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Construction of Networks by Associating with Submanifolds of Almost Hermitian Manifolds

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Abstract

The idea that data lies in a non-linear space has brought up the concept of manifold learning as a part of machine learning and such notion is one of the most important research fields of today. The main idea here is to design the data as a submanifold model embedded in a high-dimensional manifold. On the other hand, graph theory is one of the most important research areas of applied mathematics and computer science. As a result, many researchers investigate new methods for machine learning on graphs. From the above information, it is seen that the theory of submanifolds and graph theory have become two important concepts in machine learning and nowadays, the geometric deep learning research area using these two concepts has emerged. By combining these two fields, this article aims to present the relationships between submanifolds of complex manifolds with the help of graphs. In this paper, we build some directed networks by identifying with submanifolds of almost Hermitian manifolds. Moreover, we give some results and relations among holomorphic submanifolds, totally real submanifolds, CR-submanifolds, slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds, and bi-slant submanifolds in almost Hermitian manifolds in terms of graph theory.

1. Introduction

Graph theory can be used to model computer networks, social networks, communications networks, information networks, software design, transportation networks, biological networks, etc. So this theory is applicable in many real-world mathematical modelling. Therefore, this theory is the most active area of mathematical research.

On the other hand, one of the most active research areas of differential geometry is the submanifold theory of complex manifolds. A submanifold of an almost Hermitian manifold is characterized by the behavior of tangent space of the submanifold of almost Hermitian manifold under the complex structure of the ambient manifold. In this way, we have various submanifolds titled as holomorphic, totally real, CR, slant, semi-slant, hemi-slant, bi-slant for almost Hermitian manifolds. In fact, the theory of submanifolds of almost Hermitian manifolds is still the main active area of complex differential geometry, see: [1]-[8] for recent results.

Manifold learning method is one of the most exciting developments in machine learning recently. Manifold learning has been applied in utilizing semi-supervised learning [9]. Moreover, manifolds also play an important role in public health. Fiorini has defined the Riemannian manifold, which is isomorphic to traditional information geometry Riemannian manifold, for noise reduction in theoretical computerized tomography providing many competitive computational advantages over the traditional Euclidean approach [10]. Besides, Monti et al. have introduced a general framework, geometric deep learning, enabling the design of convolutional deep architectures on manifolds and graphs [11]. Moreover, Shahzad et al. have simplified the complex chemical reaction by reducing it from a high dimension to the low by applying three well-established techniques based on

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manifolds [12], and they have investigated the different completion routes of reaction and overall reaction for dehydrogenation of butane to further extend towards the surfaces using the slow invariant manifold comparison [13].
Also, Carriazo and Fernandez [14] have constructed a relation between slant surface and graph theory. Later, they have related graph theory with vector spaces of even dimension [15, 16]. Their work was restricted to slant submanifolds. We believe that further use of graph theory is possible in the theory of submanifolds.
By considering vast literature of graph theory and submanifold theory, one expects more relations between these research areas. In this direction, the aim of this paper is to examine the relation among various submanifolds of almost Hermitian manifolds by using graph theory. We note that our approach is different from the approach considered in [14] and [16]. They only considered adapted frames of slant surface and they used them to characterize CR-submanifolds by means of trees. Later they have extended this approach for weakly associated graphs. In this paper, we give relations between submanifolds of Hermitian manifolds in terms of graph theory notions.

2. Preliminaries

In this section, we are going to recall certain notions used in graph theory to be used in this paper from [17]-[25]. For those who are not familiar with the theory of graphs (especially for readers working with the submanifolds theory), we specifically recall the basic definitions from graph theory.
A graph $G=(V,E)$ consists of a nonempty set V of vertices and a set E of edges. Each edge has either one or two vertices connected with it, called its endpoints. An edge connects its endpoints. Two distinct vertices u,v in a graph G are called adjacent (or neighbors) in G if there is an edge e between u and v, where the edge e is called incident with the vertices u and v and e connects u and v. The set of all neighbors of a vertex v of $G=(V,E)$ is denoted by $N(v)$. If $A\subset V$, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. The degree of a vertex in a graph is the number of edges incident with it. The degree of the vertex v is denoted by $d(v)$ and $d(v)=|N(v)|$. The graph theory can be divided into two branches as undirected and directed graphs [24].
A directed graph (digraph) D is a finite nonempty set of objects called vertices together with a set of ordered pairs of distinct vertices of D called directed edges or arcs. For a digraph $D=(V,A)$, the vertex set of D is denoted by $V(D)$ or simply V and the arc set of D is denoted by $A(D)$ or A. Each arc is an ordered pair of vertices. The arc (u,v) is said to start at u and end at v. The in-degree of a vertex v, $d^-(v)$, is the number of edges which end at v. The out-degree of v, $d^+(v)$, is the number of edges with v as their initial vertex. Also, for a vertex $v\in V(D)$, $N_D^-(v)$ and $N_D^+(v)$ are respectively called out-neighbors and in-neighbors where $N_D^-(v)=\{u|(u,v)\in A(D),u\in V(D)\}$ and $N_D^+(v)=\{u|(v,u)\in A(D),u\in V(D)\}$ [19, 22, 24, 26].
In a digraph $D=(V,A)$, given a pair of vertices u and v, whether or not there is a path from u to v in the digraph is useful to know. The transitive closure of D is to construct a new digraph, $D^*=(V,A^*)$, such that there is an arc (u,v) in D^* if and only if there is a path from u to v in D [23].
A walk $W=x_1a_1x_2a_2x_3...x_{k-1}a_{k-1}x_k$ is a sequence of vertices x_i and arcs a_j in D such that the tail and head of a_i is x_i and x_{i+1} for $\forall i<k$, respectively. The set of vertices and arcs of the walk W are denoted $V(W)$ and $A(W)$, respectively. W is denoted without arcs as $x_1x_2...x_k$ and shortly (x_1,x_k)-walk. If $x_1=x_k$ then W is a closed walk, and otherwise w is an open walk. If W is an open walk, the vertices x_1 and x_k are end-vertices and named as the initial and the terminal vertex of W, respectively. The length of a walk is the number of its arcs and the walk W above has length $k-1$ [19].
A trail is a walk in which all arcs are distinct. W is called a path, if the vertices of a trail $V(W)\subset V(D)$ are distinct. If the vertices $x_1,x_2,...,x_{k-1}$ are distinct, $k\geq 3$ and $x_1=x_k$, then W is a cycle. The longest path in D is a path of maximum length in D [19].

Proposition 2.1. *[19] Let D be a digraph and let x,y be a pair of distinct vertices in D. If D has an (x,y)-walk W, then D contains an (x,y)-path P such that $A(P)\subseteq A(W)$. If D has a closed $(x,x)-$walk W, then D contains a cycle C through x such that $A(C)\subseteq A(W)$.*

An oriented graph is a digraph with no cycle of length two [19]. For a digraph D, the underlying graph of D is the undirected graph engendered utilizing all vertices in $V(D)$, and superseding all of the arcs in $A(D)$ with undirected edges [21].
If a digraph D has an (x,y)-walk, then the vertex y is reachable from the vertex x. Every vertex is reachable from itself specifically. By Proposition 2.1, y is reachable from x if and only if D contains an (x,y)-path. If every pair of vertices in digraph D is mutually reachable then D is strongly connected (or shortly strong). A strong component of digraph D is a maximal induced strong subdigraph in D. If $D_1,...,D_t$ are the strong components of D, then precisely $V(D_1)\cup...\cup V(D_t)=V(D)$. If a digraph D is not strongly connected and if the underlying graph of D is connected, then D is said to be weakly connected [19, 26].
Pseudograph is a graph having parallel edges and loops, and multigraph is a pseudograph with no loops. If every pair of distinct vertices are adjacent in a multigraph then the multigraph is complete.
A multigraph H is called as $p-$partite if there is a partition into p sets $V(H)=V_1\cup V_2\cup...\cup V_p$ where $V_i\cap V_j=\emptyset$ for every $i\neq j$. In particular, when $p=2$ the graph is called a bipartite graph. A bipartite graph B is denoted by $B=(V_1,V_2;E)$. If the edge (x,y) is in $p-$partite multigraph H where all $x\in V_i$, $y\in V_j$ for $i\neq j$ then H is complete $p-$partite [19].
A digraph $D=(V,A)$ is symmetric if arc $(x,y)\in A$ implies arc $(y,x)\in A$. A matching M is an arc set having no common end-vertices and loops in D. Also, the arcs of M are independent if M is a matching. If a matching M implicates the highest number of arcs in D, then M is maximum. Besides, a maximum matching is perfect if it has $\frac{|A(D)|}{2}$ arcs. A set Q of vertices in

a directed pseudograph H is independent if there are no arcs between vertices in Q. The independence number of H is the size of the independent set having maximum cardinality in H. A coloring of a digraph H is a partition of $V(H)$ into disjoint independent sets. The minimum number of independent sets in the coloring of H is the chromatic number of H. A simple directed graph is a digraph that has no multiple arcs or loops. If a digraph contains no cycle, then it is acyclic and called acyclic digraph [19].

The eccentricity $e(v)$ of a vertex v is the distance from v to the farthest vertex from itself. The radius (rad) of D is the minimum eccentricity, and the diameter ($diam$) is the maximum eccentricity. Besides, a vertex v is central if $e(v) = rad(D)$, and v is peripheral if $e(v) = diam(D)$ [27].

Let $D = (V,A)$ be a digraph, $V(D) = n$ and $S \subset V(D)$. S is a dominating set of D if each vertex $v \in V(D) - S$ is dominated by at least a vertex in S. A dominating set of D having the smallest cardinality is called the minimum dominating set of D. Also, the cardinality of the minimum dominating set is called the domination number of D [28, 29].

Let r be a root vertex in D. A directed spanning tree T starting from r is a subdigraph of D such that the undirected form of T is a tree and there is a directed unique (r,v)-path in T for each $v \in V(T) - r$ [19].

The vertex-integrity of a digraph D is defined by $I(D) = min\{|F| + m(D-F) : F \subseteq V(D)\}$, where $m(D-F)$ indicates the maximum order of a strong component of $D-F$. If $I(D) = |F| + m(D-F)$ then F is called as an I-set of D. In addition, the arc-integrity of a digraph D, shortly $I^{'}(D)$, is described as the minimum value of $\{|F| + m(D-F) : F \subseteq A(D)\}$. The set F is called as an $I^{'}$-set of D if $I^{'}(D) = |F| + m(D-F)$ [30].

Proposition 2.2. *[30] If S is a subdigraph of D then $I(S) \leq I(D)$ and $I^{'}(S) \leq I^{'}(D)$.*

3. Networks built among submanifolds of almost Hermitian manifolds

Let (M,g) be a Riemannian manifold. (M,g) is called an almost Hermitian manifold if there is a (1,1) tensor field on M such that $J^2 = -I$, where I is the identity map on the tangent bundle of M, and $g(JX,JY) = g(X,Y)$ for vector fields X,Y on M. Moreover, if J is parallel with respect to any vector field X, then (M,J,g) is called a Kaehler manifold [31]. There are various submanifolds of an almost Hermitian manifold based on the behavior of the tangent space of the submanifold at a point under the almost complex structure J. Let N be a submanifold of an almost Hermitian manifold and T_pN the tangent space at a point p belongs to N. Then, if T_pN is invariant with respect to J_p for any point p, then N is called holomorphic (or complex) submanifold [31]. We denote the normal space at p by $T_pN^{\perp}$. A submanifold of an almost Hermitian manifold is called an anti-invariant submanifold if $JT_pN \subseteq T_pN^{\perp}$ [31]. As a generalization of holomorphic submanifold and anti-invariant submanifolds, a submanifold M of a Kaehler manifold N is called CR-submanifold [32] if there are two orthogonal complementary distributions $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $\mathscr{D}_1$ is invariant with respect to J and $\mathscr{D}_2$ is anti-invariant with respect to J for every point $p \in M$. It is clear that if $\mathscr{D}_1 = \{0\}$, then a CR-submanifold becomes an anti-invariant submanifold. If $\mathscr{D}_2 = \{0\}$, then M becomes a holomorphic submanifold. Another generalization of holomorphic submanifolds and anti-anti-invariant submanifolds is slant submanifolds. Let N be a submanifold of an almost Hermitian manifold M. The submanifold N is called slant [33] if for each non-zero vector X tangent to N the angle $\theta(X)$ between JX and T_pN is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pN$. θ is called the slant angle. It is clear that if $\theta(X) = 0$ then N becomes a holomorphic submanifold. If $\theta(X) = \pi/2$, N becomes an anti-invariant submanifold. We will use the v_1, v_2, v_3, and v_4 to represent the submanifolds holomorphic, CR, anti-invariant and slant, respectively.

Digraph $D_1 = (V,A)$ has four vertices, $V(D_1) = \{v_1,v_2,v_3,v_4\}$, and four arcs, $A(D_1) = \{(v_2,v_1),(v_2,v_3),(v_4,v_1),(v_4,v_3)\}$ in Fig. 3.1. D_1 has the maximum length of one as the longest path. D_1 has 2 vertices (v_2 and v_4) which are not reachable. Topological sort of D_1 is $v_4 - v_2 - v_3 - v_1$. $rad(D_1) = 1$, the radius of D_1 is $v_2 \rightarrow v_1$. $diam(D_1) = 1$, the diameter of D_1 is the same as the radius. Also, in D_1, there is no center vertex, but two peripheral vertices such as v_2 and v_4.

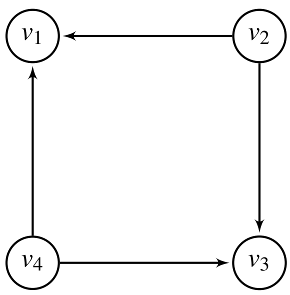

Figure 3.1: Digraph D_1 built by submanifolds holomorphic, CR, anti-invariant and slant

Theorem 3.1. *Let D_1 be a digraph constructed by the four submanifolds holomorphic, CR, anti-invariant and slant considering as the vertices v_1, v_2, v_3, and v_4, respectively. Then D_1 holds the following properties:*

1. *D_1 is a bipartite digraph as well as a complete bipartite digraph.*
2. *D_1 has a perfect matching.*

3. *The independence number of D_1 is 2.*
4. *The chromatic number of D_1 is 2.*
5. *D_1 has no directed spanning tree.*
6. *The domination number of D_1 is 2.*

Proof. 1. There exists a partition V_1 and V_2 of $V(D_1)$ into two partite sets for the submanifolds in D_1: $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$. Owing to $V(D_1) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, then D_1 is a bipartite digraph.
Besides, for every submanifold, $x \in V_1, y \in V_2$, a connection from x to y (i.e. an arc (x, y)) is in D_1. Therefore, D_1 is a complete bipartite digraph.

2. There is a matching $M = \{(v_2, v_1), (v_4, v_3)\} \subset A(D_1)$ in D_1. Each element (arc or connection between two submanifolds) in M is independent, i.e. no common vertices, and M is maximum. Also, M is perfect so that $|M| = \frac{|A(D_1)|}{2}$. It is obvious that D_1 has a perfect matching.
3. The subset $\widetilde{V} = \{v_2, v_4\} \subset V(D_1)$ is one of the independent sets having maximum cardinality and the size of maximum independent submanifolds set is 2. This also means that there is no relation between submanifolds v_2 and v_4. Then, the independence number of D_1 is 2.
4. $V_1 = \{v_2, v_4\}$ and $V_2 = \{v_1, v_3\}$ are two subsets of $V(D_1)$. $V_i (i = 1, 2)$ are all independent sets providing the minimum number of cardinality at the same time. Hence, the minimum number of independent sets of D_1 is 2. Then, the chromatic number of D_1 is 2.
5. There is no root vertex where a subdigraph T of D_1 contains a directed path from the root to any other vertex in $V(D_1)$. Then, D_1 has no directed spanning tree.
6. There is a subset $\widetilde{V} = \{v_2, v_4\} \subset V(D_1)$ that including minimum cardinality of vertices in D_1. Considering this subset, for each vertex $v \in \widetilde{V}$ and $u \in V(D_1) - \widetilde{V}$, (v, u) is an arc in D_1. The domination number is 2, because of no smaller cardinality of dominating sets in D_1.

□

Corollary 3.2. *In the submanifold network represented by D_1 in Fig. 3.1, the submanifolds, CR (v_2) and slant (v_4), cannot be derived by the other submanifolds, because the in-degree of these vertices (submanifolds) are zero in D_1, $d^-(v_2) = d^-(v_4) = 0$. In addition, whereas CR and slant subamnifolds cannot be mutually derived as between holomorphic (v_1) and anti-invariant (v_3), holomorphic and anti-invariant submanifolds can be derived separately from CR and slant from $N^-_{D_1}(v_1) = N^-_{D_1}(v_3) = \{v_2, v_4\}$.*

We now recall the notion of hemi-slant submanifolds of an almost Hermitian manifold. Let M be an almost Hermitian manifold and N a real submanifold of M. Then we say that N is a hemi-slant submanifold [34]-[37] if there exist two orthogonal distributions $\mathscr{D}^\perp$ and $\mathscr{D}^\theta$ on N such that

1. TN admits the orthogonal direct decomposition $TN = \mathscr{D}^\perp \oplus \mathscr{D}^\theta$.
2. The distribution $\mathscr{D}^\perp$ is an anti-invariant distribution, i.e., $J\mathscr{D}^\perp \subset TM^\perp$.
3. The distribution $\mathscr{D}^\theta$ is slant with slant angle θ.

It is easy to see that if $\mathscr{D}^\perp = \{0\}$, N becomes a slant submanifold with a slant angle θ. If $\mathscr{D}^\theta = \{0\}$, then N becomes an anti-invariant submanifold. Moreover if $\theta = 0$, then N becomes a CR-submanifold. Furthermore, if $\mathscr{D}^\perp = \{0\}$ and $\theta = 0$, then N becomes a holomorphic submanifold. We denote hemi-slant submanifolds by v_6.
Digraph $D_2 = (V, A)$ is an extension of D_1, and has five vertices, $V(D_2) = \{v_1, v_2, v_3, v_4, v_6\}$, and seven arcs, $A(D_2) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_6, v_1), (v_6, v_2), (v_6, v_3)\}$ in Fig. 3.2. D_2 has the maximum length of two as the longest path. It has 2 vertices (v_4 and v_6) which are not reachable. Topological sort of D_2 is $v_6 - v_4 - v_2 - v_3 - v_1$. $rad(D_2) = 1$, the radius of D_2 is $v_2 \to v_1$. $diam(D_2) = 1$, the diameter of D_2 is the same as the radius. Also, there is no center vertex but three peripheral vertices such as v_2, v_4 and v_6.

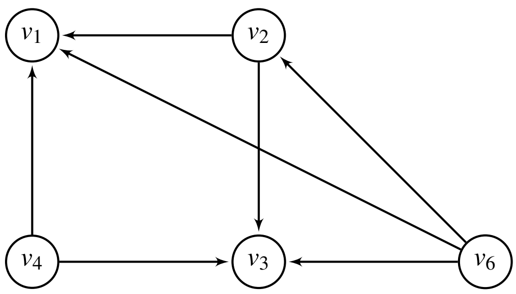

Figure 3.2: Digraph D_2 built by submanifolds in D_1 and the hemi-slant submanifold

Theorem 3.3. *Let D_2 be a digraph built by adding the hemi-slant submanifolds as vertex v_6 to the D_1. Then, D_2 satisfies the following properties:*

1. *D_2 is a three-partite digraph.*
2. *The maximum matching is 2.*
3. *The independence number is 2.*
4. *The chromatic number is 3.*
5. *D_2 has no directed spanning tree.*
6. *The domination number is 2.*

Proof.

1. There exists a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2\}$ and $V_3 = \{v_4, v_6\}$ of $V(D_2)$. These three subsets are three partite sets because of following attributes: $V(D_2) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). Then, D_2 is a three-partite digraph.
2. There is an arc subset $M = \{(v_6, v_1), (v_4, v_3)\}$ in D_2, and $|M| = 2$. In M, there is no common vertices and loops, that is M is a matching. Also, there is no arc subset having greater cardinality than M. Therefore, M is maximum matching in D_2.
3. The maximum independent set and independence number of D_2 is the same as D_1. See Theorem 3.1-iii.
4. The minimum number of disjoint independent sets of D_2 is three: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2\}$ and $V_3 = \{v_4, v_6\}$. Then, chromatic number of D_2 is 3.
5. No root vertex that contains a directed path from the root to any other vertex in $V(D_2)$. Then, D_2 has no directed spanning tree.
6. There is a subset $\widetilde{V} = \{v_4, v_6\} \subset V(D_2)$. Considering this subset, that including the minimum cardinality of vertices in D_2 as a dominating set, for each vertex $v \in \widetilde{V}$ and $u \in V(D_2) - \widetilde{V}$, (v, u) is an arc in D_2. Clearly, the domination number is 2.

□

Corollary 3.4. *In the submanifold network represented by D_2 in Fig. 3.2, the submanifolds, slant (v_4) and hemi-slant (v_6), cannot be derived by the other submanifolds, because $d^-(v_4) = d^-(v_6) = 0$ in D_2. Also, holomorphic (v_1) and anti-invariant (v_3) submanifolds can be derived separately by CR (v_2), slant and hemi-slant since $N^-_{D_2}(v_1) = N^-_{D_2}(v_3) = \{v_2, v_4, v_6\}$.*

To remind the notion of semi-slant submanifolds of an almost Hermitian manifold, let M be an almost Hermitian manifold and N a real submanifold of M. Then we say that N is a semi-slant submanifold [38] if there exist two orthogonal distributions $\mathscr{D}$ and $\mathscr{D}^\theta$ on N such that

1. TN admits the orthogonal direct decomposition $TN = \mathscr{D} \oplus \mathscr{D}^\theta$.
2. The distribution $\mathscr{D}$ is an invariant distribution, i.e., $J(\mathscr{D}) = \mathscr{D}$.
3. The distribution $\mathscr{D}^\theta$ is slant with slant angle θ.

It is easy to see that if $\mathscr{D} = \{0\}$, N becomes a slant submanifold with a slant angle θ. If $\mathscr{D}^\theta = \{0\}$, then N becomes a holomorphic submanifold. Moreover if $\theta = \frac{\pi}{2}$, then N becomes a CR-submanifold. Furthermore, if $\mathscr{D} = \{0\}$ and $\theta = \frac{\pi}{2}$, then N becomes an anti-invariant submanifold. We denote semi-slant submanifolds by v_5.
Digraph $D_3 = (V, A)$ is another extension of D_1, and has five vertices, $V(D_3) = \{v_1, v_2, v_3, v_4, v_5\}$, and seven arcs, $A(D_3) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_2), (v_5, v_3), (v_5, v_4)\}$ in Fig. 3.3. D_3 has the maximum length of two as the longest path. It has a vertex (v_5) which is not reachable. Using transitive closure, D_3 has only one new direct connection such as $v_5 \to v_1$. Topological sort of D_3 is $v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_3) = 1$, the radius of D_3 is $v_2 \to v_1$. $diam(D_3) = 2$, the diameter of D_3 is $v_5 \to v_2 \to v_1$. Also, in D_3, there are two center vertices as v_2 and v_4, and one peripheral vertex as v_5.

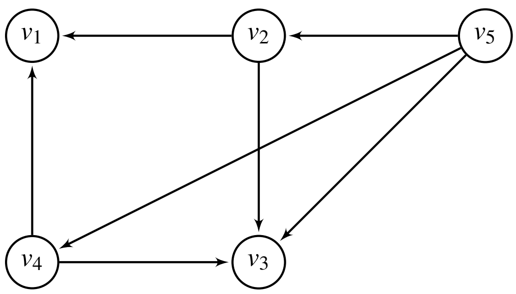

Figure 3.3: Digraph D_3 built by submanifolds in D_1 and the semi-slant submanifold

Theorem 3.5. *Let D_3 be a digraph created by adding the semi-slant submanifolds as vertex v_5 to the D_1. Then, D_3 holds the followings:*

1. *D_3 is a three-partite digraph.*
2. *The maximum matching is 2.*

3. *The independence number is 2.*
4. *The chromatic number is 3.*
5. *D_3 has a directed spanning tree.*
6. *The domination number is 2.*

Proof. 1. There exists a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5\}$ of $V(D_3)$ as three partite sets in D_3, and the subsets provide following properties: $V(D_3) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). In that case, D_3 is a three-partite digraph.

2. There is an arc subset $M = \{(v_2, v_1), (v_4, v_3)\}$ in D_3, and $|M| = 2$. Because of no common vertices and no loops in M, M is a matching. Furthermore, M has the maximum cardinality so that M is the maximum matching in D_3.
3. The maximum independent set and independence number of D_3 is the same as D_1. See Theorem 3.1-iii.
4. The minimum number of disjoint independent sets of D_3 is 3: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5\}$. It follows that the chromatic number of D_3 is 3.
5. D_3 has a unique directed spanning tree of length 4 and rooted at v_5 such as in Fig. 3.4. It also means that there is a transformation from submanifolds v_5 to all other submanifolds in D_3.

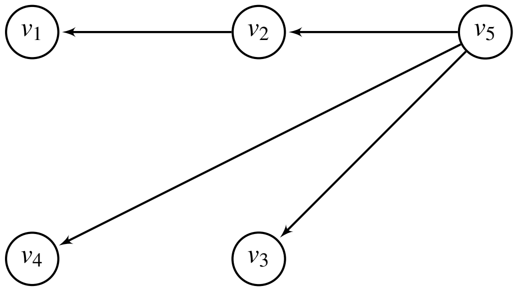

Figure 3.4: Directed spanning tree in D_3

6. There is a subset $\widetilde{V} = \{v_4, v_5\} \subset V(D_3)$. According to this subset, that having the minimum cardinality, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_3) - \widetilde{V}$, (v, u) is an arc in D_3, the domination number is 2.

□

Corollary 3.6. *In the submanifold network represented by D_3 in Fig. 3.3, while no submanifolds can be transformed to semi-slant (v_5) submanifold since $N^-_{D_3}(v_5) = \emptyset$, all other submanifolds (holomorphic (v_1), CR (v_2), anti-invariant (v_3) and slant (v_4)) can be obtained from semi-slant submanifold because of existence of a directed spanning tree with a root vertex v_5 (Fig. 3.4).*

Digraph $D_4 = (V, A)$ has six vertices, $V(D_4) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and 10 arcs, $A(D_4) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_2), (v_5, v_3), (v_5, v_4), (v_6, v_1), (v_6, v_2), (v_6, v_3)\}$ in Fig. 3.5. D_4 has the maximum length of two as the longest path. It has 2 vertices (v_5 and v_6) which are not reachable. Using transitive closure, D_4 has only one new direct connection such as $v_5 \rightarrow v_1$. The topological sort of D_4 is $v_6 - v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_4) = 1$, the radius of D_4 is $v_2 \rightarrow v_1$. $diam(D_4) = 2$, the diameter of D_4 is $v_5 \rightarrow v_2 \rightarrow v_1$. Also, in D_4, there are three center vertices as v_2, v_4 and v_6, and one peripheral vertex as v_5.

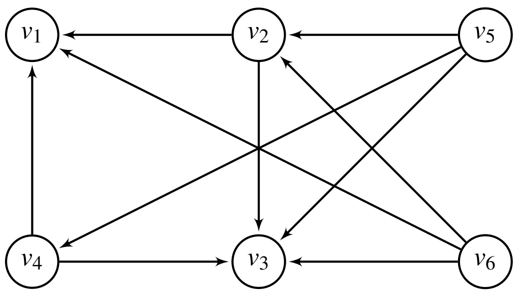

Figure 3.5: Digraph D_4 built by submanifolds in D_3 and the hemi-slant submanifold

Theorem 3.7. *Let D_4 be a digraph obtained by adding the hemi-slant submanifolds as vertex v_6 to the D_3. Then, D_4 provides the following properties:*

1. *D_4 is a three-partite digraph.*
2. *D_4 has a perfect matching.*
3. *The independence number is 2.*

4. *The chromatic number is 3.*
5. *D_4 has no directed spanning tree.*
6. *The domination number is 2.*

Proof. 1. There exists a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5, v_6\}$ of $V(D_4)$ as three subsets, and these subsets provide that $V(D_4) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). Under these conditions, D_4 is a three-partite digraph.

2. There is an arc subset $M = \{(v_2, v_1), (v_5, v_4), (v_6, v_3)\}$ in D_4, and $|M| = 3$. On conditions that no common vertices and no loops in M and $|M| = \frac{|A(D_4)|}{2}$, M is perfect matching that's why D_4 has a matching also perfect.
3. The maximum independent set and the independence number of D_4 is the same as D_1. See Theorem 3.1-iii.
4. The minimum number of disjoint independent sets of D_4 is 3: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5, v_6\}$. Then, the chromatic number of D_4 is 3.
5. No root vertex that contains a directed path from the root to any other vertex in $V(D_4)$. Then, D_4 has no directed spanning tree.
6. There is a subset $\widetilde{V} = \{v_5, v_6\} \subset V(D_4)$. According to this subset, that having the minimum cardinality, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_4) - \widetilde{V}$, (v, u) is an arc in D_4 so that the domination number is 2.

□

Corollary 3.8. *In the submanifold network represented by D_4 in Fig. 3.5, semi-slant (v_5) and hemi-slant (v_6) submanifolds cannot be obtained by any other submanifolds because $d^-(v_5) = d^-(v_6) = 0$. Besides, no submanifolds can be derived from holomorphic (v_1) and anti-invariant (v_3) submanifolds since $N^-_{D_4}(v_1) = N^-_{D_4}(v_3) = \emptyset$.*

Let M be an almost Hermitian manifold and N a real submanifold of M. Then we say that N is a bi-slant submanifold [34] if there exist two orthogonal distributions $\mathscr{D}^{\theta_1}$ and $\mathscr{D}^{\theta_2}$ on N such that

1. TN admits the orthogonal direct decomposition $TN = \mathscr{D}^{\theta_1} \oplus \mathscr{D}^{\theta_2}$.
2. The distributions $\mathscr{D}^{\theta_1}$ and $\mathscr{D}^{\theta_2}$ are slant distributions with slant angles θ_1 and θ_2.

It is easy to see that if $\mathscr{D}^{\theta_1} = \{0\}$ (or $\mathscr{D}^{\theta_2} = \{0\}$), N becomes a slant submanifold with a slant angle θ_1. If $\theta = \theta_1 = \theta_2 = \{0\}$, then N becomes a holomorphic submanifold. If $\theta = \theta_1 = \theta_2 = \frac{\pi}{2}$, then N becomes an anti-invariant submanifold. Moreover, if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, then N becomes a CR-submanifold. Furthermore, $\theta_1 = \frac{\pi}{2}$ and $\theta_1 = 0$, then N becomes a hemi-slant submanifold and semi-slant submanifold, respectively. We denote bi-slant submanifolds by v_7.
Digraph $D_5 = (V, A)$ has seven vertices, $V(D_5) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, and 12 arcs, $A(D_4) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_2), (v_5, v_3), (v_5, v_4), (v_6, v_1), (v_6, v_2), (v_6, v_3), (v_7, v_5), (v_7, v_6)\}$ in Fig. 3.6. D_5 has the maximum length of three as the longest path. It has a vertex (v_7) which is not reachable. Using transitive closure, D_5 has five new direct connections such as $v_5 \to v_1$, $v_7 \to v_1$, $v_7 \to v_2$, $v_7 \to v_3$ and $v_7 \to v_4$. Topological sort of D_5 is $v_7 - v_6 - v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_5) = 1$, the radius of D_5 is $v_2 \to v_1$. $diam(D_5) = 2$, the diameter of D_5 is $v_5 \to v_2 \to v_1$. Also, in D_5, there are three center vertices as v_2, v_4 and v_6, and two peripheral vertices as v_5 and v_7.

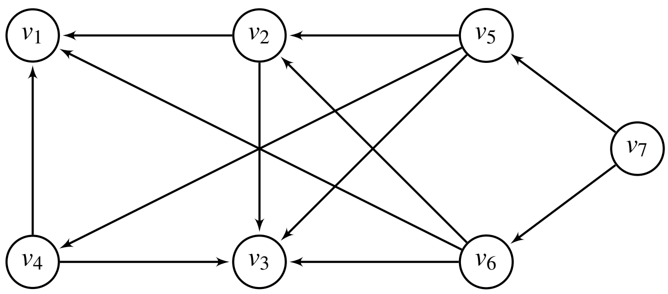

Figure 3.6: Digraph D_5 built by submanifolds in D_4 and the bi-slant submanifold

Theorem 3.9. *Let D_5 be a digraph constructed by adding the bi-slant submanifolds as vertex v_7 to the D_4. Then, D_5 holds the followings:*

1. *D_5 is a three-partite digraph.*
2. *The maximum matching is 3.*
3. *The independence number is 3.*
4. *The chromatic number is 3.*
5. *D_5 has a directed spanning tree.*
6. *The domination number is 3.*

Proof. 1. There is a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4, v_7\}$ and $V_3 = \{v_5, v_6\}$ of $V(D_5)$ as three subsets, and these subsets support that $V(D_4) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). Then, D_5, containing the subsets, is actually a three-partite digraph.

2. $M = \{(v_2, v_1), (v_5, v_4), (v_6, v_3)\}$ is an arc subset in D_5, and $|M| = 3$. According to this, M, that includes no common vertices and no loops, is a matching. Since no other subset greater cardinality than M, D_5 has a maximum matching called M.
3. The subset $\widetilde{V} = \{v_2, v_4, v_7\}$ is an independent set having maximum cardinality. It also means that there is no direct relationship between any two elements, i.e. submanifolds, in $\widetilde{V}$. Then, the independence number of D_5 is 3, because $|\widetilde{V}| = 3$.
4. The minimum number of disjoint independent sets of D_5 is 3: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4, v_7\}$ and $V_3 = \{v_5, v_6\}$. According to that, three different colors are needed to coloring D_5 and that's why the chromatic number of D_5 is 3.
5. D_5 has a directed spanning tree of length 6 and root at v_7 such as in Fig. 3.7. It also means that there is a transformation from submanifolds v_7 to all other submanifolds in D_5 at most two-step.

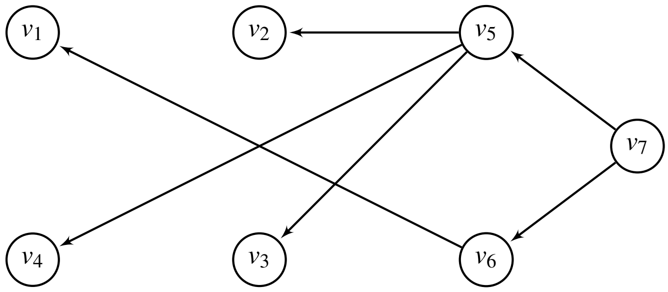

Figure 3.7: A directed spanning tree in D_5

6. There is a subset $\widetilde{V} = \{v_5, v_6, v_7\} \subset V(D_5)$. According to this subset, that having the minimum cardinality, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_5) - \widetilde{V}$, (v, u) is an arc in D_5. The domination number is 3.

□

Corollary 3.10. *In the submanifold network represented by D_5 in Fig. 3.6, all other submanifolds can be derivated from bi-slant (v_7) submanifold since v_7 is the root vertex of the directed spanning tree of D_5 and $N^+_{D_5}(v_7) = \{v_5, v_6\}$ in Fig. 3.7. Also, no submanifolds can be transformed to bi-slant because $N^-_{D_5}(v_7) = \emptyset$.*

Digraph $D_6 = (V, A)$ has also seven vertices as well as D_5, $V(D_6) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, and 12 arcs, $A(D_6) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_1), (v_5, v_2), (v_5, v_3), (v_5, v_4), (v_6, v_1), (v_6, v_2), (v_6, v_3), (v_6, v_4), (v_7, v_5), (v_7, v_6)\}$ in Fig. 3.8. D_6 has the maximum length of three as the longest path. It has a vertex (v_7) which is not reachable. Using transitive closure, D_6 has four new direct connections such as $v_7 \to v_1$, $v_7 \to v_2$, $v_7 \to v_3$ and $v_7 \to v_4$. Topological sort of D_6 is $v_7 - v_6 - v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_6) = 1$, the radius of D_6 is $v_2 \to v_1$. $diam(D_6) = 2$, the diameter of D_6 is $v_7 \to v_5 \to v_1$. Also, in D_6, there are four center vertices as v_2, v_4, v_5 and v_6, and one peripheral vertex as v_7.

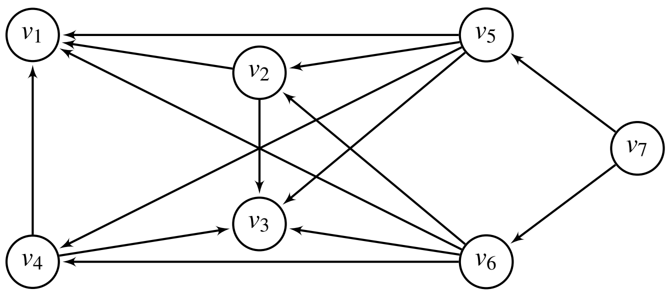

Figure 3.8: Digraph D_6 built by D_5 with arcs (v_5, v_1) and (v_6, v_4)

Theorem 3.11. *Let D_6 be a digraph created by adding two more relations from semi-slant to holomorphic and from hemi-slant to slant as arcs to the D_5. Then, D_6 satisfies the following properties:*

1. *D_6 is a three-partite digraph.*
2. *The maximum matching is 3.*
3. *The independence number is 3.*
4. *The chromatic number is 3.*
5. *D_6 contains a directed spanning tree.*

6. *The domination number is 2.*

Proof. The properties i, ii, iii and iv are clear from Theorem 3.9.

v. D_6 has a directed spanning tree having the same structure as in Fig. 3.7 (see Theorem 3.9-v).
vi. There is a subset $\widetilde{V} = \{v_5, v_7\} \subset V(D_6)$. According to that, the subset has the minimum cardinality while dominating all other vertices, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_6) - \widetilde{V}$, (v, u) is an arc in D_6. The domination number is 2.

□

Corollary 3.12. *In the most comprehensive submanifold network represented by D_6 in Fig. 3.7, just two submanifolds, holomorphic (v_1) and anti-invariant (v_3), are not generative since $d^+(v_1) = d^+(v_3) = 0$. Besides, bi-slant (v_7) is the most productive submanifold owing to transforming to all other submanifolds.*

Using the seven submanifolds, named as holomorphic, CR, anti-invariant, slant hemi-slant, semi-slant and bi-slant, it is constructed six digraphs, called D_1, D_2, D_3, D_4, D_5 and D_6, whose vertices are submanifolds and arcs are connections among submanifolds from one to another.

Theorem 3.13. *Let $D \in \{D_1, D_2, D_3, D_4, D_5, D_6\}$ be a digraph. D provides the following properties:*

1. *Simple directed graph.*
2. *Directed acyclic graph.*
3. *Weakly connected.*

Proof. 1. In digraph D, there is no more than one relationship between any two submanifolds and no transformations from a submanifold to itself. According to that, D is a simple directed graph.
2. Given a transition list among submanifolds such as $v_1 v_2 ... v_k$, meaning that v_1 is the source submanifold and v_k is the sink submanifold. Because D doesn't have any transition list including the same submanifold is both source and also sink, D is acyclic. That's why D is a directed acyclic digraph.
3. D has one pair of submanifolds as a relation at least that they can not mutually be transformed from one to another submanifold. Hence, D is not strongly connected. However, when D, that considered as without direction of transformations, is connected, named connectedness of underlying graph because there are no isolated submanifolds. For this reason, D is weakly connected.

□

Corollary 3.14. *Among all digraphs D_1, D_2, D_3, D_4, D_5, and D_6, the digraph D_6 has*

- *the maximum vertex-integrity, and*
- *the maximum edge-integrity*

as well as the maximum size by Proposition 2.2.

Example 3.15. *Let H be a directed graph having vertex set $V(H) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and arc set $A(H) = \{(u_1, u_2), (u_1, u_4), (u_1, u_5), (u_2, u_4), (u_2, u_7), (u_3, u_2), (u_3, u_4), (u_3, u_7), (u_5, u_4), (u_5, u_7), (u_6, u_3), (u_6, u_7)\}$ as modeled in Fig. 3.9. Suppose that H' indicates an induced subgraph of H when we consider as $V(H') = U$ where $U = \{u_1, u_2, u_3, u_4, u_5, u_7\}$ is a vertex subset of $V(H)$. Accordingly, we attain that H' is isomorphic to the network created by submanifolds called as holomorphic, CR, anti-invariant, slant, hemi-slant and semi-slant. Thus, H' provides the same properties as D_4. This means that H contains bounds for some parameters such as independence, domination and chromatic numbers.*

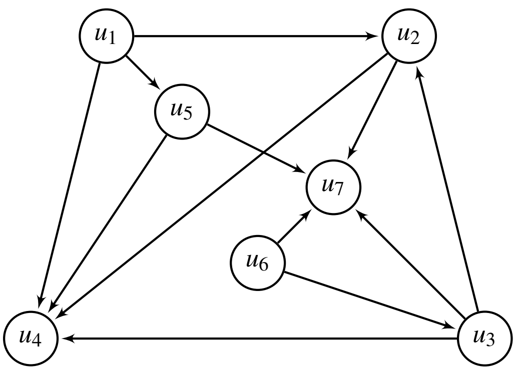

Figure 3.9: A sample network

4. Conclusion

Manifold learning plays an important role in analyzing data lying on a non-linear space as a part of machine learning. Moreover, the geometric deep learning yields using the concepts of manifolds and graphs together in building convolutional deep structures. In this paper, using holomorphic submanifolds, anti-invariant submanifolds, CR-submanifolds, slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds and bi-slant submanifolds in almost Hermitian manifolds, it is given relations among them, six different digraphs are created as a network of these submanifolds, and main properties of them are first examined in terms of digraphs. Accordingly, some directed networks by identifying with submanifolds of almost Hermitian manifolds are established. We note that there is a much wider class that includes slant submanifolds. This class was first defined in [6] by Etayo as quasi-slant submanifolds. Later, these submanifolds were called pointwise slant submanifolds in [7] by Chen and Garay. Although we have excluded such submanifolds in this article, our next research will be to examine the connections between these submanifolds and graph theory.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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