in the R_2^4. Similarly, we examine $(1, 3)$-evolute curve of a given timelike curve with respect to the causal characteristics of the $(1, 3)$-normal plane that are spanned by the principal normal and the second binormal of the vector fields in the same space.

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2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves in semi-Euclidean space R_2^4 are briefly presented in this section.

Let R_2^4 denote 4-dimensional semi-Euclidean space with 2-index provided with standard flat metric given by

$$g = -da_1^2 - da_2^2 + da_3^2 + da_4^2,$$

where (a_1, a_2, a_3, a_4) is a rectangular coordinate system of R^4. A vector w in R_2^4 is called a spacelike, timelike or null (lightlike) if hold $g(w, w) > 0$, $g(w, w) < 0$ or $g(w, w) = 0$ and $w \neq 0$, respectively. The norm of a vector w is given by $\|w\| = \sqrt{|g(w, w)|}$. Therefore, w is a unit vector if $g(w, w) = \pm 1$. Two vectors u and w are said to be orthogonal if $g(u, w) = 0$ [13]. Also, let u and w be two timelike vectors in R_2^4. An arbitrary curve $\gamma = \gamma(s)$ in R_2^4 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null (lightlike). The velocity of the curve γ is given by $\|\gamma'\|$. Thus, a timelike curve γ is said to be parametrized by arc length function s if $g(\gamma', \gamma') = -1$ [13].

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ denotes the moving Frenet frame along γ in the semi-Euclidean space R_2^4, then $T(s), N(s), B_1(s)$ and $B_2(s)$ are called the tangent, the principal normal, the first binormal, and the second binormal vector fields of γ, respectively.

A unit speed curve γ is said to be a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. Let γ be a C^∞ special timelike Frenet curve with timelike principal normal, spacelike both first binormal and second binormal vector fields in R_2^4, parametrized by arc length function s. Moreover, non-zero C^∞ scalar functions κ_1, κ_2 and κ_3 be the first, second, and third curvatures of γ, respectively. Then for a C^∞ special nonnull Frenet curve γ, the Frenet formula is given by

$$\begin{aligned}
T' &= \varepsilon \kappa_1 N \\
N' &= -\varepsilon \kappa_1 T + \kappa_2 B_1 \\
B_1' &= \kappa_2 N + \kappa_3 B_2 \\
B_2' &= -\kappa_3 B_1,
\end{aligned} \tag{2.1}$$

where T, N, B_1 and B_2 mutually orthogonal vector fields satisfying

$$g(T, T) = g(N, N) = \varepsilon, g(B_1, B_1) = g(B_2, B_2) = -\varepsilon$$

and $\varepsilon = \begin{cases} +1; \ \gamma \ spacelike \\ -1; \ \gamma \ timelike \end{cases}$ (for the semi-Euclidean space E_ν^{n+1}, see [1, 4, 15]).

Definition 2.1 ([3]). Let any curves γ and γ^* be given with coordinate neighborhoods (I, γ) and (I, γ^*), respectively. Let's denote the Frenet frames at the points $\gamma(s), \gamma^*(s)$ corresponding to s by $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. If $g(T, T^*) = 0$ for $\forall s \in I$, the curve pair (γ, γ^*) is the involute-evolute curve pair. Then the curve γ^* is called the involute of the curve γ and the curve γ is called the evolute of the curve γ^* [16].

Definition 2.2 ([3]). The plane spanned by $\{T, B_1\}$ is called the $(0, 2)$-tangent plane of γ at any point of γ,. The plane spanned by $\{N, B_2\}$ is called the $(1, 3)$-normal plane of γ [8].

Definition 2.3 ([3]). Let $\gamma : I \to R_2^4$ and $\gamma^* : I \to R_2^4$ be two regular curves in R_2^4, where s is the arc-length parameter of γ. Denote $s^* = \varphi(s)$ to be the arc-length parameters of γ^*. For $\forall s \in I$, if the $(0, 2)$-tangent plane of γ at $\gamma(s)$ of coincides with the $(1, 3)$-normal plane at $\gamma^*(s)$ of γ^*, then γ^* is called the $(0, 2)$-involute curve of γ and γ is called the $(1, 3)$-evolute curve of γ^* in the R_2^4.

3. The (0,2)-Involute Curve of a Timelike Curve in E_2^4

In this section, we proceed to study the existence and expression of the $(0, 2)$-involute curve of a given timelike curve in R_2^4.

Let $\gamma : I \to R_2^4$ be a regular timelike curve with arc-length parameter s and κ_1, κ_2 and κ_3 never vanish. Suppose that $\gamma^* : I \to R_2^4$ is the $(0, 2)$-involute curve of γ. It is clear that γ^* can be timelike or spacelike. Denote T^*, N^*, B_1^*, B_2^* to be the Frenet frame along and κ_1^*, κ_2^* and κ_3^* to be the curvatures of γ^*. Then

$$span\{T, B_1\} = span\{N^*, B_2^*\}, span\{N, B_2\} = span\{T^*, B_1^*\}.$$

Moreover, γ^* can be expressed as

$$\gamma^*(s) = \gamma(s) + \lambda(s)T(s) + \mu(s)B_1, \tag{3.1}$$

where $\lambda(s)$ and $\mu(s)$ are C^∞ functions on I. Differentiating (3.1) with respect to s and using the Frenet formula (2.1), we get

$$\varphi' T^* = (1 + \lambda')T + (\mu\kappa_2 - \lambda\kappa_1)N + \mu'B_1 + \mu\kappa_3 B_2. \tag{3.2}$$

By the inner products of both-sides of (3.2) with T and B_1, respectively, we get $1 + \lambda' = 0$ and $\mu' = 0$, which implies that μ is constant and $\lambda = \lambda_0 - s$, where λ_0 is the integration constant and $\varphi' = ds^*/ds$. So (3.2) becomes

$$\varphi' T^* = (\mu\kappa_2 - \lambda\kappa_1)N + \mu\kappa_3 B_2. \tag{3.3}$$

Denote

$$\delta = \frac{\mu\kappa_2 - \lambda\kappa_1}{\varphi'}, \zeta = \frac{\mu\kappa_3}{\varphi'}.$$

Then (3.3) can be written as

$$T^* = \delta N + \zeta B_2, \zeta^2 - \delta^2 = \varepsilon. \tag{3.4}$$

Case 1: $\mu \neq 0$. In this case, $\zeta \neq 0$. Denote $\delta/\zeta = \alpha_1$. Then $\delta = \zeta\alpha_1$ and

$$\mu\kappa_2 - \lambda\kappa_1 = \alpha_1\mu\kappa_3, \varphi' = \zeta^{-1}\mu\kappa_3, \zeta^2 = \frac{\varepsilon}{1 - \alpha_1^2}. \tag{3.5}$$

Differentiating (3.4) with respect to s and using the Frenet frame (2.1) we get

$$\varepsilon\varphi'\kappa_1^* N^* = \delta\kappa\varsigma_1 T + \delta'N + (\delta\kappa_2 - \zeta\kappa_3)B_1 + \zeta'B_2. \tag{3.6}$$

By the inner products of both-sides of (3.6) with N and B_2, respectively, we get $\delta' = 0$ and $\zeta' = 0$, which implies that δ and ζ are constants. So (3.6) gives us

$$\varepsilon\varphi'\kappa_1^* N^* = \delta\kappa_1 T + (\delta\kappa_2 - \zeta\kappa_3)B_1. \tag{3.7}$$

Denote

$$\vartheta = \frac{\zeta\alpha_1\kappa_1}{\varepsilon\varphi'\kappa_1^*}, \varrho = \frac{\zeta(\alpha_1\kappa_2 - \kappa_3)}{\varepsilon\varphi'\kappa_1^*}. \tag{3.8}$$

So from the equation (3.7), we get

$$N^* = \vartheta T + \varrho B_1, \varepsilon = \varrho^2 - \vartheta^2. \tag{3.9}$$

Denote $\varrho/\vartheta = \alpha_2$. Then we find $\varrho = \alpha_2\vartheta$ and

$$\alpha_1(\kappa_2 - \alpha_2\kappa_1) = \kappa_3, \vartheta^2 = \frac{\varepsilon}{\alpha_2^2 - 1}. \tag{3.10}$$

From the first equations of (3.5) and (3.10), we have

$$\eta = \frac{\kappa_2}{\kappa_1} = \frac{\lambda/\mu - \alpha_1^2\alpha_2}{1 - \alpha_1^2}, \frac{\kappa_3}{\kappa_1} = \alpha_1(\eta - \alpha_2).$$

Denote $\zeta/\vartheta = \alpha_3$. Then $\zeta = \alpha_3\vartheta$. From (3.8), we have

$$\varepsilon\varphi'\kappa_1^* = \alpha_1\alpha_3\kappa_1, \alpha_3^2 = \frac{\alpha_2^2 - 1}{1 - \alpha_1^2}. \tag{3.11}$$

Differentiating (3.9) with respect to s and using the Frenet formula, we get

$$-\varepsilon\varphi'\kappa_1^* T^* + \varphi'\kappa_2^* B_1^* = \vartheta'T + (\varrho\kappa_2 - \vartheta\kappa_1)N + \varrho'B_1 + \varrho\kappa_3 B_2. \tag{3.12}$$

By the inner products of both-sides of (3.12) with T and B_1, respectively, we get $\vartheta' = 0$ and $\varrho' = 0$, which implies that ϑ and ϱ are constants. In this case, (3.12) turns into

$$\varphi'\kappa_2^* B_1^* = (\varrho\kappa_2 - \vartheta\kappa_1)N + \varrho\kappa_3 B_2 + \varepsilon\varphi'\kappa_1^* T^*. \tag{3.13}$$

Substituting (3.4) and (3.11) into (3.13), we obtain

$$\varphi'\kappa_2^* B_1^* = \vartheta\kappa_1(\alpha_2\eta + \alpha_3^2 - \alpha_2^2)(N + \alpha_1 B_2). \tag{3.14}$$

From (3.14), we may choose that

$$B_1^* = \zeta N + \delta B_2, \varphi'\kappa_2^* = \alpha_3^{-1}\kappa_1(\alpha_2\eta + \alpha_3^2 - \alpha_2^2).$$

Differentiating with respect to s and using the Frenet formula (2.1), we get

$$\varphi' \kappa_2^* N^* + \varphi' \kappa_3^* B_2^* = \zeta \kappa_1 T + (\zeta \kappa_2 - \delta \kappa_3) B_1,$$

from which we obtain

$$\varphi' \kappa_3^* B_2^* = -\alpha_3^{-1} \kappa_1 (\eta - \alpha_2)(\varrho T + \vartheta B_1). \tag{3.15}$$

From (3.15), we may choose that

$$B_2^* = \varrho T + \vartheta B_1, \varphi' \kappa_3^* = -\alpha_3^{-1} \kappa_1 (\eta - \alpha_2).$$

So we can give the following theorem.

Theorem 3.1. *Let $\gamma : I \to R_2^4$ be a regular timelike curve with arc-length parameter s so that κ_1, κ_2 and κ_3 never vanish. If $\gamma^* : I \to R_2^4$ is the $(0, 2)$-involute mate curve of γ, $\gamma^*(s) = \gamma(s) + (\lambda_0 - s)T(s) + \mu B_1$ with $\mu \neq 0$, then κ_1, κ_2 and κ_3 satisfy*

$$\frac{\kappa_2}{\kappa_1} = \eta, \frac{\kappa_3}{\kappa_1} = \alpha_1 (\eta - \alpha_2), \eta = \frac{\lambda_0 - s - \mu \alpha_1^2 \alpha_2}{\mu(1 - \alpha_1^2)},$$

where λ_0, μ, α_1 and α_2 are given constants. Moreover, the three curvatures of γ^* are given by

$$\kappa_1^* = \frac{\vartheta \alpha_3^2}{\varepsilon \mu (\eta - \alpha_2)}, \kappa_2^* = \frac{\vartheta(\alpha_2 \eta + \alpha_3^2 - \alpha_2^2)}{\mu \alpha_1 (\eta - \alpha_2)}, \kappa_3^* = -\frac{\vartheta}{\mu \alpha_1},$$

where $\vartheta \neq 0$. The associated Frenet Frame are given by

$$T^* = \vartheta \alpha_3 (\alpha_1 N + B_2), N^* = \vartheta(T + \alpha_2 B_1), B_1^* = \vartheta \alpha_3 (N + \alpha_1 B_2), B_2^* = \vartheta(\alpha_2 T + B_1).$$

Case 2: $\mu = 0$. In this case, (3.1) turns into

$$\gamma^*(s) = \gamma(s) + (\lambda_0 - s)T(s). \tag{3.16}$$

Differentiating (3.16) with respect to s and using the Frenet formula (2.1), we get

$$\varphi' T^* = -(\lambda_0 - s)\kappa_1 N,$$

from which we may assume that

$$\varphi' = (\lambda_0 - s)\kappa_1, T^* = -N. \tag{3.17}$$

Differentiating the second equation of (3.17) with respect to s and using the Frenet formula (2.1), we get

$$\varepsilon \varphi' \kappa_1^* N^* = -\kappa_1 T - \kappa_2 B_1.$$

Suppose that

$$N^* = \vartheta T + \varrho B_1, \vartheta = \frac{-\kappa_1}{\varepsilon \varphi' \kappa_1^*}, \varrho = \frac{-\kappa_2}{\varepsilon \varphi' \kappa_1^*}, -\vartheta^2 + \varrho^2 = \varepsilon. \tag{3.18}$$

It follows that

$$\frac{\kappa_2}{\kappa_1} = \frac{\varrho}{\vartheta}.$$

Differentiating (3.18) with respect to s, we obtain that ϑ and ϱ are constant and

$$\varphi' \kappa_2^* B_1^* = \varepsilon \varphi' \kappa_1^* T^* + (-\vartheta \kappa_1 + \varrho \kappa_2)N + e \kappa_3 B_2 = -\frac{\kappa_1}{\vartheta}(1 + \varepsilon)N + \varrho \kappa_3 B_2.$$

Suppose that

$$B_1^* = B_2, \varphi' \kappa_2^* = \varrho \kappa_3, \frac{\kappa_1}{\vartheta}(1 + \varepsilon) = 0. \tag{3.19}$$

Corollary 3.2. *Since $\kappa_1 \neq 0$, $\varepsilon = -1$ in the last equation of expression (3.19), the $(0, 2)$-involute mate curve of a timelike curve can only be timelike, for $\mu = 0$.*

Differentiating (3.19) with respect to s, we obtain

$$\varphi' \kappa_3^* B_2^* = -\varphi' \kappa_2^* N^* - \kappa_3 B_1 = -\vartheta \kappa_3 (\varrho T + \vartheta B_1).$$

We suppose that $T^* \times N^* \times B_1^* \times B_2^* = T \times N \times B_1 \times B_2$. Then

$$B_2^* = \varrho T + \vartheta B_1, \varphi' \kappa_3^* = -\vartheta \kappa_3.$$

So we can give the following theorem.

Theorem 3.3. *Let* $\gamma : I \to R_2^4$ *be a regular timelike curve with arc-length parameter* s *so that* κ_1, κ_2 *and* κ_3 *never vanish. If* $\gamma^* : I \to R_2^4$ *is the* $(0, 2)$*-involute mate curve of* γ, $\gamma^*(s) = \gamma(s) + (\lambda_0 - s)T(s)$ *with* $\mu = 0$, *then* κ_1, κ_2 *satisfy*

$$\varrho \kappa_1 - \vartheta \kappa_2 = 0,$$

where λ_0, ϑ *and* ϱ *are given constants. Moreover, the three curvatures of* γ^* *are given by*

$$\kappa_1^* = \frac{-1}{\varepsilon(\lambda_0 - s)\vartheta}, \kappa_2^* = \frac{\varrho \kappa_3}{(\lambda_0 - s)\kappa_1}, \kappa_3^* = \frac{-\vartheta \kappa_3}{(\lambda_0 - s)\kappa_1}.$$

The associated Frenet Frame are given by

$$T^* = -N, N^* = \vartheta T + \varrho B_1, B_1^* = B_2, B_2^* = \varrho T + \vartheta B_1.$$

4. The (1,3)-Evolute Curve of a Timelike Curve in E_2^4

In this section, we proceed to study the existence and expression of the $(1, 3)$-evolute curve of a given timelike curve in R_2^4.

Let $\gamma : I \to R_2^4$ be a regular timelike curve with arc-length parameter s so that κ_1, κ_2 and κ_3 never vanish. Suppose that $\gamma^* : I \to R_2^4$ is the $(1, 3)$-evolute curve of γ. It is clear that γ^* can be timelike or spacelike. Denote T^*, N^*, B_1^*, B_2^* to be the Frenet frame along and κ_1^*, κ_2^* and κ_3^* to be the curvatures of γ^*. Then

$$span\{T, B_1\} = span\{N^*, B_2^*\}, span\{N, B_2\} = span\{T^*, B_1^*\}, T \perp T^*.$$

Moreover, γ^* can be expressed as

$$\gamma^*(s) = \gamma(s) + v(s)N(s) + w(s)B_2, \tag{4.1}$$

where $v(s)$ and $w(s)$ are C^∞ functions on I. Differentiating (4.1) with respect to s and using the Frenet formula (2.1), we get

$$\varphi' T^* = (1 + v\kappa_1)T + v'N + (v\kappa_2 - w\kappa_3)B_1 + w'B_2. \tag{4.2}$$

By the inner products of both-sides of (4.2) with T and B_1, respectively, we get

$$\varphi' T^* = v'N + w'B_2, \ v = -\frac{1}{\kappa_1}, w = -\frac{\kappa_2}{\kappa_1\kappa_3}. \tag{4.3}$$

Denote

$$a = \frac{v'}{\varphi'}, b = \frac{w'}{\varphi'}. \tag{4.4}$$

Then (4.3) turns into

$$T^* = aN + bB_2, b^2 - a^2 = \varepsilon. \tag{4.5}$$

Differentiating (4.5) with respect to s and using the Frenet formula (2.1), we get

$$\varepsilon \varphi' \kappa_1^* N^* = a\kappa_1 T + a'N + (a\kappa_2 - b\kappa_3)B_1 + b'B_2. \tag{4.6}$$

Taking inner product on both-sides of (4.6) with N and B_2 respectively, we get $a' = 0$ and $b' = 0$ which implies that a and b are constants. From (4.4), we obtain

$$v = a\varphi + v_0 = -\frac{1}{\kappa_1}, w = b\varphi + w_0 = -\frac{\kappa_2}{\kappa_1\kappa_3}. \tag{4.7}$$

Moreover, (4.6) turns into

$$\varepsilon \varphi' \kappa_1^* N^* = a\kappa_1 T + (a\kappa_2 - b\kappa_3)B_1. \tag{4.8}$$

Denote

$$m = \frac{a\kappa_1}{\varepsilon \varphi' \kappa_1^*}, n = \frac{a\kappa_2 - b\kappa_3}{\varepsilon \varphi' \kappa_1^*}.$$

Then (4.8) turns into

$$N^* = mT + nB_1, \varepsilon \varphi' \kappa_1^* = m^{-1}a\kappa_1, n^2 - m^2 = \varepsilon. \tag{4.9}$$

Moreover, we have

$$n a\kappa_1 + mb\kappa_3 - ma\kappa_2 = 0.$$

Case 1: $n \neq 0$. Differentiating (4.9) with respect to s and using (2.1), we obtain

$$-\varepsilon \varphi' \kappa_1^* T^* + \varphi' \kappa_2^* B_1^* = m'T + (n\kappa_2 - m\kappa_1)N + n'B_1 + n\kappa_3 B_2. \tag{4.10}$$

By the inner products of both-sides of (4.10) with T and B_1 respectively, we get $m' = 0$ and $n' = 0$, which implies that m and n are constants. In this case, (4.10) turns into

$$\varphi' \kappa_2^* B_1^* = (\frac{a^2 - m^2}{m} \kappa_1 + n\kappa_2)N + (\frac{ab}{m} \kappa_1 + n\kappa_3)B_2. \tag{4.11}$$

Denote

$$\psi = (\varphi' \kappa_2^*)^{-1}(\frac{a^2 - m^2}{m} \kappa_1 + n\kappa_2), \omega = (\varphi' \kappa_2^*)^{-1}(\frac{ab}{m} \kappa_1 + n\kappa_3).$$

Then (4.11) turns into

$$B_1^* = \psi N + \omega B_2, \omega^2 - \psi^2 = \varepsilon. \tag{4.12}$$

Since $T^* \perp B_1^*$, it follows from (4.5) and (4.12) that $\psi / \omega = b/a$, which implies that

$$B_1^* = bN + aB_2, \varphi' \kappa_2^* = \frac{b\kappa_1}{m} + \frac{n\kappa_3}{a}. \tag{4.13}$$

Differentiating (4.13) with respect to s and using (2.1), we get

$$\varphi' \kappa_3^* B_2^* = -\varphi' \kappa_2^* N^* + b\kappa_1 T + (b\kappa_2 - a\kappa_3)B_1 = -\frac{m\kappa_3}{a}(nT + mB_1). \tag{4.14}$$

It follows that from (4.14) that

$$B_2^* = nT + mB_1, \varphi' \kappa_3^* = -\frac{m\kappa_3}{a}.$$

So we can give the following theorem.

Theorem 4.1. *Let $\gamma : I \to R_2^4$ be a regular curve with arc-length parameter s so that κ_1, κ_2 and κ_3 never vanish. Suppose that $\gamma^* : I \to R_2^4$ the $(1, 3)$-evolute mate curve of γ,*

$$\gamma^*(s) = \gamma(s) - \frac{1}{a\kappa_1}[aN(s) + bB_2] - \frac{n}{m\kappa_3}B_2,$$

with $n \neq 0$, then κ_1, κ_2 satisfy $na\kappa_1 + mb\kappa_3 - ma\kappa_2 = 0$, where a and b are given constants. Moreover, the three curvatures of are given by

$$\kappa_1^* = \frac{a\kappa_1}{\varepsilon \varphi' m}, \kappa_2^* = \frac{b\kappa_1}{\varphi' m} + \frac{n\kappa_3}{\varphi' a}, \kappa_3^* = -\frac{m\kappa_3}{\varphi' a}.$$

The associated Frenet Frame are given by

$$T^* = aN + bB_2, N^* = mT + nB_1, B_1^* = bN + aB_2, B_2^* = nT + mB_1.$$

Case 2: $n = 0$. In this case, we may suppose that

$$N^* = mT, \varepsilon \varphi' \kappa_1^* = m^{-1} a\kappa_1, a\kappa_2 - b\kappa_3 = 0, \varepsilon = -m^2. \tag{4.15}$$

Corollary 4.2. *Since $m^2 = -\varepsilon$, $\varepsilon \leq 0$, the $(1, 3)$-evolute mate curve of a timelike curve in R_2^4 can only be timelike, for $n = 0$. Also $m = \varepsilon$.*

Moreover, we have from (4.7) and the third equation of (4.15) that

$$v = a(\varphi + \varphi_0) = -\frac{1}{\kappa_1}, w = b(\varphi + \varphi_0) = -\frac{b}{a\kappa_1}.$$

Differentiating (4.15) with respect to s and using (2.1), we get

$$\varphi' \kappa_2^* B_1^* = \varepsilon \varphi' \kappa_1^* T^* - m\kappa_1 N.$$

It follows that we may choose

$$B_1^* = bN + aB_2, \varphi' \kappa_2^* = \frac{b\kappa_1}{\varepsilon}. \tag{4.16}$$

Differentiating (4.16) with respect to s, using (2.1) and the third equation of (4.15), we get

$$\varphi' \kappa_3^* B_2^* = -\varphi' \kappa_2^* N^* + (b\kappa_2 - a\kappa_3)B_1.$$

from which we obtain

$$B_2^* = mB_1, \varphi' \kappa_3^* = a^{-1}\kappa_3.$$

So we can give the following theorem.

Theorem 4.3. *Let* $\gamma : I \rightarrow R_2^4$ *be a regular curve with arc-length parameter s so that* κ_1, κ_2 *and* κ_3 *are not zero. Suppose that* $\gamma^* : I \rightarrow R_2^4$ *the (1,3)-evolute mate curve of* γ,

$$\gamma^*(s) = \gamma(s) - \frac{1}{a\kappa_1}[aN(s) + bB_2]$$

with a = 0, then κ_1, κ_2 *satisfy* $a\kappa_2 - b\kappa_3 = 0$ *where a and b are given constants. Moreover, the three curvatures of are given by*

$$\varphi'\kappa_1^* = \frac{a\kappa_1}{\varphi'}, \kappa_2^* = \frac{b\kappa_1}{\varepsilon\varphi'}, \kappa_3^* = \frac{\kappa_3}{a\varphi'}.$$

The associated Frenet Frame are given by

$$T^* = aN + bB_2, N^* = mT, B_1^* = bN + aB_2, B_2^* = mB_1.$$

5. Results

In this article a kind of generalized involute-evolute curve pair is presented in 4 dimensional semi Euclidean space with 2 index. We examine the existence of the $(0, 2)$-involute curve and the $(1, 3)$-evolute curve of the timelike curve for the different cases in the R_2^4. The curvatures and the associated Frenet Frame of this kind of generalized involute-evolute curve pair are presented for each existence state.

Conflicts of Interest

The author/authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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