Mathematical Sciences And Applications E-Notes
Volume 2 No. 1 pp. 99–105 (2014) ©MSAEN

HYPERSURFACES SATISFYING SOME CURVATURE CONDITIONS ON PSEUDO PROJECTIVE CURVATURE TENSOR IN THE SEMI-EUCLIDEAN SPACE

YUSUF DOĞRU

(Communicated by Maria FALCITELLI)

Abstract. We consider some curvature conditions on the Pseudo projective curvature tensor $\widetilde{P}$ on a hypersurface in the semi-Euclidean space E_s^{n+1}. We prove that every pseudo projectively Ricci-semisymmetric hypersurface M satisfying the condition $\widetilde{P} \cdot R = 0$ is pseudosymmetric. We also consider the condition $\widetilde{P} \cdot S = 0$ on hypersurfaces of the semi-Euclidean space E_s^{n+1}.

1. Introduction

Let (M, g) be an n-dimensional, $n \geq 3$, differentiable manifold of class C^∞. The pseudo projective curvature tensor $\widetilde{P}$ was introduced by B.Prasad [11]. According to them, a pseudo projective curvature tensor is defined by

$$\begin{aligned}\widetilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &-\frac{\kappa}{n}\left[\frac{a}{n-1}+b\right](g(Y,Z)X - g(X,Z)Y),\end{aligned}$$

where a and b are constants, S is the Ricci tensor and κ is the scalar curvature of the manifold M.

In [7], Dabrowska, Defever, Deszcz and Kowalczyk studied semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Euclidean space. Recently in [8], Özgür studied hypersurfaces satisfying some curvature conditions in the semi-Euclidean space. In [10], Özgür, Arslan and Murathan studied conharmonically semiparallel hypersurfaces in Euclidean space. In [9], Özgür and Arslan studied pseudosymmetric hypersurfaces satisfying Chen's equality in Euclidean space. In the present study, our aim is to study hypersurfaces of dimension $n \geq 4$, in $(n+1)$-dimensional semi-Euclidean space E_s^{n+1}. We show that if a pseudo projectively Ricci-semisymmetric hypersurface M satisfies the condition $\widetilde{P} \cdot R = 0$, where R denotes the curvature tensor of M, then M is pseudosymmetric.

Date: Received: October 2, 2013; Revised: January 20, 2014; Accepted: February 18, 2014.
2010 *Mathematics Subject Classification.* 53B20, 53C25, 53B30, 53C40.
Key words and phrases. Pseudo projective curvature tensor, pseudosymmetric manifold, Ricci-semisymmetric manifold.

The paper is organized as follows: In Section 2, we give a brief account of pseudo projective curvature tensor, pseudosymmetric manifolds and Kulkarni-Nomizu product. In Section 3, we give some information about hypersurfaces of semi-Euclidean space E_s^{n+1} and the main results of the study are presented.

2. **Preliminaries**

We denote by ∇, R, $\widetilde{P}$, S and κ are the Levi-Civita connection, the Riemannian-Christoffel curvature tensor, the pseudo projective curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively. Next, we define the endomorphisms $\mathcal{R}(X,Y)$ and $\widetilde{P}(X,Y)$ of $\chi(M)$ by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]\, Z - \nabla_{[X,Y]} Z$$

and

$$\widetilde{P}(X,Y)Z = a\mathcal{R}(X,Y)Z + b\,(X \wedge_S Y)\, Z \text{-} \frac{\kappa}{n}\left[\frac{a}{n-1} + b\right](X \wedge Y)\, Z, \tag{2.1}$$

respectively, where $(X \wedge Y)\,Z$ is the tensor, defined by

$$(X \wedge Y)\, Z = g(Y,Z)X - g(X,Z)Y,$$

and $X, Y, Z \in \chi(M)$.

The Riemannian-Christoffel curvature tensor R and the pseudo projective curvature tensor $\widetilde{P}$ are defined by

$$\begin{aligned} R\,(X,Y,Z,W) &= g(\mathcal{R}\,(X,Y)Z, W)\,, \\ \widetilde{P}\,(X,Y,Z,W) &= g(\widetilde{\mathcal{P}}\,(X,Y)Z, W)\,, \end{aligned}$$

respectively, where $W \in \chi(M)$. The (0,4)-tensor G is defined by $G\,(X,Y,Z,W) = g((X \wedge Y)Z, W)$.

For a $(0,k)$-tensor field T, $k \geq 1$, and a $(0,2)$-tensor field E on (M,g) we define the tensors $R \cdot T$, $\widetilde{P} \cdot T$, and $Q(E,T)$ by

$$\begin{aligned} (R(X,Y)\cdot T)(X_1,...,X_k) &= -T(\mathcal{R}(X,Y)X_1, X_2,...,X_k) \\ &\quad -... - T(X_1,...,X_{k-1}, \mathcal{R}(X,Y)X_k). \end{aligned} \tag{2.2}$$

$$\begin{aligned} (\widetilde{P}(X,Y)\cdot T)(X_1,...,X_k) &= -T(\widetilde{\mathcal{P}}(X,Y)X_1, X_2,...,X_k) \\ &\quad -... - T(X_1,...,X_{k-1}, \widetilde{\mathcal{P}}(X,Y)X_k). \end{aligned} \tag{2.3}$$

$$\begin{aligned} Q(E,T)(X_1,...,X_k;X,Y) &= -T((X \wedge_E Y)X_1, X_2,...,X_k) \\ &\quad -... - T(X_1,...,X_{k-1}, (X \wedge_E Y)X_k), \end{aligned} \tag{2.4}$$

respectively, where the tensor $X \wedge_E Y$ is defined by

$$(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y.$$

If $E = g$ then we simply denote it by $X \wedge Y$.

If the tensor $R \cdot R$ and $Q(g,R)$ are linearly dependent, then M is called *pseudosymmetric.* This is equivalent to

$$R \cdot R = L_R Q(g,R) \tag{2.5}$$

holding on the set $U_R = \{x \in M^n |\, Q(g,R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R(see[5]). If $R \cdot R = 0$, then M is called semi-symmetric (see[12]).

If the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent, then M is called *Ricci-pseudosymmetric*. This is equivalent to

$$R \cdot S = L_S Q(g, S) \tag{2.6}$$

holding on the set $U_S = \{x \in M^n \,|\, S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, where L_S is some function on U_S (see [5]).

The Kulkarni-Nomizu product $E \,\widetilde{\wedge}\, B$ is given by

$$\begin{aligned}(E\widetilde{\wedge}B)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)B(X_2, X_3) + E(X_2, X_3)B(X_1, X_4)\\ &\quad -E(X_1, X_3)B(X_2, X_4) - E(X_2, X_4)B(X_1, X_3).\end{aligned} \tag{2.7}$$

We note that if $E = B$, then we have $\bar{E} = \frac{1}{2} E \,\widetilde{\wedge}\, E$, where the (0,4)-tensor $\bar{E}$ is defined by

$$\bar{E}(X_1, X_2, X_3, X_4) = E(X_1, X_4)E(X_2, X_3) - E(X_1, X_3)E(X_2, X_4).$$

Further, for a symmetric $(0, 2)$-tensor E and a $(0, k)$-tensor T, $k \geq 2$, we define their Kulkarni-Nomizu product $E \,\widetilde{\wedge}\, T$ by

$$\begin{aligned}(E\widetilde{\wedge}T)(X_1, X_2, X_3, X_4; Y_3, ..., Y_k) &= E(X_1, X_4)T(X_2, X_3; Y_3, ..., Y_k)\\ &\quad +E(X_2, X_3)T(X_1, X_4; Y_3, ..., Y_k)\\ &\quad -E(X_1, X_3)T(X_2, X_4; Y_3, ..., Y_k)\\ &\quad -E(X_2, X_4)T(X_1, X_3; Y_3, ..., Y_k)\end{aligned} \tag{2.8}$$

(see [4]). For symmetric $(0, 2)$-tensor field E and B, we have the following identity ([4]):

$$E\widetilde{\wedge}Q(B, E) = Q(B, \bar{E}). \tag{2.9}$$

Note that

$$\bar{g} = G. \tag{2.10}$$

3. **Hypersurfaces**

Let M, $n = dimM \geq 3$, be a connected hypersurface immersed isometrically in a semi-Riemannian manifold $(N, \widetilde{g})$. We denote by g the metric tensor of M induced from the metric tensor $\widetilde{g}$. Further, we denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connections corresponding to the metric tensors $\widetilde{g}$ and g, respectively. Let ξ be a local unit vector field on M in N and let $\varepsilon = \widetilde{g}(\xi, \xi) = \pm 1$. We can present the Gauss formula and Weingarten formula of M in N in the following form:

$$\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi, \qquad \tilde{\nabla}_X \xi = -A(X)$$

respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor and A is the shape operator of M in N and $g(A(X), Y) = H(X, Y)$. Furthermore, for $k > 1$, we also have that $H^k(X, Y) = g(A^k(X), Y)$, $\mathrm{tr}(H^k) = tr(A^k)$, $k \geq 1$, $H^1 = H$ and $A^1 = A$. We denote by R and $\tilde{R}$ the Riemannian-Christoffel curvature tensors of M and N, respectively.

The Gauss equation of M in N has the following form:

$$R(X_1, X_2, X_3, X_4) = \tilde{R}(X_1, X_2, X_3, X_4) + \varepsilon \overline{H}(X_1, X_2, X_3, X_4). \tag{3.1}$$

From now on, we will assume that M is a hypersurface in a semi-Euclidean space E_s^{n+1}. So, Eq.(3.1) turns into

$$R(X_1, X_2, X_3, X_4) = \varepsilon \overline{H}(X_1, X_2, X_3, X_4), \tag{3.2}$$

where X_1, X_2, X_3, X_4 are vector fields tangent to M and $\overline{H} = \frac{1}{2} H \widetilde{\wedge} H$. From (3.2), by contraction, we get easily

$$S(X_1, X_4) = \varepsilon(\mathrm{tr}(H)H(X_1, X_4) - H^2(X_1, X_4)). \tag{3.3}$$

Moreover, by contracting (3.3), we obtain

$$\kappa = \varepsilon(\mathrm{tr}(H)^2 - \mathrm{tr}(H^2)). \tag{3.4}$$

Now we give the following Lemmas which will be used in the main results.

Lemma 3.1. [6]*Let E and D be two symmetric $(0,2)-$tensors at point x of a semi-Riemannian manifold (M, g). If the condition*

$$\alpha Q(g, E) + \gamma Q(E, D) + \beta Q(g, D) = 0; \qquad \alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0$$

is satisfied at x, then the tensors $E - \frac{1}{n} tr(E) g$ and $D - \frac{1}{n} tr(D) g$ are linearly dependent.

Lemma 3.2. [6]*Any hypersurface M, immersed isometrically in an $(n+1)$-dimensional semi-Euclidean space $E_s^{n+1}, n \geq 4$, satisfies the condition*

$$R \cdot R = Q(S, R). \tag{3.5}$$

Proposition 3.1. *Let M be a hypersurface in a semi-Euclidean space E_s^{n+1}, $n \geq 4$, then we have*

$$\widetilde{P} \cdot R = -(a+b) R \cdot R + \frac{\kappa}{n}\left[\frac{a}{n-1} + b\right] Q(g, R). \tag{3.6}$$

Proof. Let $X_h, X_i, X_j, X_k, X_l, X_m \in \chi(M)$. So using (2.3) we have

$$\begin{aligned} (\widetilde{P}(X_h, X_i) \cdot R)(X_j, X_k, X_l, X_m) &= -R(\widetilde{P}(X_h, X_i)X_j, X_k, X_l, X_m) \\ &\quad -R(X_j, \widetilde{P}(X_h, X_i)X_k, X_l, X_m) \\ &\quad -R(X_j, X_k, \widetilde{P}(X_h, X_i)X_l, X_m) \\ &\quad -R(X_j, X_k, X_l, \widetilde{P}(X_h, X_i)X_m). \end{aligned} \tag{3.7}$$

Then using (2.1), (2.4) and (2.7), we have

$$\widetilde{P} \cdot R = aH \widetilde{\wedge} Q(H^2, H) - bQ(S, R) + \frac{\kappa\varepsilon}{n}\left[\frac{a}{n-1} + b\right](H \widetilde{\wedge} Q(g, H). \tag{3.8}$$

Thus, by (2.9), Eq. (3.8) turns into

$$\widetilde{P} \cdot R = aQ(H^2, \overline{H}) - bQ(S, R) + \frac{\kappa\varepsilon}{n}\left[\frac{a}{n-1} + b\right] Q(g, \overline{H}). \tag{3.9}$$

By using (3.3), (3.2) and Lemma 3.2, the Eq. (3.9) can be rewritten as

$$\widetilde{P} \cdot R = -(a+b) R \cdot R + \frac{\kappa}{n}\left[\frac{a}{n-1} + b\right] Q(g, R). \tag{3.10}$$

This completes the proof of the proposition. □

As an immediate consequence of Proposition 3.1, we have the following theorem:

Theorem 3.1. *Let M be a hypersurface in a semi-Euclidean space E_s^{n+1}, $n \geq 4$. If the condition $\widetilde{P} \cdot R = 0$ holds on M, then M is pseudosymmetric.*

Lemma 3.3. [3]*Let M be a hypersurface in a semi-Euclidean space E_s^{n+1}, $n \geq 3$. Then M is pseudosymmetric if and only if $R \cdot R = 0$ or the second fundamental tensor H of M satisfies the condition*

$$H^2 = \alpha H + \beta g \quad \alpha, \beta \in \mathbb{R}.$$

Definition 3.1. Let M be a hypersurface in a semi-Euclidean space E_s^{n+1},n≥ 4. If $\widetilde{P} \cdot S = 0$, then M is called pseudo projectively Ricci-semisymmetric.

Lemma 3.4. *Let M be a hypersurface in a semi-Euclidean space $E_s^{n+1}, n \geq 4$. If M is pseudo projectively Ricci-semisymmetric, then there is a real valued function λ on M such that*

$$H^2 = \lambda H + \frac{1}{n}(tr(H^2) - \lambda tr(H))g. \tag{3.11}$$

Proof. Let $X_h, X_i, X_j, X_k \in \chi(M)$. So, by using (2.3), we have

$$(\widetilde{P} \cdot H)(X_h, X_i; X_j, X_k) = -H(\widetilde{P}(X_j, X_k)X_h, X_i) - H(X_h, \widetilde{P}(X_j, X_k)X_i) \tag{3.12}$$

and, similarly,

$$(\widetilde{P} \cdot H^2)(X_h, X_i; X_j, X_k) = -H^2(\widetilde{P}(X_j, X_k)X_h, X_i) - H^2(X_h, \widetilde{P}(X_j, X_k)X_i). \tag{3.13}$$

Then, by making use of (2.1), (2.4) and (3.2), we get

$$\widetilde{P} \cdot H = (a+b)\varepsilon Q(H, H^2) - \frac{\kappa}{n}\left[\frac{a}{n-1} + b\right] Q(g, H) \tag{3.14}$$

and

$$\begin{aligned} \widetilde{P} \cdot H^2 &= a\varepsilon Q(H, H^3) + b\varepsilon \mathrm{tr}(H) Q(H, H^2) \\ &\quad -\frac{\kappa}{n}\left[\frac{a}{n-1} + b\right] Q(g, H^2). \end{aligned} \tag{3.15}$$

Since M is pseudo projectively Ricci-semisymmetric, by using (3.3), we have

$$\widetilde{P} \cdot S = \varepsilon(\mathrm{tr}(H)\widetilde{P} \cdot H - \widetilde{P} \cdot H^2) = 0. \tag{3.16}$$

Thus, by substituting (3.14) and (3.15) into (3.16), we obtain

$$\begin{aligned} &a\mathrm{tr}(H)Q(H, H^2) - aQ(H, H^3) - \frac{\varepsilon\kappa}{n}\left[\frac{a}{n-1} + b\right] \mathrm{tr}(H) Q(g, H) \\ &+\frac{\varepsilon\kappa}{n}\left[\frac{a}{n-1} + b\right] Q(g, H^2) = 0. \end{aligned} \tag{3.17}$$

Hence, from (3.17), by a contraction, we have

$$\begin{aligned} H^3 &= \mathrm{tr}(H)H^2 + \left[-\mathrm{tr}(H^2) + \frac{\mathrm{tr}(H^3)}{\mathrm{tr}(H)} - \frac{\varepsilon\kappa}{a}(\frac{a}{n-1} + b)\right] H \\ &\quad +\frac{\varepsilon\kappa}{a\mathrm{tr}(H)}\left[\frac{a}{(n-1)} + b\right] H^2 \\ &\quad +\left[\left(\frac{\varepsilon\kappa \mathrm{tr}(H)}{a.n} - \frac{\varepsilon\kappa \mathrm{tr}(H^2)}{a.n\mathrm{tr}(H)}\right)\left[\frac{a}{(n-1)} + b\right]\right] g. \end{aligned} \tag{3.18}$$

So, by substituting (3.18) into (3.17), we get

$$-\frac{\varepsilon\kappa}{\mathrm{tr}(H)}\left[\frac{a}{(n-1)}+b\right]Q(H,H^2)-\frac{\varepsilon\kappa.\mathrm{tr}(H^2)}{n.\mathrm{tr}(H)}\left[\frac{a}{(n-1)}+b\right]Q(g,H)$$
$$+\frac{\varepsilon\kappa}{n}\left[\frac{a}{(n-1)}+b\right]Q(g,H^2)=0 \tag{3.19}$$

Then, by Lemma 3.1, the tensors

$$H^2-\frac{1}{n}\mathrm{tr}(H^2)g$$

and

$$H-\frac{1}{n}\mathrm{tr}(H)g$$

are linearly dependent, which proves the lemma. □

Hence, by combining Lemma 3.3 and Lemma 3.4, we have the following theorem:

Theorem 3.2. *Let M be a hypersurface in a semi-Euclidean space E_s^{n+1}, $n\geq 4$. If M is pseudo projectively Ricci-semisymmetric, then M is pseudosymmetric.*

Theorem 3.3. *Let M be a hypersurface in a semi-Euclidean space E_s^{n+1}, $n\geq 4$. If M is pseudo projectively Ricci-semisymmetric, then M is Ricci-pseudosymmetric.*

Proof. By using (2.1), (2.3)and (2.4), we have

$$(\widetilde{P}\cdot S)=a(R\cdot S)-\frac{\kappa}{n}\left[\frac{a}{n-1}+b\right]Q(g,S). \tag{3.20}$$

Since the condition $\widetilde{P}\cdot S=0$ holds on M, we get

$$R\cdot S=\frac{\kappa}{n}\left[\frac{1}{n-1}+\frac{b}{a}\right]Q(g,S).$$

This completes the proof of the theorem. □

Lemma 3.5. [2]*Let M be a hypersurface in a semi-Euclidean space $E_s^{n+1}, n\geq 4$. M satisfies the condition*

$$R\cdot S=Q(H,tr(H)H^2-H^3). \tag{3.21}$$

Theorem 3.4. *Let M be a hypersurface in a semi-Euclidean space E_s^{n+1}, $n\geq 4$. If M the condition $H^2=$tr$(H)H$ holds on M, then M is pseudo projectively Ricci-semisymmetric.*

Proof. Since $H^2=\mathrm{tr}(H)H$ and $H^k(X,Y)=g(A^k(X),Y)$, we have

$$H^3=tr(H)H^2. \tag{3.22}$$

So, by substituting (3.22) into (3.21), we get $R\cdot S=0$. Thus, Eq.(3.20) turns into

$$(\widetilde{P}\cdot S)=-\frac{\kappa}{n}\left[\frac{a}{n-1}+b\right]Q(g,S).$$

Since $H^2=\mathrm{tr}(H)H$, by using (3.22) and (3.3), we get $Q(g,S)=0$. In the proof of this theorem which proves that M is pseudo projectively Ricci-semisymmetric. □

Example 3.1. Let $\mathbf{S}^2 = \left\{p \in \mathbb{R}^3 \text{ such that } |p| = 1\right\}$ be the standard unit sphere. First we consider

$$M^4 = \mathbf{S}_1^2 \times \mathbf{S}_2^2 = \left\{(p,q) \in \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 \text{ such that } |p| = |q| = 1\right\}.$$

Next we take the cone

$$\mathbf{C}^5 = \left\{(tp, tq) \in \mathbb{R}^6 \text{ such that } |p| = |q| = 1, \quad t > 0, \ \mathrm{t} \in \mathbb{R}\right\}.$$

In [1], the authors show that the principal curvatures of $\mathbf{C}^5$ are $\left(0, \frac{1}{\sqrt{2}t}, \frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t}\right)$ and the cone $\mathbf{C}^5$ is Ricci-semisymmetric, but not semi-symmetric. It can be easily seen that the cone $\mathbf{C}^5$ satisfies the condition $\widetilde{P} \cdot S = 0$ and it is pseudosymmetric.

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Isiklar Air Force High School. Bursa-TURKEY

E-mail address: yusufdogru1@yahoo.com.tr