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RICCI SOLITONS AND GRADIENT RICCI SOLITONS ON NEARLY COSYMPLECTIC MANIFOLDS

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Abstract. In this article, a number of properties have been obtained by examining Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds.

1. Introduction

In 1959, Liberman [22] and in 1967 Blair [8] have described as odd-dimensional cosymplectic manifolds similar to Kähler manifolds. Later in 1970, nearly Kähler manifolds with the structure (M, J, g) have been introduced as almost Hermitian manifolds by Gray. Based on this study, almost complex structure's covariant derivative is skew symmetric operator according to the Levi-Civita connection. Also the covariant derivative operator satisfies

$$(\nabla_X J) X = 0,$$

for every vector field X on M [18]. Following year, Blair has defined an almost contact manifold with Killing structure tensors which is a nearly cosymplectic manifold [7]. Nearly cosymplectic manifolds have defined by the same viewpoint as cosymplectic (and also called coKähler) manifolds. Almost contact metric structure (φ, ξ, η, g) that provides the condition

$$(\nabla_X \varphi) X = 0, \tag{1.1}$$

is called a nearly cosymplectic structure. Also a smooth manifold M with nearly cosymplectic structure which endowed with almost contact metric structure (φ, ξ, η, g) is said to be nearly cosymplectic manifold. Recently, nearly cosymplectic manifolds have been studied by many researchers (e.g. [1], [15], [26], [27]). Ricci solitons have recently become an important research topic due to the Ricci flow on many manifolds. Firstly, the definition of Ricci soliton, Ricci solitons have been introduced by Hamilton [19] and can be obtained as a generalization of Einstein metrics. By the way on a manifold M, a Ricci flow is defined as

$$\partial g/\partial t = -2Ric(g)$$

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(we will use S instead of Ric) in the space of metrics.We can express that Ricci solitons move under the Ricci flow easily with difeomorphisms in the first metric, which are the static points of the Ricci flow. In a Riemannian manifold (M, g) admits a smooth vector field V, a Ricci soliton provides the following condition: [20]

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{1.2}$$

where S is the Ricci tensor, $\pounds$ is the Lie derivative and λ is a constant. Depends on the announcements of λ, that is $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, the Ricci soliton is said to be shrinking, steady or expanding. Ricci solitons have an important place not only in mathematics but also in physics. Theoretical physicists have studied the equations of Ricci soliton in relation to string theory. At the same time, in physics, metrics which satisfy some special conditions (1.2) are mainly practical and generally use as quasi-Einstein metrics (e.g. [17], [12]). The first contribution to these studies have come from Friedan, who has conducted research on some aspects of Ricci solitons [17]. According to this study, a Ricci soliton is said to be a gradient Ricci soliton (called the potential function), if the vector field Y can be replaced by the gradient of some smooth function f on M. Thus, the concept of gradient Ricci soliton emerged and equation (1.2) consider as the form

$$\nabla\nabla f = S + \lambda g. \tag{1.3}$$

After the study of Ricci solitons and gradient Ricci solitons on contact metric manifolds [23], Ricci solitons and gradient Ricci solitons have been studied on several manifolds. Some of those are: in 2008, Sharma [25] studied Ricci solitons in $K-$contact manifolds with the structure field ξ is killing. In [14],Ricci solitons in $P-$Sasakian manifolds have been studied by De and recently in [5], Barua and De have studied Ricci solitons in Riemannian manifolds. Subsequent studies have been on nearly contact manifolds as nearly Sasakian and nearly Kenmotsu manifolds. In 2019, Ayar and Yıldırım have studied Ricci solitons and gradient Ricci solitons on nearly Kenmotsu manifolds and they have reached important results ([3], [4]). In the light of these studies, by taking these works into consideration, we study Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds.In this study, after the intoduction section, the definition and basic curvature properties of the nearly cosymplectic manifolds are given. In the next section, according to the conditions provided by Ricci soliton and Gradient Ricci soliton, manifolds where a nearly cosymplectic manifold is locally isomorphic are processed and we have shown that if a metric of a nearly cosymplectic manifold is a Ricci soliton, then either it is an Einstein or a $\eta-$Einstein manifold. Finaly we give some important results and theorems related to this topic.

2. Nearly Cosymplectic Manifolds

In this section, first of all, let us give some information about the nearly cosymplectic manifolds that we examined on Ricci solitons and gradient Ricci solitons. Let $(M, \varphi, \xi, \eta, g)$ be an $(n = 2m + 1)-$ dimensional almost contact Riemannian manifold, an endomorphism φ of tangent bundle of $\Gamma(M)$, a vector field ξ, called structure vector field, η dual form of ξ and g is the Riemannian metric. Under the above condition almost contact structure (φ, ξ, η, g) satisfies following: [6],

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(X) = g(X,\xi), \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

for any X, Y tangent on M. Almost contact manifold is called nearly cosymplectic manifold if the equality

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0, \tag{2.4}$$

$X, Y \in \Gamma(M)$.It is known that in a nearly cosymplectic manifold with the Reeb vector field ξ is Killing and satisfies the $\nabla_\xi \xi = 0$ and $\nabla_\xi \eta = 0$ conditions. Also the tensor field h of type $(1,1)$ defined by

$$\nabla_X \xi = hX, \tag{2.5}$$

is skew symmetric and anti-commutative with φ. Also h providing $h\xi = 0$, $\eta \circ h = 0$ features and

$$(\nabla_\xi \varphi)X = \varphi h X = \frac{1}{3}(\nabla_\xi \varphi)X.$$

In a nearly cosymplectic manifolds some formulas given by ([15], [16]):

$$g((\nabla_X \varphi)Y, hZ) = \eta(Y)g(h^2X, \varphi Z) - \eta(X)g(h^2Y, \varphi Z), \tag{2.6}$$

$$(\nabla_X h)Y = g(h^2X, Y)\xi - \eta(Y)h^2X, \tag{2.7}$$

$$tr(h^2) = constant, \tag{2.8}$$

$$R(Y,Z)\xi = \eta(Y)h^2Z - \eta(Z)h^2Y, \tag{2.9}$$

$$S(\xi, Z) = -\eta(Z)tr(h^2), \tag{2.10}$$

$$S(\varphi Y, Z) = S(Y, \varphi Z), \quad \varphi Q = Q\varphi, \tag{2.11}$$

$$S(\varphi Y, \varphi Z) = S(Y,Z) + \eta(Y)\eta(Z)tr(h^2). \tag{2.12}$$

where R is Riemann curvature tensor and S is Ricci tensor.

Definition 2.1. An $n-$dimensional Riemann manifold (M,g) is said to be Einstein Manifold if the Ricci Tensor satisfies;

$$S(X,Y) = \rho g(X,Y), \tag{2.13}$$

for every $X, Y \in \chi(M)$, where $\rho : M \to R$ is a function [11].

Definition 2.2. Let M be a nearly cosymplectic manifold, for every $X, Y \in \chi(M)$, if M satisfies the condition [11].

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y), \tag{2.14}$$

then M is an $\eta-$Einstein manifold where $\alpha, \beta : M \longrightarrow R$ is a function.

Lemma 2.3. *If M a $n-$dimensional $\eta-$Einstein nearly cosymplectic manifold, the $\eta-$Einstein condition for nearly cosymplectic manifolds is characterized by the following equality* [2]*,*

$$S(X,Y) = \left\{ \frac{r + tr(h^2)}{n-1} \right\} g(X,Y) + \left\{ \frac{-n\ tr(h^2) - r}{n-1} \right\} \eta(X)\eta(Y). \tag{2.15}$$

Proposition 1. ([15])Let $(M, \varphi, \xi, \eta, g)$ be a nearly cosymplectic manifold. Then $h = 0$ if and only if M is locally isometric to the Riemannian product $\mathbb{R} \times N$, where N is a nearly Kähler manifold.

3. Ricci Solitons and Gradient Ricci Solitons on Nearly Cosymplectic Manifolds

Theorem 3.1. *In a nearly cosymplectic manifold if the metric g is a Ricci soliton and Y is point-wise collinear with ξ, then Y is a constant multiple of ξ provided $\lambda = tr(h^2)$.*

Proof. In particular, let Y be point-wise collinear with ξ i.e. $V = f\xi$ in where f is a function on the nearly cosymplectic manifold Then

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0 \tag{3.1}$$

which states that

$$g(\nabla_X f\xi, Y) + g(\nabla_Y f\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.2}$$

Using (2.5) and (2.2)

$$X(f)\eta(Y) + Y(f)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.3}$$

Putting ξ in the equation instead of Y in (3.3) we get

$$X(f) + \xi(f)\eta(X) + 2S(X, Y) + 2\lambda\eta(X) = 0,$$

$$X(f) + \xi(f)\eta(X) - 2tr(h^2) + 2\lambda\eta(X) = 0. \tag{3.4}$$

Again putting $X = \xi$ in (3.4)

$$2\xi(f) - 2tr(h^2) + 2\lambda = 0,$$

$$\xi(f) = tr(h^2) - \lambda, \tag{3.5}$$

from (3.4) and (3.5)

$$X(f) = (tr(h^2) - \lambda)\eta(X).$$

Consequently we get $\lambda = tr(h^2)$ if and only if $X(f) = 0$,under the condition $X \notin \ker(\eta)$. □

Theorem 3.2. *In a nearly cosymplectic manifold with the Ricci soliton metric g and vector field Y which is point-wise collinear with ξ, then the manifold is an Einstein and Ricci soliton depends on $tr(h^2)$ which is*

i) if $tr(h^2) < 0$ then Ricci soliton is shrinking,
ii) if $tr(h^2) = 0$ then Ricci soliton is steady,
iii) if $tr(h^2) > 0$ then Ricci soliton is expanding.

Proof. In particular, let $Y = \xi$ in (3.1), then from (3.3)

$$S(X, Y) = -\lambda g(X, Y). \tag{3.6}$$

Putting $Y = \xi$ in (3.6) ,

$$S(X, \xi) = -\lambda\eta(X), \tag{3.7}$$

from (2.10)

$$-trh^2\eta(X) = -\lambda\eta(X),$$

then we get

$$\lambda = tr(h^2).$$

Hence we get desired results. Also, using (3.6) equation takes the form

$$S(X,Y) = tr(h^2)g(X,Y),$$

that is, an Einstein manifold. □

Corollary 1. Let a nearly cosymplectic manifold with the Ricci soliton metric g and vector field Y which is point-wise collinear with ξ. If the Ricci soliton is steadily then the manifold is locally isometric to the Riemannian product $R \times N$, where N is a nearly Kähler manifold.

Proof. From Theorem 3.2 and Theorem 1 the proof is clear. □

Theorem 3.3. *An $\eta-$Einstein nearly cosymplectic manifold admits a Ricci soliton $(g, \xi, tr(h^2))$ and Ricci soliton is shrinking.*

Proof. Let M be an $\eta-$Einstein nearly cosymplectic manifold then,

$$S(X,Y) = \left\{\frac{r + tr(h^2)}{n-1}\right\} g(X,Y) + \left\{\frac{-n\ tr(h^2) - r}{n-1}\right\} \eta(X)\eta(Y). \tag{3.8}$$

Taking $Y = \xi$ in (3.1) and from (3.3) we get

$$2S(X,Y) + 2\lambda g(X,Y) = 0,$$

and

$$\left\{\frac{r + tr(h^2)}{n-1}\right\} g(X,Y) + \left\{\lambda + \frac{-n\ tr(h^2) - r}{n-1}\right\} \eta(X)\eta(Y) = 0. \tag{3.9}$$

Putting $Y = \xi$ in (3.9)

$$\frac{r + tr(h^2)}{n-1} + \lambda + \frac{-n\ tr(h^2) - r}{n-1} = 0,$$

so we have $\lambda = tr(h^2)$. Since $tr(h^2) < 0$, $\lambda < 0$ so the Ricci soliton is shrinking. □

Theorem 3.4. *If an $\eta-$Einstein nearly cosymplectic manifold admits a gradient Ricci soliton then the manifold is locally isometric to the Riemann product $\mathbb{R} \times N$,where N is a nearly Kähler manifold.*

Proof. When the vector field Y is the gradient of a potential function $-f$, then we can call g as a gradient Ricci soliton. (1.2) And here we can give the following equation,

$$\nabla\nabla f = S + \lambda g. \tag{3.10}$$

From (3.10)

$$\nabla_Y Df = QY + \lambda Y, \tag{3.11}$$

in where D symbolize the gradient operator of g. From (3.11) it is obviously that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{3.12}$$

Putting $X = \xi$ with g this implies

$$g(R(\xi,Y)Df, \xi) = g((\nabla_\xi Q)Y, \xi) - g((\nabla_Y Q)\xi, \xi). \tag{3.13}$$

Then

(3.14)
$$g(QX,Y)=S(X,Y)=\left\{\frac{r+tr(h^2)}{n-1}\right\}g(X,Y)+\left\{\frac{-n\ tr(h^2)-r}{n-1}\right\}\eta(X)\eta(Y),$$

from above equation

$$QX=\left\{\frac{r+tr(h^2)}{n-1}\right\}X+\left\{\frac{-n\ tr(h^2)-r}{n-1}\right\}\eta(X)\xi, \tag{3.15}$$

taking $\frac{r+tr(h^2)}{n-1}=A$, $\frac{-n\ tr(h^2)-r}{n-1}=f$ and also using (2.1),(2.2) and (2.5),

$$(\nabla_Y Q)X=fg(X,\nabla_Y\xi)\xi+f\eta(X)\nabla_Y\xi, \tag{3.16}$$

$$(\nabla_Y Q)X=fg(X,hY)\xi+f\eta(X)hY. \tag{3.17}$$

Putting $Y=\xi$ in (3.17) we have

$$(\nabla_\xi Q)(X)=0, \tag{3.18}$$

and putting $X=\xi$ in (3.17) we have

$$(\nabla_Y Q)\xi=fhY. \tag{3.19}$$

Furthermore from (3.13) we get

$$g(R(\xi,Y)\nabla f,\xi)=0.$$

Since $g(R(\xi,Y)\nabla f,\xi)=g(R(Df,\xi)\xi,Y)$ one can easily obtain that

$$g(R(Df,\xi)\xi,Y)=g(\eta(Df)h^2\xi-\eta(\xi)h^2Df,Y)=0,$$

so, we have

$$g(h^2Df,Y)=0.$$

Consequently we get $tr(h^2)=0$ and and it is clear from here $h=0$. So the manifold is locally isometric to the Riemann product $\mathbb{R}\times N$, where N is a nearly Kähler manifold. The proof is complete. □

4. Conclusion

Ricci solitons have an important application for many sciences such as physics and mathematical physics. Researchers have increased studies on this field from different areas in recent years. In this paper, the idea of examining Ricci solitons and gradient Ricci solitons on nearly cosymplectic manifolds is emphasized. The works on this subject will be useful tools for the applications of Ricci Solitons on different manifolds.

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