KMU Journal of Engineering and Natural Sciences
https://dergipark.org.tr/en/pub/kmujens
3(1), 15-23, (2021) © KMUJENS
e-ISSN: 2687-5071

Konumsal Uyarlanmış Çatıya Göre Özel Smarandache Eğrilerinin Ürettiği Yörüngeler

Trajectories Generated by Special Smarandache Curves According to Positional Adapted Frame

Kahraman Esen ÖZEN[1*] , Murat TOSUN[2]

[1]Sakarya, Turkey

[2]Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya, Turkey

(Received: 11 June 2021; Accepted: 17 June 2021)

Özet. Diferansiyel geometride eğriler teorisi önemli bir yere sahiptir. Eğriler üzerinde tanımlanan hareketli çatı kavramı bu teorinin önemli bir parçasıdır. Yakın geçmişte, Özen ve Tosun, 3 boyutlu Öklid uzayında sıfırlanmayan açısal momentuma sahip yörüngeler için yeni bir hareketli çatı tanıttı (J. Math. Sci. Model. 4(1), 2021). Bu çatı $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}\}$ile gösterilir ve konumsal uyarlanmış çatı olarak adlandırılır. Bu çalışmada konumsal uyarlanmış çatıya göre **TM**, **TY** ve **MY**−Smarandache eğrilerinin ürettiği özel yörüngeleri E^3 de araştırdık ve bu yörüngelerin Serret-Frenet elemanlarını hesapladık. Daha sonra, spesifik bir eğriyi ele aldık ve bu eğri için, daha önce belirtilen özel yörüngelerin parametrik denklemlerini elde ettik. Son olarak elde edilen bu özel yörüngelerin Mathematica programıyla çizilmiş grafiklerini verdik. Burada elde edilen sonuçlar alana yeni birer katkıdır. Bu sonuçların gelecekte diferansiyel geometri ve parçacık kinematiğinin bazı özel uygulamalarında faydalı olacağını umuyoruz.

Anahtar Kelimeler: Açısal momentum, parçacık kinematiği, hareketli çatı, Smarandache eğrileri.

Abstract. In differential geometry, the theory of curves has an important place. The concept of moving frame defined on curves is an important part of this theory. Recently, Özen and Tosun have introduced a new moving frame for the trajectories with non-vanishing angular momentum in 3-dimensional Euclidean space (J. Math. Sci. Model. 4(1), 2021). This frame is denoted by $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}\}$ and called as positional adapted frame. In the present study, we investigate the special trajectories generated by **TM**, **TY** and **MY**−Smarandache curves according to positional adapted frame in E^3 and we calculate the Serret-Frenet apparatus of these trajectories. Later, we consider a specific curve and obtain the parametric equations of the aforesaid special trajectories for this curve. Finally, we give the graphics of these obtained special trajectories which were drawn with the Mathematica program. The results obtained here are new contributions to the field. We expect that these results will be useful in some specific applications of differential geometry and particle kinematics in the future.

Keywords: Angular momentum, kinematics of a particle, moving frame, Smarandache curves.

1. Introduction and Preliminaries

The local theory of space curves plays an important role in differential geometry. The concept of moving frames is one of the most important concepts in the local theory of space curves. Despite its long history, it is still a field of interest. The discovery of the Serret-Frenet frame was a milestone for

Corresponding Author Email: kahraman.ozen1@ogr.sakarya.edu.tr.

This paper is the completed version of the paper titled "Trajectories Generated by Special Smarandache Curves According to Positional Adapted Frame", which was presented as a poster presentation at the 5th International Conference on Computational Mathematics and Engineering Sciences. The conference was held in Van Yüzüncü Yıl University between 8-10 June 2021 and the abstract was published in the abstract book.

the researchers interested in this topic. Until now, many researchers have carried out many interesting studies on the local theory of space curves by using Serret-Frenet frame. The readers are referred to some of these studies [1, 2, 3, 4, 5, 6]. There is a very close relationship between the kinematics of a moving particle and the differential geometry of the trajectory which is the oriented curve traced out by this particle. As a result of this case, Serret-Frenet frame has been used to investigate the kinematics of a moving particle, as well.

Assume that a point particle of constant mass m moves in the 3-dimensional Euclidean space E^3 which is taken into account with the standard scalar product $\langle \mathbf{G}, \mathbf{H} \rangle = g_1h_1 + g_2h_2 + g_3h_3$. Here $\mathbf{G} = (g_1, g_2, g_3)$, $\mathbf{H} = (h_1, h_2, h_3)$ are any vectors in E^3. The norm of $\mathbf{G}$ is given as $\|\mathbf{G}\| = \sqrt{\langle \mathbf{G}, \mathbf{G} \rangle}$. If a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ satisfies $\left\| \frac{d\alpha}{ds} \right\| = 1$ for all $s \in I$, it is called a unit speed curve. In that case, s is said to be arc-length parameter of α. A differentiable curve is called as regular curve if its derivative is not equal to zero along the curve. An arbitrary regular curve can be reparameterized by the arc-length of itself [7]. Throughout the paper, the differentiation with respect to the arc-length parameter s will be shown with a prime.

Let the unit speed parameterization for the trajectory of the moving particle be denoted by $\alpha = \alpha(s)$. In that case, the vectors $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$ compose an orthonormal moving frame for $\alpha = \alpha(s)$ which is called Serret-Frenet frame. $\mathbf{T}(s), \mathbf{N}(s)$ and $\mathbf{B}(s)$ are called the unit tangent, unit principal normal and unit binormal vectors, respectively. Serret-Frenet formulas are given as in the following:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ is the torsion function [7].

From past to present, many researchers have developed new moving frames which have a common base vector with the Serret-Frenet frame (see [8, 9, 10] for some examples). One of the newest of them is the study [5] presented by Özen and Tosun. They introduced the Positional Adapted Frame (PAF) for the trajectories with non-vanishing angular momentum in E^3.

The most important thing for the construction of PAF is the angular momentum vector of the moving particle about the origin. This vector has an important place in Newtonian mechanics. It is determined by vector product of the position vector $\mathbf{x} = \langle \alpha(s), \mathbf{T}(s) \rangle \mathbf{T}(s) + \langle \alpha(s), \mathbf{N}(s) \rangle \mathbf{N}(s) + \langle \alpha(s), \mathbf{B}(s) \rangle \mathbf{B}(s)$ and linear momentum vector $\mathbf{p}(t) = m\left(\frac{ds}{dt}\right)\mathbf{T}(s)$ of the moving particle where t indicates the time. It always lies on the instantaneous normal plane $Sp\{\mathbf{N}(s), \mathbf{B}(s)\}$ of the trajectory $\alpha = \alpha(s)$ and it is expressed as $\mathbf{H}^O = m\langle \alpha(s), \mathbf{B}(s) \rangle \left(\frac{ds}{dt}\right)\mathbf{N}(s) - m\langle \alpha(s), \mathbf{N}(s) \rangle \left(\frac{ds}{dt}\right)\mathbf{B}(s)$. Suppose that this vector does not equal to zero vector along $\alpha = \alpha(s)$. This assumption ensures that the functions $\langle \alpha(s), \mathbf{N}(s) \rangle$ and $\langle \alpha(s), \mathbf{B}(s) \rangle$ do not equal to zero simultaneously during the motion of the moving particle. So, it can be said that the tangent line of $\alpha = \alpha(s)$ never passes through the origin. Then, there exists PAF shown with $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ along $\alpha = \alpha(s)$. Take into consideration the vector whose starting point is the foot of the perpendicular (from origin to instantaneous rectifying plane) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane). The equivalent of it at the point $\alpha(s)$ determines the vector $\mathbf{Y}(s)$. Thus, $\mathbf{Y}(s)$ is given as in the following (see [5] for more details):

$$\mathbf{Y}(s) = \frac{\langle -\alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{B}(s).$$

On the other hand, the vector $\mathbf{M}(s)$ is obtained by vector product $\mathbf{Y}(s) \wedge \mathbf{T}(s)$ as follows:

$$\mathbf{M}(s) = \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}}\mathbf{B}(s).$$

Because $\mathbf{T}(s)$ is mutual in both PAF and Serret-Frenet frame, $\mathbf{N}(s)$, $\mathbf{B}(s)$, $\mathbf{M}(s)$ and $\mathbf{Y}(s)$ lie on the same plane. Therefore, there is a relation between the Serret-Frenet frame and PAF as in the following:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Omega(s) & -\sin\Omega(s)\\ 0 & \sin\Omega(s) & \cos\Omega(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{pmatrix}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{B}(s)$ and $\mathbf{Y}(s)$ which is positively oriented from $\mathbf{B}(s)$ to $\mathbf{Y}(s)$(see Figure 1). Also, the derivative formulas of PAF are given as [5]:

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{M}'(s)\\ \mathbf{Y}'(s)\end{pmatrix}=\begin{pmatrix}0 & k_1(s) & k_2(s)\\ -k_1(s) & 0 & k_3(s)\\ -k_2(s) & -k_3(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{M}(s)\\ \mathbf{Y}(s)\end{pmatrix}$$

where

$$\begin{aligned}k_1(s) &= \kappa(s)\cos\Omega(s)\\ k_2(s) &= \kappa(s)\sin\Omega(s)\\ k_3(s) &= \tau(s)-\Omega'(s).\end{aligned}$$

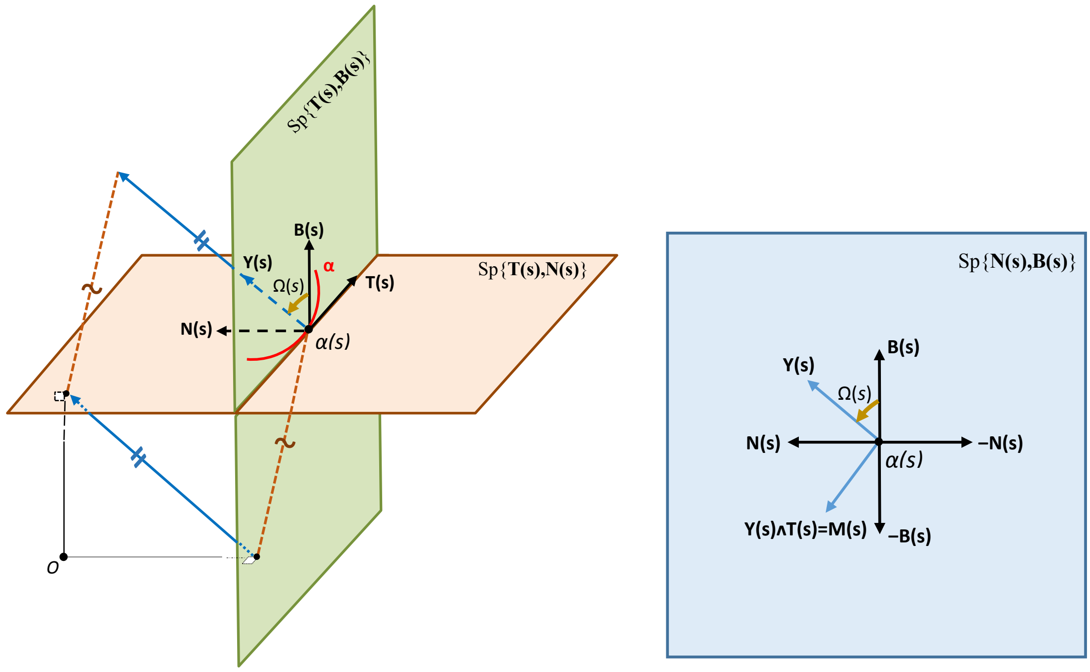

Figure 1. An illustration for the Positional Adapted Frame (PAF)

The aforesaid angle $\Omega(s)$ is calculated as follows:

$$\Omega(s)=\begin{cases}\arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right) & if\ \ \langle\alpha(s),\,\mathbf{B}(s)\rangle>0\\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right)+\pi & if\ \ \langle\alpha(s),\,\mathbf{B}(s)\rangle<0\\ -\frac{\pi}{2}\ \ if\ \ \langle\alpha(s),\,\mathbf{B}(s)\rangle=0\ ,\ \ \langle\alpha(s),\,\mathbf{N}(s)\rangle>0 & \\ \frac{\pi}{2}\ \ if\ \ \langle\alpha(s),\,\mathbf{B}(s)\rangle=0\ ,\ \ \langle\alpha(s),\,\mathbf{N}(s)\rangle<0. & \end{cases}$$

Any element of the set $\{\mathbf{T}(s),\mathbf{M}(s),\mathbf{Y}(s),k_1(s),k_2(s),k_3(s)\}$ is called PAF apparatus of $\alpha=\alpha(s)$ [5].

This paper is organized as follows. In Section 2, we study the special trajectories generated by **TM**, **TY** and **MY**−Smarandache curves according to PAF in three-dimensional Euclidean space and we calculate the Serret-Frenet apparatus of them. In Section 3, we provide an example involving illustrative figures for the obtained results.

2. Some Special Trajectories Generated by Smarandache Curves According to PAF

In the study [4], A. T. Ali defined special Smarandache curves in the Euclidean space. He took into consideration a unit speed regular curve $\gamma = \gamma(s)$ with its Serret-Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and expressed $\mathbf{TN}$, $\mathbf{NB}$, $\mathbf{TNB}-$ Smarandache curves as in the following:

$$\begin{aligned}\beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N})\\ \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B})\\ \beta(s^*) &= \frac{1}{\sqrt{3}}(\mathbf{T}+\mathbf{N}+\mathbf{B}),\end{aligned}$$

respectively. For this topic, the readers are referred to the studies [4, 6, 11, 12, 13, 14, 15] which can be found in the literature.

In this section, we continue to consider any moving point particle satisfying the aforesaid assumption and to denote the unit speed parameterization of the trajectory by $\alpha = \alpha(s)$. We will investigate special trajectories generated by Smarandache curves according to PAF in E^3.

We must emphasize that $\{\mathbf{T}_\alpha(s), \mathbf{M}_\alpha(s), \mathbf{Y}_\alpha(s), k_1(s), k_2(s), k_3(s)\}$ will show the PAF apparatus of $\alpha = \alpha(s)$ throughout the paper. Finally, note that we will follow similar steps given in [16] in this section.

Definition 1. *The special trajectories generated by* $\mathbf{T}_\alpha\mathbf{M}_\alpha-$*Smarandache curves may be defined as*

$$\sigma_1(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\alpha + \mathbf{M}_\alpha\right). \tag{1}$$

For convenience, they are said to be $\mathbf{T}_\alpha\mathbf{M}_\alpha-$*Smarandache trajectories.*

Now, we investigate Serret-Frenet apparatus of $\mathbf{T}_\alpha\mathbf{M}_\alpha-$Smarandache trajectories. Differentiating the equation (1) with respect to s, we obtain

$$\sigma_1{}' = \frac{d\sigma_1}{ds^*}\frac{ds^*}{ds} = \mathbf{T}_{\sigma_1}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right). \tag{2}$$

From the equation (2),

$$\frac{ds^*}{ds} = \sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}} \tag{3}$$

can be found. Therefore, the equation (2) can be rewritten as

$$\mathbf{T}_{\sigma_1}\sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right). \tag{4}$$

The equation (4) yields the tangent vector of σ_1:

$$\mathbf{T}_{\sigma_1} = \frac{1}{\sqrt{2{k_1}^2 + (k_2+k_3)^2}}\left(-k_1\mathbf{T}_\alpha + k_1\mathbf{M}_\alpha + (k_2+k_3)\,\mathbf{Y}_\alpha\right).$$

Differentiating the last equation with respect to s, we get

$$\frac{d\mathbf{T}_{\sigma_1}}{ds^*}\frac{ds^*}{ds} = \left(2{k_1}^2 + (k_2+k_3)^2\right)^{-3/2}\left(\mu_1\mathbf{T}_\alpha + \mu_2\mathbf{M}_\alpha + \mu_3\mathbf{Y}_\alpha\right) \tag{5}$$

where

$$\begin{aligned}\mu_1 &= -2{k_1}^4 + \left[k_1k'_2 + k_1k'_3 - {k_1}^2k_2 - {k_1}^2k_3 - k'_1\,(k_2+k_3) - k_2\left(2{k_1}^2 + (k_2+k_3)^2\right)\right](k_2+k_3)\\ \mu_2 &= -2{k_1}^4 - \left[k_1k'_2 + k_1k'_3 + {k_1}^2k_2 + {k_1}^2k_3 - k'_1\,(k_2+k_3) + k_3\left(2{k_1}^2 + (k_2+k_3)^2\right)\right](k_2+k_3)\\ \mu_3 &= 2{k_1}^2\left[k'_2 + k'_3 + k_1k_3 - k_1k_2\right] - \left[2k'_1 + {k_2}^2 - {k_3}^2\right]k_1\,(k_2+k_3)\,.\end{aligned}$$

Taking into consideration the equation (3) in the equation (5), we obtain

$$\frac{d\mathbf{T}_{\sigma_1}}{ds^*} = \sqrt{2}\left(2{k_1}^2 + (k_2+k_3)^2\right)^{-2}\left(\mu_1\mathbf{T}_\alpha + \mu_2\mathbf{M}_\alpha + \mu_3\mathbf{Y}_\alpha\right).$$

In this case, the curvature and principal normal vector of σ_1 are obtained as follows:

$$\kappa_{\sigma_1} = \left\| \frac{d\mathbf{T}_{\sigma_1}}{ds^*} \right\| = \frac{\sqrt{2\left({\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2\right)}}{\left(2{k_1}^2 + (k_2 + k_3)^2\right)^2}$$

and

$$\mathbf{N}_{\sigma_1} = \frac{1}{\sqrt{{\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2}} \left(\mu_1 \mathbf{T}_\alpha + \mu_2 \mathbf{M}_\alpha + \mu_3 \mathbf{Y}_\alpha\right).$$

Finally, we can immediately find the binormal vector of σ_1 as

$$\mathbf{B}_{\sigma_1} = \frac{(k_1\mu_3 - k_2\mu_2 - k_3\mu_2)\mathbf{T}_\alpha + (k_2\mu_1 + k_3\mu_1 + k_1\mu_3)\mathbf{M}_\alpha - (k_1\mu_2 + k_1\mu_1)\mathbf{Y}_\alpha}{\sqrt{\left(2{k_1}^2 + (k_2 + k_3)^2\right)\left({\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2\right)}}$$

by vector product $\mathbf{T}_{\sigma_1} \wedge \mathbf{N}_{\sigma_1}$. The torsion of σ_1 can be obtained similarly. We leave that to the readers.

Definition 2. *The special trajectories generated by* $\mathbf{T}_\alpha \mathbf{Y}_\alpha-$*Smarandache curves may be defined by*

$$\sigma_2(s^*) = \frac{1}{\sqrt{2}} \left(\mathbf{T}_\alpha + \mathbf{Y}_\alpha\right).$$

For convenience, they are called as $\mathbf{T}_\alpha \mathbf{Y}_\alpha-$*Smarandache trajectories.*

Definition 3. *The special trajectories generated by* $\mathbf{M}_\alpha \mathbf{Y}_\alpha-$*Smarandache curves can be given by*

$$\sigma_3(s^*) = \frac{1}{\sqrt{2}} \left(\mathbf{M}_\alpha + \mathbf{Y}_\alpha\right).$$

For convenience, they are said to be $\mathbf{M}_\alpha \mathbf{Y}_\alpha-$*Smarandache trajectories.*

By following the similar steps given above, one can easily find

$$\begin{aligned}
\mathbf{T}_{\sigma_2} &= \frac{1}{\sqrt{2{k_2}^2 + (k_1 - k_3)^2}} \left(-k_2 \mathbf{T}_\alpha + (k_1 - k_3)\mathbf{M}_\alpha + k_2 \mathbf{Y}_\alpha\right) \\
\kappa_{\sigma_2} &= \frac{\sqrt{2\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}{\left(2{k_2}^2 + (k_1 - k_3)^2\right)^2} \\
\mathbf{N}_{\sigma_2} &= \frac{1}{\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}} \left(\upsilon_1 \mathbf{T}_\alpha + \upsilon_2 \mathbf{M}_\alpha + \upsilon_3 \mathbf{Y}_\alpha\right) \\
\mathbf{B}_{\sigma_2} &= \frac{(k_1\upsilon_3 - k_3\upsilon_3 - k_2\upsilon_2)\mathbf{T}_\alpha + (k_2\upsilon_1 + k_2\upsilon_3)\mathbf{M}_\alpha - (k_2\upsilon_2 - k_3\upsilon_1 + k_1\upsilon_1)\mathbf{Y}_\alpha}{\sqrt{\left(2{k_2}^2 + (k_1 - k_3)^2\right)\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}} \\
\mathbf{T}_{\sigma_3} &= \frac{1}{\sqrt{2{k_3}^2 + (k_1 + k_2)^2}} \left((-k_1 - k_2)\mathbf{T}_\alpha - k_3 \mathbf{M}_\alpha + k_3 \mathbf{Y}_\alpha\right) \\
\kappa_{\sigma_3} &= \frac{\sqrt{2\left({\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2\right)}}{\left(2{k_3}^2 + (k_1 + k_2)^2\right)^2} \\
\mathbf{N}_{\sigma_3} &= \frac{1}{\sqrt{{\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2}} \left(\xi_1 \mathbf{T}_\alpha + \xi_2 \mathbf{M}_\alpha + \xi_3 \mathbf{Y}_\alpha\right) \\
\mathbf{B}_{\sigma_3} &= \frac{-(k_3\xi_3 + k_3\xi_2)\mathbf{T}_\alpha + (k_3\xi_1 + k_2\xi_3 + k_1\xi_3)\mathbf{M}_\alpha - (k_1\xi_2 + k_2\xi_2 - k_3\xi_1)\mathbf{Y}_\alpha}{\sqrt{\left(2{k_3}^2 + (k_1 + k_2)^2\right)\left({\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2\right)}}
\end{aligned}$$

for σ_2 and σ_3 where

$$\begin{aligned}
\upsilon_1 &= -2k_2{}^4 - \left[k_2\left(k'_1 - k'_3\right) - 2k_1k_2{}^2 + k'_2\left(k_3 - k_1\right) + k_2{}^2\left(k_3 - k_1\right) - k_1(k_3 - k_1)^2\right]\left(k_3 - k_1\right)\\
\upsilon_2 &= 2k_2{}^2\left[k'_1 - k'_3 - k_1k_2 - k_2k_3\right] - \left[2k'_2 + k_1{}^2 - k_3{}^2\right]k_2\left(k_1 - k_3\right)\\
\upsilon_3 &= -2k_2{}^4 + \left[-k_2\left(k'_1 - k'_3\right) + 2k_3k_2{}^2 - k'_2\left(k_3 - k_1\right) + k_2{}^2\left(k_3 - k_1\right) + k_3(k_3 - k_1)^2\right]\left(k_1 - k_3\right)\\
\xi_1 &= 2k_3{}^2\left[k_1k_3 - k_2k_3 - k'_1 - k'_2\right] - \left[k_2{}^2 - 2k'_3 - k_1{}^2\right]k_3\left(k_1 + k_2\right)\\
\xi_2 &= -2k_3{}^4 - \left[2k_1k_3{}^2 - k_3\left(k'_1 + k'_2\right) + k'_3\left(k_1 + k_2\right) + k_3{}^2\left(k_1 + k_2\right) + k_1(k_1 + k_2)^2\right]\left(k_1 + k_2\right)\\
\xi_3 &= -2k_3{}^4 - \left[k_3\left(k'_1 + k'_2\right) + 2k_2k_3{}^2 - k'_3\left(k_1 + k_2\right) + k_3{}^2\left(k_1 + k_2\right) + k_2(k_1 + k_2)^2\right]\left(k_1 + k_2\right).
\end{aligned}$$

Finally, note that the obtained results here are in accordance with the results obtained in [16]. The reason of that arises from the similarity between PAF derivative formulas and PAFORS derivative formulas. To avoid misunderstanding, we recommend the readers to take into consideration the differences between the PAF apparatus and PAFORS apparatus.

In the next section, we will consider a point particle P moving on a specific circular helix $\alpha = \alpha(s)$ and we will provide examples to $\mathbf{T}_\alpha\mathbf{M}_\alpha$-Smarandache trajectory, $\mathbf{T}_\alpha\mathbf{Y}_\alpha$-Smarandache trajectory and $\mathbf{M}_\alpha\mathbf{Y}_\alpha$-Smarandache trajectory.

3. Applications

Example 1. *In E^3, suppose that a point particle P moves on the trajectory*

$$\alpha : \left(0, 15\sqrt{50}\right) \to E^3, \ \ \alpha(s) = \left(7\cos\frac{s}{\sqrt{50}}, 7\sin\frac{s}{\sqrt{50}}, \frac{s}{\sqrt{50}}\right)$$

which is a unit speed curve (see $\alpha = \alpha(s)$ in Figure 2).

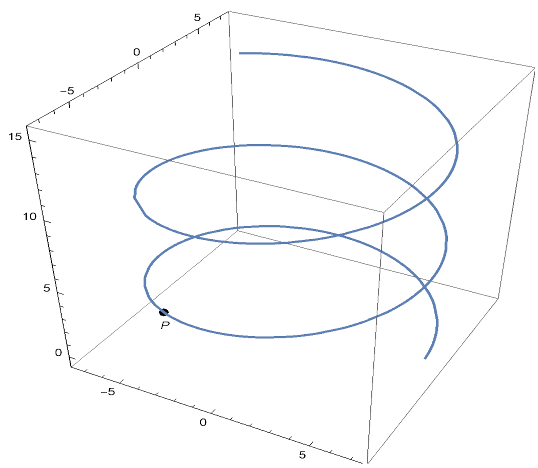

Figure 2. The trajectory of the moving point particle P

In the light of the information given in the first section, PAF apparatus of this trajectory are obtained as

$$\begin{aligned}
k_1(s) &= \frac{7}{50}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\\
k_2(s) &= \frac{7}{50}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\\
k_3(s) &= \frac{1}{50} + \frac{50}{2500 + s^2}
\end{aligned} \tag{6}$$

$$
\begin{aligned}
\mathbf{T}_\alpha(s) &= \left(\frac{-7}{\sqrt{50}}\sin\frac{s}{\sqrt{50}}, \frac{7}{\sqrt{50}}\cos\frac{s}{\sqrt{50}}, \frac{1}{\sqrt{50}}\right)\\
\mathbf{M}_\alpha(s) &= \left(\begin{array}{r}
-\cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}},\\
-\cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}},\\
-\frac{7}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)
\end{array}\right) \qquad (7)\\
\mathbf{Y}_\alpha(s) &= \left(\begin{array}{r}
-\sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}},\\
-\sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}},\\
\frac{7}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)
\end{array}\right).
\end{aligned}
$$

Let us show $\mathbf{T}_\alpha\mathbf{M}_\alpha$, $\mathbf{T}_\alpha\mathbf{Y}_\alpha$, $\mathbf{M}_\alpha\mathbf{Y}_\alpha$−Smarandache trajectories with σ_1, σ_2, σ_3, respectively. In that case, the parametric equation of σ_1 can be easily given as follows:

$$
\sigma_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{r}
\frac{-7}{\sqrt{50}}\sin\frac{s}{\sqrt{50}} - \cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}},\\
\frac{7}{\sqrt{50}}\cos\frac{s}{\sqrt{50}} - \cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}},\\
\frac{1}{\sqrt{50}} - \frac{7}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)
\end{array}\right).
$$

See σ_1 in Figure 3.

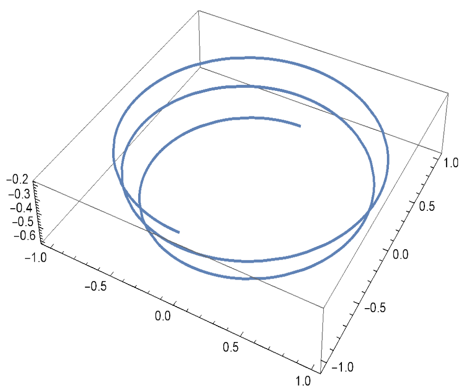

Figure 3. $\mathbf{T}_\alpha\mathbf{M}_\alpha$-Smarandache trajectory

Similarly, the parametric equation of σ_2 can be immediately given as

$$
\sigma_2 = \frac{1}{\sqrt{2}}\left(\begin{array}{r}
\frac{-7}{\sqrt{50}}\sin\frac{s}{\sqrt{50}} - \sin\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}} + \frac{1}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}},\\
\frac{7}{\sqrt{50}}\cos\frac{s}{\sqrt{50}} - \sin\left(\arctan\left(\frac{50}{s}\right)\right)\sin\frac{s}{\sqrt{50}} - \frac{1}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)\cos\frac{s}{\sqrt{50}},\\
\frac{1}{\sqrt{50}} + \frac{7}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)
\end{array}\right).
$$

See σ_2 in Figure 4.

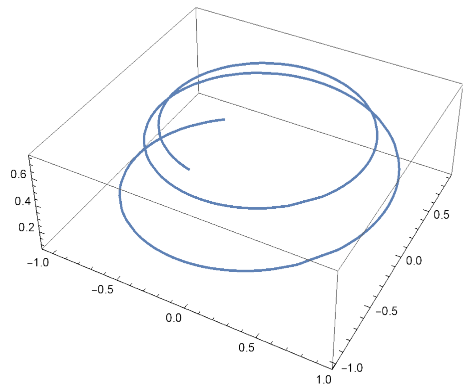

Figure 4. $\mathbf{T}_\alpha\mathbf{Y}_\alpha$-Smarandache trajectory

Finally, we obtain the parametric equation of σ_3 *as*

$$\sigma_3=\left(\begin{array}{c}\dfrac{\cos\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}}\sin\frac{s}{\sqrt{50}}-\cos\frac{s}{\sqrt{50}}\right)-\sin\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}}\sin\frac{s}{\sqrt{50}}+\cos\frac{s}{\sqrt{50}}\right)}{\sqrt{2}},\\ \dfrac{\sin\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}}\cos\frac{s}{\sqrt{50}}-\sin\frac{s}{\sqrt{50}}\right)-\cos\left(\arctan\left(\frac{50}{s}\right)\right)\left(\frac{1}{\sqrt{50}}\cos\frac{s}{\sqrt{50}}+\sin\frac{s}{\sqrt{50}}\right)}{\sqrt{2}},\\ \dfrac{-\frac{7}{\sqrt{50}}\sin\left(\arctan\left(\frac{50}{s}\right)\right)+\frac{7}{\sqrt{50}}\cos\left(\arctan\left(\frac{50}{s}\right)\right)}{\sqrt{2}}\end{array}\right).$$

See σ_3 *in Figure 5.*

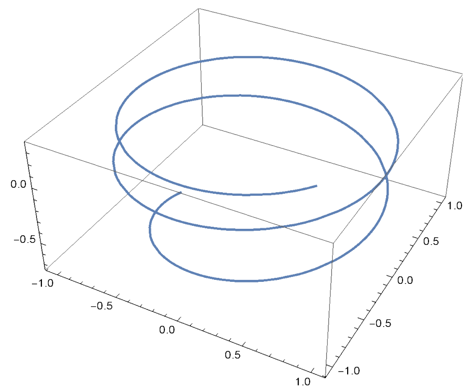

Figure 5. $\mathbf{M}_\alpha\mathbf{Y}_\alpha$-Smarandache trajectory

In the light of the information given in the previous section, one can immediately see $\mathbf{T}_{\sigma_i}$, $\mathbf{N}_{\sigma_i}$, $\mathbf{B}_{\sigma_i}$, κ_{σ_i}, $(i=1,2,3)$ *by using the equations* (6) *and* (7)*.*

4. Conclusion

For a particle moving in E^3, there is a very close relationship between the kinematics of the particle and the differential geometry of its trajectory. As a result of this case, the differential calculations for the curves which are the trajectories of moving particles play an important role in particle kinematics. Moving frames defined on these trajectories have been used as very useful tools in these differential calculations. Positional Adapted Frame (PAF) has been recently developed for the trajectories having non-zero angular momentum in three-dimensional Euclidean space by using the own position vector of the moving particle in [5]. Due to the relations of PAF with the position vector and angular momentum vector of the moving particle, we expect that PAF will enable more convenient observation environment of the researchers studying on inverse kinematics and robotics. Also, it is expected that this frame will be widely preferred to discuss many special topics in particle kinematics and differential geometry. The present paper can be seen as the first step of these future studies.

In this paper, the trajectories generated by **TM**, **TY**, **MY**−Smarandache curves are defined according to positional adapted frame in three-dimensional Euclidean space and Serret-Frenet apparatus of these trajectories are investigated. Also, illustrative examples are provided for these trajectories.

In the future study, we plan to investigate the **T**−magnetic, **M**−magnetic and **Y**−magnetic curves according to positional adapted frame in Euclidean 3-space.

Acknowledgement

The authors thank Assistant Professor Furkan Semih Dündar for his help in drawing the figures in Section 3.

References

[1] Chen B.Y., When does the position vector of a space curve always lie in its rectifying plane?, Am. Math. Mon., 110(2), 147-152, (2003).

[2] Erkan E., and Yüce S., Serret-Frenet frame and curvatures of Bézier curves, Mathematics, 6(12), 321, (2018).

[3] İlarslan K., and Nešović E., Some characterizations of osculating curves in the Euclidean spaces, Demonstr. Math., 41(4), 931-939, (2008).

[4] Ali A.T., Special Smarandache curves in the Euclidian space, International J. Math. Combin., 2, 30-36, (2010).

[5] Özen K.E., and Tosun M., A new moving frame for trajectories with non-vanishing angular momentum, J. Math. Sci. Model., 4(1), 7-18, (2021).

[6] Şenyurt S., and Çalışkan A., Smarandache curves of Bertrand curves pair according to Frenet frame, Boletim da Sociedade Paranaense de Matemática, 39(5), 163-173, (2021).

[7] Shifrin T., Differential Geometry: A First Course in Curves and Surfaces, University of Georgia, Preliminary Version (2008).

[8] Bishop R.L., There is more than one way to frame a curve, Am. Math. Mon., 82, 246-251, (1975).

[9] Yılmaz S., and Turgut M., A new version of Bishop frame and an application to spherical images, J. Math. Anal. Appl., 371(2), 764-776, (2010).

[10] Soliman M.A., Abdel-All N.H., Hussien R.A., and Youssef T., Evolution of space curves using Type-3 Bishop frame, Caspian J. Math. Sci., 8(1), 58-73, (2019).

[11] Turgut M., and Yılmaz S., Smarandache curves in Minkowski space-time, International J. Math. Combin., 3, 51-55, (2008).

[12] Bektaş Ö., and Yüce S., Smarandache curves according to Darboux frame in E^3, Romanian Journal of Mathematics and Computer Science, 3(1), 48-59, (2013).

[13] Çetin M., Tuncer Y., and Karacan M.K., Smarandache curves according to Bishop frame in Euclidean 3-space, Gen. Math. Notes, 20, 50-66, (2014).

[14] Taşköprü K., and Tosun M., Smarandache curves on S^2, Boletim da Sociedade Paranaense de Matematica, 32(1), 51-59, (2014).

[15] Şenyurt S., Altun Y., and Cevahir C. Smarandache curves according to Sabban frame belonging to Mannheim curves pair, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(1), 500-513, (2019).

[16] Özen K.E., and Tosun M., A new moving frame for trajectories on regular surfaces, Ikonion Journal of Mathematics, 3(1), 20-34, (2021).