Fundamental Journal of Mathematics and Applications, 4 (3) (2021) 210-220
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.951273

Tubular Surfaces Around a Null Curve and Its Spherical Images

Fatma Ateş

Department of Mathematics-Computer Sciences, Necmettin Erbakan University, Konya, Turkey

Article Info

Keywords: *Asymptotic curve, Geodesic curve, Images, Singular points, Tubular surface*
2010 AMS: *53A04, 53A55*
Received: *11 June 2021*
Accepted: *14 September 2021*
Available online: *15 September 2021*

Abstract

In this study, we define tubular surfaces whose center curves are null curves and their spherical images in Minkowski $3-$space. Firstly, we give the interior properties of the surfaces and calculate their invariant curvatures. Then, we obtain some special characterizations for the parameter curves of the surfaces. Finally, we demonstrate the theory via example and give their visualizations with the help of Mathematica.

1. Introduction

Minkowski space is defined as the basic model of quantum physics. Many notions in Euclidean space are different in this space. There are three spheres such as de Sitter $2-$space, hyperbolic $2-$space, and lightcone. Moreover, curves are divided into three groups due to the casual characters of their tangent vectors in the Minkowski space. An arbitrary curve is called as a spacelike curve, a timelike curve or a null (lightlike) curve, if its tangent vector is a spacelike vector, a timelike vector or a null (lightlike) vector, respectively. Similarly, a surface is called a timelike, spacelike, or lightlike surface if its normal vector lies on the de Sitter $2-$space, hyperbolic $2-$space, or null cone, respectively. Null curves have different properties than spacelike and timelike curves. So, the author [1] has defined Cartan frame as the most useful frame, and he used this frame to study null curves. Also, studies in the differential geometry are examined in two classes as null and non-null structures (see [2]- [4]).

A canal surface is defined as an envelope of one parameter family of spheres centered by a space curve. A tubular surface is a canal surface with constant radius. Many authors have studied on the characaterizations of tubular and canal surfaces [5]- [10]. The authors [4] have studied some characterizatons of the tubular surfaces generated by non-null curves in Minkowski $3-$space. Blaga [11] has presented a new approach to the tubular surfaces and provided CAD applications. Arslan *et.al.* [12] have obtained a medical application of the tube surfaces. In [13], they have examined a new type of the canal surface.

A tubular surface is one of the fundamental objects in geometric modelling. It appears in many application areas such as the networks of blood vessels and the neurons in medicine, hose systems, surface modeling in CAGD and CAD/CAM systems. On the other hand, null curves are important curves in general relativity. The surfaces produced by these curves provide good models for the study of different horizon types. In this study, we indicate the tubular surface around a null curve since they are generated by parabolas. To find geodesics on tubular surfaces are important to found the shortest distances between two points on a surface. Asymptotic curve on a surface whose osculating plane at each point coincides with the tangent plane to the surface at that point. Therefore, we have obtained characterizations of these curves on the surface. Also, we have examined the singular points of the tubular surface and the condition of the tubular surface being a Weingarten surface. Finally, we have investigated the tubular surfaces formed by spherical images of the null curve.

Email address and ORCID number: fgokcelik@erbakan.edu.tr, 0000-0002-3529-1077

2. Preliminaries

The standard metric of the Minkowski 3−space is

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

where x_i and y_j $(i,j=1,2,3)$ are shown the coefficients of the vectors x and y, respectively [14]. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $u \in E_1^3$ has three categories: if $\langle u,u\rangle > 0$ or $u=0$ it is a spacelike vector, if $\langle u,u\rangle < 0$ it is a timelike vector and if $\langle u,u\rangle = 0$ and $u \neq 0$ it is a null (lightlike) vector. Also, an arbitrary curve α in E_1^3 is called as spacelike, timelike or null (lightlike) curves according to casual character of the tangent vector. Cartan [1] has defined a frame $\{L(s)=\alpha'(s), N(s), W(s)\}$ similar to the Frenet frame for a null curve $\alpha(s)$, called Cartan frame satisfying

$$\langle L,L\rangle = \langle N,N\rangle = 0, \langle L,N\rangle = 1,$$
$$\langle W,L\rangle = \langle W,N\rangle = 0, \langle W,W\rangle = 1,$$

with $L\times N = W, W\times L = L$ and $N\times W = N$. The Cartan equations are

$$\begin{aligned} L^{'}(s) &= k_1(s)W(s), \\ N^{'}(s) &= k_2(s)W(s), \\ W^{'}(s) &= -k_2(s)L(s) - k_1(s)N(s), \end{aligned}$$

where $k_1(s) = \langle \alpha'', \alpha''\rangle^{1/2}$ and $k_2(s) = \langle N'(s), W(s)\rangle$ are Cartan curvature functions [15].
For investigate the interior geometry of the parametric surface $X(s,\theta)$ at the point $X(s_0,\theta_0)$, we use the first fundamental form. The coefficients of the first fundamental form are calculated as $e=\langle X_s,X_s\rangle$, $f=\langle X_s,X_\theta\rangle$, $g=\langle X_\theta,X_\theta\rangle$. The Gauss map of the surface $X(s,\theta)$ is U such that $\{X_s,X_\theta,U\}$ is an orthogonal frame along the surface. Let ε be a sign function of the Gauss map U, this is used to determine the causal character of the surface. If $\varepsilon = 1$ or $\varepsilon = -1$, then the surface is the timelike surface or the spacelike surface, respectively. The coefficients of the second fundamental form are $\ell = \langle X_{ss},U\rangle$, $m=\langle X_{s\theta},U\rangle$, $n=\langle X_{\theta\theta},U\rangle$. The invariant curvatures K and H of the surface are calculated as:

$$K := \frac{\varepsilon\left(\ell n - m^2\right)}{eg-f^2} \text{ and } H := \frac{\varepsilon\left(en-2fm+g\ell\right)}{2\left(eg-f^2\right)} \tag{2.1}$$

where K, H are called as Gaussian curvature and mean curvature of the surface, respectively. A surface in Minkowski 3−space is called as linear Weingarten surface if its invariant curvatures is satisfied the equation $2aH+bK=c$, where a,b,c are real numbers and $(a,b,c)\neq(0,0,0)$ [16].
The parametric equation of the canal surface is given by

$$X(s,\theta) = \alpha(s) + r(s)(\cos\theta N(s) + \sin\theta B(s))$$

where $N(s)$ and $B(s)$ are the Frenet normal vectors of the spine curve α. There are three kind of the tubular surface with respect to causal characters of the non-null curves in Minkowski 3−space [4].

3. Tubular surface around a null curve

In this section, we will analyze the properties of the tubular surface whose center curve is a null curve α and characterize some special curves on this surface.
Lopez [14] defined that the orbit of a point lies in the null plane is a parabola. The parabola in the null plane play the same role as the circle in Euclidean ambient. In [17], the authors is defined the tubular surface around the null curve $\alpha(s)$ as follows:

$$X(s,\theta) = \alpha(s) + \theta N(s) + \theta^2 W(s)$$

where $N(s)$ and $W(s)$ are the Cartan frame vectors of the null curve α and the parameter θ is characterized the parabola lies on the null plane spanned by the vectors $N(s)$ and $W(s)$. The coefficients of the first fundamental form are given by

$$\begin{aligned} &e = -2\theta^2 k_1(s)(1-\theta^2 k_2(s)) + \theta^2 k_2^2(s), \\ &f = 1+\theta^2 k_2(s), \quad g = 4\theta^2. \end{aligned} \tag{3.1}$$

The Gauss map U of the tubular surface $X(s,\theta)$ is calculated as

$$U = \frac{1}{\sqrt{A}}\{-2\theta(1-\theta^2k_2(s))L(s) - (2\theta^3 k_1(s) + \theta k_2(s))N(s) + (1-\theta^2 k_2(s))W(s)\}$$

and also, the coefficients of the second fundamental form are

$$\begin{aligned}\ell&=\tfrac{1}{\sqrt{A}}\{2\theta^5(k_1k_2'-k_1'k_2)+2\theta^4k_1k_2^2+2\theta^3k_1'+\theta^2(k_2^3-k_1k_2)+\theta k_2'+k_1\},\\ m&=\tfrac{1}{\sqrt{A}}\{\theta^2k_2^2(s)+4\theta^2k_1(s)+k_2(s)\},\quad n=\tfrac{2(1-\theta^2k_2(s))}{\sqrt{A}},\end{aligned}\tag{3.2}$$

where $A=\varepsilon(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)$ and the derivatives are taken by the parameter s.

Proposition 3.1. *The tubular surface $X(s,\theta)$ generated by the null curve α is a regular surface if and only if it satisfies the following conditions $\theta^2\neq\frac{1}{k_2(s)}$ and $\theta^2\neq\frac{\left(-3k_2(s)\pm\sqrt{9k_2^2(s)-32k_1(s)}\right)}{16k_1(s)}$ for $k_2^2(s)\geq\frac{32k_1(s)}{9}$.*

Proof. The condition $eg-f^2\neq 0$ provide for every regular surface at the point (s,θ). By using the equation (3.1), we obtain $-(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)\neq 0$ for the surface $X(s,\theta)$ and this equation gives us the desired conditions. □

Remark 3.2. *The tubular surface $X(s,\theta)$ has singular points at (s_0,θ_0) if and only if the equation $(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)=0$ is satisfied for the points (s_0,θ_0).*

Using the equation (2.1), the invariant curvatures of the surface $X(s,\theta)$ can be computed as

$$K=-\frac{\varepsilon}{A^2}\left\{\begin{array}{c}(1-\theta^2k_2)(2\theta^3k_1+\theta k_2)(2k_2'\theta^2-2k_2^2\theta-16k_1k_2\theta^3)\\ +(1-\theta^2k_2)^2(4\theta^3k_1'+2k_1+2k_2'\theta-16k_1^2\theta^4-8k_1k_2\theta^2-k_2^2)\\ -4k_2^2\theta^2(2\theta^3k_1+\theta k_2)^2\end{array}\right\}$$

and

$$H=-\frac{\varepsilon}{A^{3/2}}\left\{\begin{array}{c}(1-\theta^2k_2)(-4\theta^2k_1-k_2+4\theta^5k_1'-2\theta^4k_1k_2+2\theta^3k_2')\\ +(2\theta^3k_1+\theta k_2)(-2\theta k_2+2\theta^4k_2')\end{array}\right\}$$

where $A=\varepsilon(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)$.

Theorem 3.3. *The $s-$parameter curves of $X(s,\theta)$ are the geodesic curves if and only if the condition is satisfied*

$$k_2'(s)(k_2(s)+\theta^2k_1(s))-k_1'(s)(1-\theta^2k_2(s))=0$$

in terms of the Cartan curvatures of the null curve α.

Proof. If the normal vector of the surface and second derivative of a curve lying on the surface are linearly dependent, then the curve is called the geodesic curve of the surface [8]. Based on this definition, we obtain the following system of equations for the $s-$parameter curves on the regular tubular surface $X(s,\theta)$:

$$\left\{\begin{array}{c}(1-\theta^2k_2)[-k_2'\theta^2+k_2(2\theta^3k_1+\theta k_2)-2\theta k_1(1-\theta^2k_2)]=0,\\ (1-\theta^2k_2)(2\theta^3k_1^2-\theta^2k_1')+(2\theta^3k_1+\theta k_2)(-\theta^2k_1k_2+\theta k_2')=0,\\ (1-\theta^2k_2)(-2\theta^3k_1'-2\theta^2k_1k_2)+(2\theta^3k_1+\theta k_2)(\theta^2k_2'+\theta k_2^2)=0.\end{array}\right.\tag{3.3}$$

Since $X(s,\theta)$ is the regular surface, we have $-k_2'\theta^2+k_2(2\theta^3k_1+\theta k_2)-2\theta k_1(1-\theta^2k_2)=0$. If this equation is solved together with the last two equations in equation (3.3), we get

$$k_2'(s)(k_2(s)+\theta^2k_1(s))-k_1'(s)(1-\theta^2k_2(s))=0.$$

□

Corollary 3.4. *The $s-$parameter curves of the surface with the Cartan curvatures $k_1(s)=\frac{1}{1-(as^n+b)\theta^2}(\frac{a^2s^{2n}}{2}+abs^n+c)$ and $k_2(s)=as^n+b,(n\geq 1$ and a,b,c*

are constants) are the geodesic curves on the tubular surface.

Proof. If we consider $k_2(s)=as^n+b$ for the constants a, b and substituting this equation into the equation $k_2'(s)(k_2(s)+\theta^2k_1(s))-k_1'(s)(1-\theta^2k_2(s))=0$, then the following differential equation is obtained

$$k_1'(s)-\frac{ans^{n-1}\theta^2}{1-(as^n+b)\theta^2}k_1(s)=\frac{ans^{n-1}(as^n+b)}{1-(as^n+b)\theta^2}.$$

From solution of the ODE according to function $k_1(s)$, the first Cartan curvature is found as $k_1(s)=\frac{1}{1-(as^n+b)\theta^2}(\frac{a^2s^{2n}}{2}+abs^n+c)$. □

Theorem 3.5. *The $s-$parameter curves of $X(s,\theta)$ are the asymptotic curves if and only if the condition is fulfilled the following equation*

$$2\theta^5(k_1k_2'-k_1'k_2)+2\theta^4k_1k_2^2+2\theta^3k_1'+\theta^2(k_2^3-k_1k_2)+\theta k_2'+k_1=0$$

where $k_1(s)$ and $k_2(s)$ are the Cartan curvatures of the curve $\alpha(s)$.

Proof. If the normal vector of the surface is tangent to second derivative of a curve lying on the surface, this curve is called as the asymptotic curve of the surface, that is $\ell=0$ [8]. The desired result is obtained from the expression of $\ell=0$ in the equation (3.2) for the $s-$parameter curves on the regular tubular surface $X(s,\theta)$. □

Theorem 3.6. *The $\theta-$parameter curves of the regular surface $X(s,\theta)$ are neither geodesic curve nor asymptotic curve.*

Proof. For $\theta-$ parameter curves to be geodesic, it must provide the condition $U\times X_{\theta\theta}=0$. From this condition, we obtain

$$\frac{1}{\sqrt{A}}\{-4\theta(1-\theta^2k_2(s))L(s)+2(2\theta^3k_1(s)+\theta k_2(s))N(s)\}=0.$$

Since $1-\theta^2k_2(s)=0$ conflicts with the regularity condition of the surface $X(s,\theta)$, the $\theta-$ parameter curves cannot be geodesic curves. If the $\theta-$ parameter curves are to be asymptotic curve, then the coefficient of the second fundamental form n in equation (3.2) must be equal to zero. This condition conflicts with the regularity condition of the surface $X(s,\theta)$. So, the $\theta-$ parameter curves cannot be an asymptotic curve. □

4. Tubular surfaces around the spherical images of the null curve

In this section, we introduce tubular surfaces formed by spherical images of the null curve α. First, we will give definitions of the spheres in the Minkowski $3-$space. There are three kinds of spheres in E_1^3: de Sitter $2-$space, hyperbolic $2-$space, and lightlike cone. These are respectively:

$$S_1^2=\left\{p\in E_1^3\,|\ \langle p,p\rangle=1\right\},H_0^2=\left\{p\in E_1^3\,|\ \langle p,p\rangle=-1\right\}$$
$$\text{and } Q^2=\left\{p\in E_1^3\,|\ \langle p,p\rangle=0\right\}.$$

Now, we will give the definitions of the spherical images of the null curve. The null Cartan vector field L of the curve α is located at the center of the lightcone, the geometric location of this vector with respect to each point s indicates a curve on the lightcone Q^2, which is called the spherical (L) image of the curve α. In this definition, the spherical (N) image of the curve is defined by taking the null Cartan vector N instead of L. The spherical (W) image of the null curve is defined by the geometric location of the spacelike vector W on the de Sitter $2-$space S_1^2.
Note: Unless stated otherwise, the parameter θ given for each surface is different from each other.

4.1. Tubular surface around the spherical (L) and (N) images of the null curve

Let the spine curves of the tubular surfaces is respectively the spherical (L) and (N) images of the null curve α, that is, $\beta_i(s_i)=i(s)$ where the function s_i is the arc length parameter of the (i) image curve and $s_i=\int_0^s k_j(s)ds$ where indices are respectively $i=L$, $j=1$ and $i=N$, $j=2$. In [18], the author defined the Darboux frame $\left\{\beta_i(s_i),t_i(s_i)=\frac{d\beta}{ds_i},y_i(s_i)=\beta_i(s_i)\times t_i(s_i),\varkappa_i(s_i)\right\}$ of the spacelike curve β_i on the lightcone Q^2. The Darboux frame apparatus are calculated as follows:

$$\beta_L(s_L)=L(s),\ t_L(s_L)=W(s),\ y_L(s_L)=N(s) \text{ and } \varkappa_L(s_L)=-\tfrac{k_2(s)}{k_1(s)},$$

and

$$\beta_N(s_N)=N(s),\ t_N(s_N)=W(s),\ y_N(s_N)=L(s) \text{ and } \varkappa_N(s_N)=-\tfrac{k_1(s)}{k_2(s)}.$$

The Darbox equations are given by

$$\begin{aligned}\beta_i{}'(s_i) &= t_i(s_i),\\ t_i{}'(s_i) &= \varkappa_i(s_i)\beta_i(s_i)-y_i(s_i),\\ y_i{}'(s_i) &= -\varkappa_i(s_i)t_i(s_i),\end{aligned}$$

where $y_i\times t_i=y_i$, $\beta_i\times y_i=t_i$ and $t_i\times\beta_i=\beta_i$. The **spacelike** tubular surface around the curve β is

$$\mathscr{X}(s_i,\theta)=(1+\theta)\beta_i(s_i)+\theta^2y_i(s_i)$$

with the Gauss map $\mathscr{U}=\frac{1}{2\sqrt{|\theta|}}(\beta_i(s_i)-2\theta y_i(s_i))$. The coefficients of the first and second fundamental forms of $\mathscr{X}(s,\theta)$ are found by

$$\begin{aligned}&\mathscr{E}=(1+\theta-\theta^2\varkappa_i(s_i))^2,\quad \mathscr{F}=0,\quad \mathscr{G}=4\theta,\\ &\mathscr{L}=-\tfrac{(1+\theta-\theta^2\varkappa_i(s_i))(1+2\theta\varkappa_i(s_i))}{2\sqrt{|\theta|}},\quad \mathscr{M}=0,\quad \mathscr{N}=\tfrac{1}{\sqrt{|\theta|}}.\end{aligned} \tag{4.1}$$

Proposition 4.1. *The tubular surface $\mathscr{X}(s_i,\theta)$ is a regular surface if and only if it has the condition $\theta \neq \frac{1\mp\sqrt{1+4\varkappa_i(s_i)}}{2\varkappa_i(s_i)}$.*

Proof. The condition $\mathscr{E}\mathscr{G}-\mathscr{F}^2 \neq 0$ must be provided for a regular surface. By using the equation (4.1), we obtain $1+\theta-\theta^2\varkappa_i(s_i)\neq 0$ for the surface $\mathscr{X}(s_i,\theta)$ and the desired condition is obtained from the solution of this equation with respect to θ. □

Remark 4.2. *The surface $\mathscr{X}(s_i,\theta)$ has the singular points at the points $(s_0,\theta_0 = \frac{1\mp\sqrt{1+4\varkappa_i(s_0)}}{2\varkappa_i(s_0)}$.*

From equation (2.1) and $\varepsilon=-1$, the curvatures of the surface $\mathscr{X}(s_i,\theta)$ are calculated as follows:

$$K = -\frac{(1+2\theta\varkappa_i(s_i))}{8\theta^2(1+\theta-\theta^2\varkappa_i(s_i))} \ \ \textit{and} \ \ H = -2\theta^{3/2}\left(\frac{1}{16\theta^3}+K\right).$$

Theorem 4.3. *The $s-$parameter curves of $\mathscr{X}(s_i,\theta)$ are the geodesic curves if and only if the Darboux curvature $\varkappa_i(s_i)=1/2\theta$ is a constant, this means that the image curve (i) is a planar curve.*

Proof. The $s-$parameter curves on the regular tubular surface $\mathscr{X}(s_i,\theta)$ are the geodesic curves if and only if $\mathscr{X}_{s_is_i}\times\mathscr{U}=0$. From the last equation, we obtain

$$\frac{1}{2|\ \theta\ |^{1/2}}(1-2\theta\varkappa_i)(1+\theta-\theta^2\varkappa_i)=0.$$

Since the surface is the regular, then $1+\theta-\theta^2\varkappa_i\neq 0$. So the curvature $\varkappa_i(s_i)=1/2\theta$ is obtained as a constant. □

Theorem 4.4. *The $s-$parameter curves of $\mathscr{X}(s_i,\theta)$ are the asymptotic curves if and only if the image curve (i) is a planar curve.*

Proof. The $s-$parameter curves on the regular tubular surface $\mathscr{X}(s_i,\theta)$ are the asymptotic curves if and only if $\langle\mathscr{X}_{s_is_i},\mathscr{U}\rangle=0$. From here, we get $\varkappa_i(s_i)=-1/2\theta$. Since the parameter θ is a constant for the $s-$parameter curves, the curvature $\varkappa_i(s_i)$ is a constant. □

Theorem 4.5. *The $\theta-$parameter curves on the regular surface $X(s,\theta)$ are neither geodesic curve nor asymptotic curve.*

Proof. For $\theta-$ parameter curves, $\mathscr{X}_{\theta\theta}\times\mathscr{U}\neq 0$ and $\mathscr{N}\neq 0$ are satisfied, so the $\theta-$ parameter curves cannot be a geodesic curve and an asymptotic curve. □

Since the proofs of the theorems and propositions involving the properties of the tubular surfaces consisting of $W-$ image curve of the null curve are similar to the proofs given above, the following theorems and propositions will be given without proof.

4.2. Tubular surface around the spherical (W) image of the null curve

Let γ be the (W) image curve of the null curve α. In this subsection, the spine curve of the surface $\mathbf{X}(s_W,\theta)$ will take as the curve γ, that is, $\gamma(s_W)=W(s)$ where the function $s_W=\int_0^s\sqrt{2\mid k_1(s)k_2(s)\mid}ds$ is the arc length parameter of the (W) image curve. The Darboux frame apparatus are given by

$$\begin{aligned}
\gamma(s_W) &= W(s),\\
t_W(s_W) &= -\frac{1}{\sqrt{2\mid k_1(s)k_2(s)\mid}}(k_2(s)L(s)+k_1(s)N(s)),\\
y_W(s_W) &= \frac{1}{\sqrt{2\mid k_1(s)k_2(s)\mid}}(-k_2(s)L(s)+k_1(s)N(s)),\\
\varkappa_W(s_W) &= \frac{k_1'(s)k_2(s)-k_1(s)k_2'(s)}{(2\mid k_1(s)k_2(s)\mid)^{3/2}}.
\end{aligned}$$

There are two cases here: $k_1(s)k_2(s)\neq 0$ and $k_1(s)k_2(s)=0$.
Case 1: $\mathbf{k_1(s)k_2(s)\neq 0}$. We will examine this situation as two sub-cases.
Case 1.1: If $\mathbf{k_1(s)k_2(s)>0}$, then the curve γ is a spacelike curve on de Sitter $2-$space S_1^2. In [14], the Darboux equations are

$$\begin{aligned}
\gamma'(s_W) &= t_W(s_W),\\
t_W{}'(s_W) &= -\gamma(s_W)+\varkappa_W(s_W)y_W(s_W),\\
y_W{}'(s_W) &= \varkappa_W(s_W)t_W(s_W),
\end{aligned}$$

where $y_W\times t_W=\gamma$, $\gamma\times y_W=t_W$ and $t_W\times\gamma=-y_W$. The **timelike** tubular surface around the curve γ is

$$\mathbf{X}(s_W,\theta)=(1+r\cosh\theta)\gamma(s_W)-r\sinh\theta\, y_W(s_W)$$

with the Gauss map $\mathbf{U}=\cosh\theta\gamma(s_W)-\sinh\theta y_W(s_W)$. The coefficients of the first and second fundamental forms of $\mathbf{X}(s,\theta)$ are found by

$$\mathbf{E}=(1+r\cosh\theta-r\varkappa_W(s_W)\sinh\theta)^2,\quad \mathbf{F}=0,\quad \mathbf{G}=-r^2,$$

$$\mathbf{L}=(\varkappa_W(s_W)\sinh\theta-\cosh\theta)(1+r\cosh\theta-r\varkappa_W(s_W)\sinh\theta),\quad \mathbf{M}=0,\quad \mathbf{N}=r.$$

Proposition 4.6. *The tubular surface $\mathbf{X}(s_W,\theta)$ is a regular surface if and only if it has the condition $1+r\cosh\theta-r\varkappa_W(s_W)\sinh\theta\neq 0$.*

Remark 4.7. *The surface $\mathbf{X}(s_W,\theta)$ has the singular points satisfying the equation $1+r\cosh\theta-r\varkappa_W(s_W)\sinh\theta=0$.*

From equation (2.1) and $\varepsilon=1$, the curvatures of the surface $\mathbf{X}(s_W,\theta)$ are calculated as follows:

$$\mathbf{K}=\frac{(\cosh\theta-\varkappa_W(s_W)\sinh\theta)}{r(1+r\cosh\theta-r\varkappa_W(s_W)\sinh\theta)}\quad \textit{and}\quad \mathbf{H}=-\frac{1}{2}\left(\frac{1}{r}+rK\right).$$

Remark 4.8. *Since the surface $\mathbf{X}(s_W,\theta)$ has the condition $2\mathbf{H}+r\mathbf{K}=-\dfrac{1}{r}$, the surface $\mathbf{X}(s_W,\theta)$ is a linear Weingarten surface.*

Theorem 4.9. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the geodesic curves if and only if the Darboux curvature $\varkappa_W(s_W)=\tanh\theta$ is a constant.*

Theorem 4.10. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the asymptotic curves if and only if the curvature $\varkappa_W$ is a constant and equal to* $\coth\theta$.

Theorem 4.11. *The $\theta-$parameter curves on the regular surface $\mathbf{X}(s_W,\theta)$ are always a geodesic curve and cannot be an asymptotic curve.*

Case 1.2: If $\mathbf{k_1(s)k_2(s)<0}$, then the curve γ is a timelike curve on de Sitter $2-$space S_1^2. In [14], the Darboux equations are

$$\begin{aligned}\gamma'(s_W)&=t_W(s_W),\\ t_W{}'(s_W)&=\gamma(s_W)+\varkappa_W(s_W)y_W(s_W),\\ y_W{}'(s_W)&=\varkappa_W(s_W)t_W(s_W),\end{aligned}$$

where $y_W\times t_W=-\gamma,\ \gamma\times y_W=t_W$ and $t_W\times\gamma=-y_W$. The timelike tubular surface around the curve γ is

$$\mathbf{X}(s_W,\theta)=(1+r\cos\theta)\gamma(s_W)+r\sin\theta y_W(s_W)$$

with the spacelike Gauss map $\mathbf{U}=\cos\theta\gamma(s_W)+\sin\theta y_W(s_W)$. The coefficients of the first and second fundamental forms of $\mathbf{X}(s,\theta)$ are found by

$$\mathbf{E}=-(1+r\cos\theta+r\varkappa_W(s_W)\sin\theta)^2,\quad \mathbf{F}=0,\quad \mathbf{G}=r^2,$$

$$\mathbf{L}=(\cos\theta+\varkappa_W(s_W)\sin\theta)(1+r\cos\theta+r\varkappa_W(s_W)\sin\theta),\quad \mathbf{M}=0,\quad \mathbf{N}=-r.$$

Proposition 4.12. *The tubular surface $\mathbf{X}(s_W,\theta)$ is a regular surface if and only if it has the condition $1+r\cos\theta+r\varkappa_W(s_W)\sin\theta\neq 0$.*

Remark 4.13. *The surface $\mathbf{X}(s_W,\theta)$ has the singular points satisfying the equation $1+r\cos\theta+r\varkappa_W(s_W)\sin\theta=0$.*

The invariant curvatures of the surface $\mathbf{X}(s_W,\theta)$ are calculated as follows:

$$\mathbf{K}=\frac{(\cos\theta+\varkappa_W(s_W)\sin\theta)}{r(1+r\cos\theta+r\varkappa_W(s_W)\sin\theta)}\quad \textit{and}\quad \mathbf{H}=-\frac{1}{2}\left(\frac{1}{r}+r\mathbf{K}\right).$$

Since the surface $\mathbf{X}(s_W,\theta)$ has the condition $2\mathbf{H}+r\mathbf{K}=-\dfrac{1}{r}$, the surface $\mathbf{X}(s_W,\theta)$ is a linear Weingarten surface.

Theorem 4.14. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the geodesic curves if and only if the Darboux curvature $\varkappa_W(s_W)=\tan\theta$ is a constant.*

Theorem 4.15. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the asymptotic curves if and only if the curvature of the (W) image curve is a constant.*

Theorem 4.16. *The $\theta-$parameter curves on the regular surface $\mathbf{X}(s_W,\theta)$ are always the geodesic curves and they cannot be the asymptotic curves.*

Case 2: $\mathbf{k_1(s)k_2(s)=0}$. If $k_1(s)=0$, the null curve α is a planar line. So, we will examine the case of $k_2(s)=0$. For $k_1(s)\neq 0$ and $k_2(s)=0$, the curve α is called a generalized null cubic curve in [15] and it is given by

$$\alpha(s)=\left(\frac{1}{\sqrt{2}}\left((s+\frac{\phi(s)}{2}\right),\frac{1}{\sqrt{2}}\left(s-\frac{\phi(s)}{2}\right),\psi(s)\right)$$

where $\phi\prime(s)=(\psi\prime(s))^2$. The third Cartan vector of the curve α is $W(s)=\left(\frac{\psi'(s)}{\sqrt{2}},-\frac{\psi'(s)}{\sqrt{2}},1\right)$. To find the tubular surface around the (W) image curve of the generalized null cubic curve, we calculate the Cartan frame of the (W) image curve as follows: $\overline{L}=\left(\frac{\psi''(s)}{\sqrt{2}},-\frac{\psi''(s)}{\sqrt{2}},0\right),\overline{N}(s)=\left(-\frac{1}{\sqrt{2}\psi''(s)},-\frac{1}{\sqrt{2}\psi''(s)},0\right),\overline{W}=(0,0,1)$.
From these vectors, we obtain $\psi''(s)=\sqrt{2}a$, where a is a constant. The surface consisting of the (W) image curve of the generalized null cubic curve α can be written as

$$\mathbf{X}(s_W,\theta)=W(s)+\theta\overline{N}(s)+\theta^2\overline{W}(s).$$

This tubular surface is degenerated to a plane.

5. Visualization

In this section, we give the tubular surfaces whose center curves are a null curve α and its spherical images. Then, we calculate the some special curves on these surfaces and find the singular points of them. Also, we visualize the our calculations with Mathematica.
Let $\alpha=\alpha(s)$ be a null curve is defined by

$$\alpha(s)=\left(s,\frac{1}{5}\sin(5s+4)+1,-\frac{1}{5}\cos(5s+4)-1\right)$$

with the Cartan frame apparatus

$$\begin{aligned}L(s)&=(1,\cos(5s+4),\sin(5s+4)),\\N(s)&=\tfrac{1}{2}(-1,\cos(5s+4),\sin(5s+4)),\\W(s)&=(0,-\sin(5s+4),\cos(5s+4)),\end{aligned}$$

$k_1(s)=5$ and $k_2(s)=5/2$. The parametric form of the tubular surface $X(s,\theta)$ is given as follows

$$X(s,\theta)=\alpha(s)+\theta N(s)+\theta^2W(s),$$

$$\begin{aligned}X(s,\theta)=\Big(s-\frac{\theta}{2},1+\frac{\theta}{2}\cos(5s+4)+\frac{1}{5}\sin(5s+4)-\theta^2\sin(5s+4),\\-1+\frac{\theta}{2}\sin(5s+4)-\frac{1}{5}\cos(5s+4)+\theta^2\cos(5s+4)\Big).\end{aligned}$$

We calculate the singular points on the surface $X(s,\theta)$, then we obtain two curves consisting of singular points in Figure (5.1). Since the Cartan curvatures of the curve α are constants, the $s-$parameter curves in Figure (5.2) of the tubular surface $X(s,\theta)$

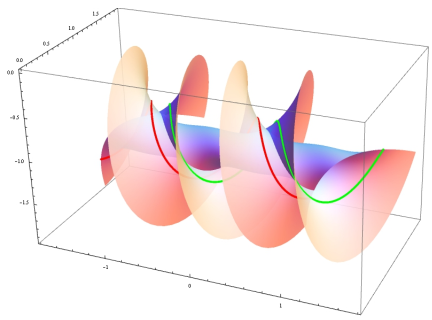

Figure 5.1: Tubular surface generated by the null curve α for $s\in[-2\pi/5,2\pi/5]$, $\theta\in[-1,1]$ and its singular curves for $\theta=\sqrt{\frac{2}{5}}$ (red), $\theta=-\sqrt{\frac{2}{5}}$ (green).

are always the geodesic curves. Since the condition given in the Theorem (3.5) is not satisfied in this example, the $s-$parameter curves are not the asymptotic curves. Also, we have shown in Theorem (3.6) that the $\theta-$parameter curves in Figure (5.3) are neither geodesic curves nor asymptotic curves.

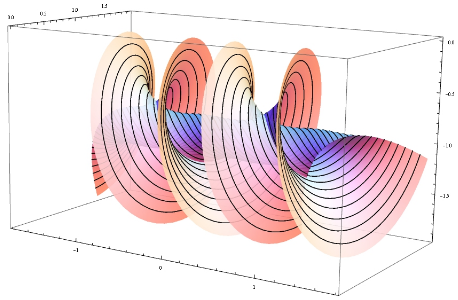

Figure 5.2: Geodesic $s-$parameter curves on the tubular surface $X(s,\theta)$

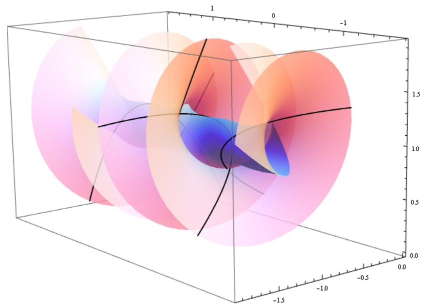

Figure 5.3: The $\theta-$parameter curves on the tubular surface $X(s,\theta)$

The tubular surface around (L) image curve is given by

$$\mathscr{X}(s_L,\theta) = (1+\theta)L(s) + \theta^2 N(s),$$

$$\mathscr{X}(s_L,\theta) = \left(1+\theta-\frac{\theta^2}{2}, \left(1+\theta+\frac{\theta^2}{2}\right)\cos(5s+4), \left(1+\theta+\frac{\theta^2}{2}\right)\sin(5s+4)\right)$$

where $s_L = 5s$. The tubular surface $\mathscr{X}(s_L,\theta)$ has no singular points. The $s-$parameter curves are the geodesic curves for $\theta = -1$ in Figure (5.4) (red) and are the asymptotic curves for $\theta = 1$ in Figure (5.4) (green). The $\theta-$parameter curves on the tubular surface $\mathscr{X}(s_L,\theta)$ are shown in Figure (5.5).

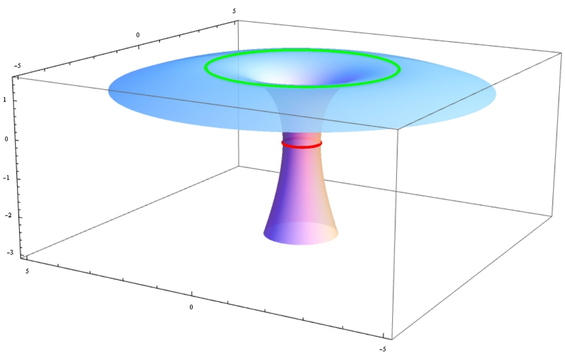

Figure 5.4: Geodesic $s-$parameter curves on $\mathscr{X}(s_L,\theta)$ for $\theta = -1$ (red) and $\theta = 1$ (green).

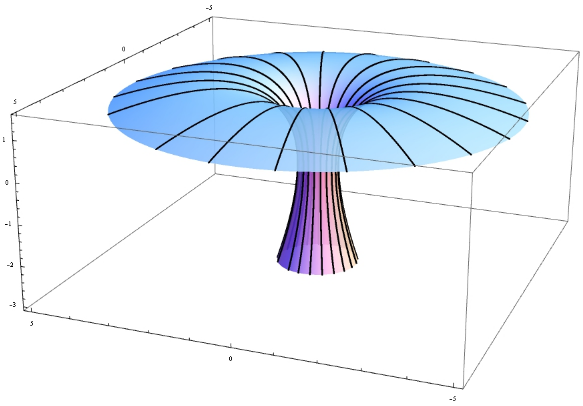

Figure 5.5: The $\theta-$parameter curves on $\mathscr{X}(s_L,\theta)$.

The tubular surface around (N) image curve is given by

$$\mathscr{X}(s_N,\theta)=(1+\theta)N(s)+\theta^2L(s),$$

$$\mathscr{X}(s_N,\theta)=\left(-\frac{1}{2}-\frac{\theta}{2}+\theta^2,\left(\frac{1}{2}+\frac{\theta}{2}+\theta^2\right)\cos(5s+4),\left(\frac{1}{2}+\frac{\theta}{2}+\theta^2\right)\sin(5s+4)\right)$$

where $s_N=5s/2$. The tubular surface $\mathscr{X}(s_N,\theta)$ has no singular points. The $s-$parameter curves are the geodesic curves for $\theta=-1/4$ in Figure (5.6) (red) and the asymptotic curves for $\theta=1/4$ in Figure (5.6) (green). The $\theta-$parameter curves on the tubular surface $\mathscr{X}(s_L,\theta)$ are shown in Figure (5.7).

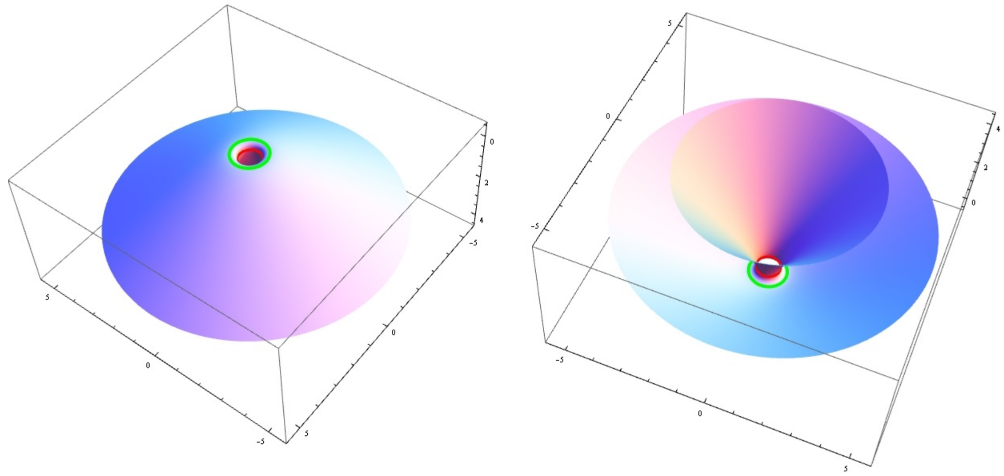

Figure 5.6: Geodesic $s-$parameter curves on $\mathscr{X}(s_N,\theta)$ for $\theta=-0.25$ (red) and $\theta=0.25$ (green).

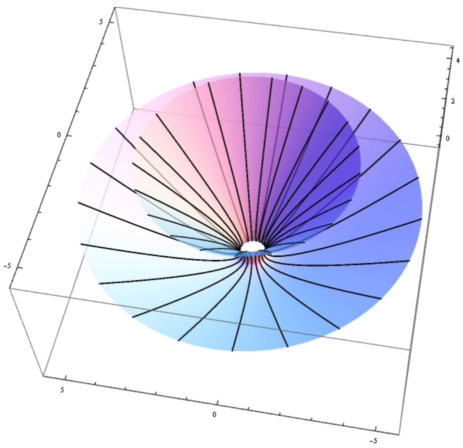

Figure 5.7: The $\theta-$parameter curves on $\mathscr{X}(s_N,\theta)$.

Since $\mathbf{k_1(s)k_2(s)} > \mathbf{0}$, the tubular surface generated by (W) image curve is

$$\mathbf{X}(s_W,\theta) = (1+r\cosh\theta)W(s) - r\sinh\theta\left(-\frac{1}{2}L(s)+N(s)\right),$$

$$\mathbf{X}(s_W,\theta) = (r\sinh\theta, -(1+r\cosh\theta)\sin(5s+4), (1+r\cosh\theta)\cos(5s+4))$$

where $s_W = 5s$. Since $\varkappa_W(s_W) = 0$ and $r > 0$, the equation in Remark (4.7) has no real root. So, there is no singular points on the tubular surface $\mathbf{X}(s_W,\theta)$. Some special curves on $\mathbf{X}(s_W,\theta)$ are shown in Figure (5.8).

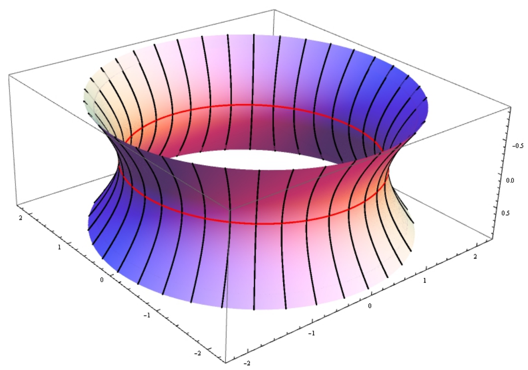

Figure 5.8: Geodesic $s-$parameter curve for $\theta = 0$ (red) and geodesic $\theta-$parameter curves (black) for $r = 0.8$ on the surface $\mathbf{X}(s_W,\theta)$.

6. Conclusion

This study is important in terms of finding tubular surface formed by the null curve and its image curves on the Minkowski spheres. Their singular points are characterized in terms of Cartan frame and Darboux frame apparatus. It is also noteworthy that to use the Darboux frame instead of the Frenet frame, this is provided an opportunity to examine the expressions in their simplest form.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] E. Cartan, *La Theorie Des Groupes Finis et Continus et la Geometrie Differentielle*, Gauthier-Villars, Paris, 1937.
[2] A. Bejancu, *Lightlike curves in Lorentz manifolds*, Publ. Math. Debrecen, **44**(1) (1994), 145–155.
[3] F. Gökçelik, I. Gök, *Null $W-$slant helices in E_1^3*, J. Math. Anal. Appl., **420** (2014), 222–241.
[4] M. K. Karacan, Y. Yaylı, *On the geodesics of tubular surfaces in Minkowski $3-$space*, Bull. Malays. Math. Sci. Soc., **31**(2) (2008), 1-10.
[5] F. Ateş, E. Kocakuşaklı, İ. Gök, N. Ekmekci, *Tubular surfaces formed by semi-spherical indicatrices in E_1^3*, Mediterr. J. Math., **17** (127) (2020). https://doi.org/10.1007/s00009-020-01561-z
[6] F. Doğan, Y. Yaylı, *On the curvatures of tubular surface with Bishop frame*, Commun. Fac. Sci. Univ. Ank. Ser. A1, **60**(1) (2011), 59–69
[7] F. Doğan, Y. Yaylı, *Tubes with darboux frame*, Int. J. Contemp. Math. Sci., **7**(16) (2012), 751 - 758.
[8] F. Doğan, Generalized canal surfaces, Ph.D. Thesis, Ankara University, 2012.
[9] M. K. Karacan, Y. Tunçer, *Tubular surfaces of Weingarten types in Galilean and pseudo-Galilean*, Bull. Math. Anal. Appl., **5**(2) (2013), 87-100.
[10] J. Qian J, M. Su, X. Fu, S. D. Jung, *Geometric characterizations of canal surfaces in Minkowski 3-space II* , Mathematics, **7**(8) (2019), 703. https://doi.org/10.3390/math7080703
[11] P. A. Blaga, *On tubular surfaces in computer graphics*, Stud. Univ. Babeş-Bolyai Inform., **50**(2) (2005), 81-90.
[12] B. Yildiz, K. Arslan, H. Yildiz, C. Özgür, *A geometric description of the ascending colon of some domestic animals*, Ann Anat., **183** (2001), 555-557.
[13] M. Peternell, H. Potmann, *Computing rational parametrizations of canal surfaces*, J. Symb. Comput., **23** (1997), 255-266.
[14] R. Lopez, *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Int. Electron. J. Geom., **7**(1) (2014), 44-107.
[15] K. L. Duggal, D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.
[16] R. Lopez, *Rotational linear Weingarten surfaces of hyperbolic type*. Isr. J. Math., **167** (2008), 283–302.
[17] P. Tekin, F. N. Ekmekci, *On Weingarten tube surfaces with null curve in Minkowski 3-space*, New Trends in Math. Sci., **3**(3) (2015), 168-174.
[18] H. Liu, *Curves in the lightlike cone*, Beitr. Algebra Geom., **45**(1) (2004), 291-303.