International Electronic Journal of Geometry
Volume 15 No. 1 Page 116–131 (2022)
DOI: https://doi.org/10.36890/iejg.963159

Some New Properties of Surfaces Generated by Null Cartan Curves

Ayşe Yavuz* and Melek Erdoğdu

(Communicated by Donghe Pei)

Abstract

In this paper, some special types of surfaces with null Cartan base curve are investigated. The generating lines of the surfaces are chosen as a linear combination of Cartan frame fields with non-constant differentiable functions. Firstly, the surfaces whose generating lines have the same direction of Cartan frame fields B, N and T are examined respectively. As a special case, Gaussian and mean curvatures of one parameter family of Bertrand curves of a given null Cartan curve and the singular points of this type of surface are stated. Furthermore, an example is also stated to explain the obtained results. Then, the surfaces with null Cartan base curve are investigated where generating lines lie on the planes spanned by $\{N, B\}$, $\{T, B\}$ and $\{T, N\}$, respectively. Finally, some differential geometric properties of these surface are given mainly in three different cases.

Keywords: *Cartan frame, Bertrand curve, null Cartan curve.*

AMS Subject Classification (2020): *Primary: 53A04, ; Secondary: 53A05, 53A10.*

1. Introduction

It is known that there are many different situations encountered in curves and surfaces in everyday life. These situations can be interpreted differently in different branches of science. For example, curves arises naturally in the motion of a particle in the time t for physicists. Most of these fields of science require the differential geometry of curves and surfaces. It is possible to consider the differential geometry of curves and surfaces from two different perspectives. The first is based on the foundations of calculus and is called classical differential geometry. In general, classical differential geometry is built on the local properties of curves and surfaces. What is meant by local properties is actually the behavior of the curve or the surface in a neighborhood of a point. The second one is called global differential geometry. Here, the effects of local properties on the behavior of the entire curve or surface are investigated. From both perspectives, important results can be revealed and other fields of science can be shed light on. In this study, both perspectives will take place together and the differential geometry of some surfaces obtained with the help of Cartan Frame of a given null Cartan curve will be examined.

The distribution parameter of a timelike ruled surface are obtained which is generated by a timelike straight line in Frenet trihedron moving along a space-like curve in [11]. Furthermore, the classification of ruled surfaces is given in a three-dimensional Minkowski space in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature in [6]. Then, the ruled surfaces are examined in Minkowski 3-spaces which satisfy some algebraic equations in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature in [7]. For ruled surfaces with lightlike ruling in Minkowski 3-space, using elementary methods, some characterizations of B-scrolls and the classification (also construction) of finite type surfaces and surfaces with finite type Gauss map are given in [8]. Considering the relationship between ruled surfaces and quaternions and even dual quaternions, different studies have emerged. For example in [2], an quaternionic operator is derived and it is proved that each ruled surface in Euclidean 3 space is obtained by this operator.

Received : *06–07–2021,* ***Accepted :*** *07–10–2021*
**** Corresponding author***

On the other hand, the ruled null surfaces of the principal normal indicatrix of a null Cartan curve in de Sitter 3-space, an important vacuum solution to Einstein's equations of general relativity with cosmological terms are investigated in [10]. Moreover, description of the geometry of null curves (Cartan frame, pseudo-arc parameter, pseudo-torsion, pairs of associated curves) in terms of the curvature of the corresponding plane curves in [9].

In this study, the surfaces which are obtained with null Cartan curves will be examined. These surfaces are parameterized as follows

$$M(s,t) = \alpha(s) + t\left(a(s)T(s) + b(s)N(s) + c(s)B(s)\right) \tag{1.1}$$

where $a, b, c : I \to \mathbb{R}$ are some differentiable functions and $\alpha : I \to \mathbb{E}^3_1$ is a null Cartan curve. The generating lines are linear combinations of its Cartan frame fields and non-constant vector fields. We will investigate some special cases which are summarized in the following table.

Case	Generating lines	Type of generating lines	Parametrization of the surface
$a(s)$=0 $b(s)$=0	$X_1(s)$=$\lambda_1(s)B(s)$	null	$M^1(s,t)$=$\alpha(s)$+$t\lambda_1(s)B(s)$
$a(s)$=0 $c(s)$=0	$X_2(s)$=$\lambda_2(s)N(s)$	spacelike	$M^2(s,t)$=$\alpha(s)$+$t\lambda_2(s)N(s)$
$b(s)$=0 $c(s)$=0	$X_3(s)$=$\lambda_3(s)T(s)$	null	$M^3(s,t)$=$\alpha(s)$+$t\lambda_3(s)T(s)$
$a(s)$=0	$X_4(s)$=$N(s)$+$\lambda_4(s)B(s)$	unit spacelike	$M^4(s,t)$=$\alpha(s)$+$t(N(s)$+$\lambda_4(s)B(s))$
$b(s)$=0	$\widetilde{X}_5(s)$=$\lambda_5(s)T(s)$+$\frac{1}{2\lambda_5(s)}B(s)$	unit spacelike	$\widetilde{M}^5(s,t)$=$\alpha(s)$+$t(\lambda_5(s)T(s)$+$\frac{1}{2\lambda_5(s)}B(s))$
$b(s)$=0	$\overline{X}_5(s)$=$\lambda_5(s)T(s)$-$\frac{1}{2\lambda_5(s)}B(s)$	unit timelike	$\overline{M}^5(s,t)$=$\alpha(s)$+$t(\lambda_5(s)T(s)$-$\frac{1}{2\lambda_5(s)}B(s))$
$c(s)$=0	$X_6(s)$=$\lambda_6(s)T(s)$+$N(s)$	unit spacelike	$M^6(s,t)$=$\alpha(s)$+$t(\lambda_6(s)T(s)$+$N(s))$

Table 1: Surfaces with null Cartan base curve

The aim of this study is to investigate surfaces with null Cartan base curve where generating lines are some linear combination of Cartan frame fields with non-constant differentiable functions. There cases correspond to some special family of surfaces. For each case, we obtain characterization (timelike, spacelike or degenerate) of these surfaces. It is seen that the casual characters of these surfaces sometimes depend on the choice of non-constant functions generating lines. Firstly, the surfaces with null Cartan base curve are examined whose generating lines have the same direction of Cartan frame fields B, N and T, respectively. As a special case, Gaussian and mean curvatures of one parameter family of Bertrand curves of a given null Cartan curve and the singular points of this type of surface are stated. Moreover, an example is also given to explain the obtained results. Then, the surfaces with null Cartan base curve are investigated where generating lines lie on the planes spanned by $\{N, B\}$, $\{T, B\}$ and $\{T, N\}$, respectively. The observation of the differential geometric properties of these surfaces are done mainly in three different cases. The Gaussian and mean curvatures of the surfaces are obtained for each case.

2. Preliminaries

In this section, we will give the necessary informations to understand the main subject of the study.

Minkowski 3-space is the Euclidean space provided with Lorentzian product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = -u_1v_1 + u_2v_2 + u_3v_3 \tag{2.1}$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. By definition, this product is not positively defined. Instead, this product classifies the vectors in $\mathbb{E}^3_1$ as follows:

If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle > 0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle < 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = 0$, then $\overrightarrow{u}$ is called a spacelike, timelike or null vector, respectively. For each $\overrightarrow{u} \in \mathbb{E}_1^3$, the norm of $\overrightarrow{u}$ vector is defined

$$\|\overrightarrow{u}\| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle|}. \tag{2.2}$$

If $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0$, then $\overrightarrow{u}$ and $\overrightarrow{v}$ vectors are said to be orthogonal [12].

Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by the pseudo arc length parameter. Then

$$T(s) = \alpha'(s) \tag{2.3}$$

is the tangent vector of α. Thus $N(s) = \alpha''(s)$ is the spacelike vector. The binormal vector field B is the unique null vector field that is orthogonal to N such that

$$\langle T(s), B(s) \rangle = 1. \tag{2.4}$$

If α is a straight line, then $\kappa(s) = 0$ and in other cases $\kappa(s) = 1$. In addition,

$$\tau(s) = \left\langle N'(s), B(s) \right\rangle \tag{2.5}$$

[9].

Theorem 2.1. *Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by pseudo arclength and $\{T, N, B\}$ be the pseudo orthonormal frame of the curve α. Then we have the derivative formulas:*

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \tau(s) & 0 & -\kappa(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.6}$$

where T, N, B satisfy following relations:

$$\langle T(s), T(s) \rangle = \langle B(s), B(s) \rangle = 0, \tag{2.7}$$

$$\langle T(s), N(s) \rangle = \langle N(s), B(s) \rangle = 0, \tag{2.8}$$

$$\langle N(s), N(s) \rangle = \langle T(s), B(s) \rangle = 1 \tag{2.9}$$

and κ is the curvature function and τ is the torsion function of the curve α [5].

In this and next sections, we will assume that the given null Cartan curve $\alpha : I \to \mathbb{E}_1^3$ is not a straight line i.e. the curvature of the curve α is equal to 1. In that case, the characterization of a null Cartan curve $\alpha : I \to \mathbb{E}_1^3$ given by pseudo arc length parameter is investigated in terms of its torsion function τ.

Definition 2.1. Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve and the curve $\alpha^* : I \to \mathbb{E}_1^3$ be given with the same domain of α. If the line joining the points $\alpha^*(s)$ and $\alpha(s)$ contains both principal normal vectors of α and α^* for each $s \in I$, then it is said that the curve α forms a Bertrand curve couple with the curve α^*.

Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by pseudo arclength parameter. If the curve α^* indicates a Bertrand curve couple of α, then the curve α^* can be given in the form

$$\alpha^*(s) = \alpha(s) + \lambda N(s) \tag{2.10}$$

where λ is a nonzero real constant. Differentiating of both sides of the above equation with respect to the pseudo arclength parameter s yields:

$$\frac{d}{ds}\alpha^*(s) = (1 + \lambda\tau(s))\, T(s) - \lambda\kappa(s)\, B(s). \tag{2.11}$$

Then we obtain

$$\left\langle \frac{d}{ds}\alpha^*(s), \frac{d}{ds}\alpha^*(s) \right\rangle_L = -2\lambda\,(1 + \lambda\tau(s)). \tag{2.12}$$

Therefore, it is seen that there are three different casual character of α^* depending on the value of the torsion function τ of the curve α [4].

Theorem 2.2. *Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by pseudo arclength parameter. If Bertrand curve α^* of α is a timelike curve, then the followings are hold*

$$T^*(s) = \frac{1+\lambda\tau}{\sqrt{2\lambda(1+\lambda\tau)}}T(s) - \frac{\lambda}{\sqrt{2\lambda(1+\lambda\tau)}}B(s), \tag{2.13}$$

$$N^*(s) = sign(1+2\lambda\tau)N(s), \tag{2.14}$$

$$B^*(s) = \frac{(1+\lambda\tau)}{\sqrt{2\lambda(1+\lambda\tau)}}T(s) + \frac{\lambda}{\sqrt{2\lambda(1+\lambda\tau)}}B(s), \tag{2.15}$$

and

$$\kappa^*(s) = \frac{|1+2\lambda\tau|}{2\lambda(1+\lambda\tau)}, \tag{2.16}$$

$$\tau^*(s) = -\frac{sign(1+2\lambda\tau)}{2\lambda(1+\lambda\tau)} \tag{2.17}$$

where $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are the pseudo orthonormal frame of the curve α and Frenet frame of the curve α^* [4].

Theorem 2.3. *Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by pseudo arclength parameter. If Bertrand curve α^* of α is a spacelike curve, then the followings are hold*

$$T^*(s) = \frac{1+\lambda\tau}{\sqrt{-2\lambda(1+\lambda\tau)}}T(s) - \frac{\lambda}{-\sqrt{2\lambda(1+\lambda\tau)}}B(s), \tag{2.18}$$

$$N^*(s) = sign(1+2\lambda\tau)N(s), \tag{2.19}$$

$$B^*(s) = \frac{(1+\lambda\tau)}{\sqrt{-2\lambda(1+\lambda\tau)}}T(s) + \frac{\lambda}{\sqrt{-2\lambda(1+\lambda\tau)}}B(s), \tag{2.20}$$

and

$$\kappa^*(s) = \frac{|1+2\lambda\tau|}{-2\lambda(1+\lambda\tau)}, \tag{2.21}$$

$$\tau^*(s) = \frac{sign(1+2\lambda\tau)}{-2\lambda(1+\lambda\tau)}. \tag{2.22}$$

where $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are the pseudo orthonormal frames of the curve α and the curve α^*, respectively [4].

Corollary 2.1. *All types (timelike, spacelike or null Cartan) Bertrand couples of a given null Cartan curve with constant curvature functions have also constant curvature functions [4].*

3. Surfaces whose generating lines have same direction of Cartan frame fields

In this section, we will investigate the surfaces with null Cartan base curve whose generating lines have the same direction of Cartan frame fields B, N and T, respectively. These are the first three cases which are given in the Table 1.

3.1. Surface with binormal generating direction

In this subsection, we will investigate the surface whose base curve is the null Cartan curve $\alpha : I \to \mathbb{E}_1^3$ and the generating line is the null vector field with the same direction of the Cartan frame field B. In this case, the surface has the following parametric representation

$$M^1(s,t) = \alpha(s) + t\lambda_1(s)B(s) \tag{3.1.1}$$

where $\lambda_1 : I \to \mathbb{R}$ is a differentiable function.

Theorem 3.1. *Gaussian curvature K and mean curvature H of timelike surface $M^1 = M^1(s,t)$ are obtained as*

$$K(s,t) = \tau^2(s), \tag{3.1.2}$$
$$H(s,t) = -\tau(s) \tag{3.1.3}$$

respectively.

Proof. We have found the coefficients of first fundamental forms of the $M^1(s,t)$ as $E = t^2\tau^2(s)\lambda_1(s)^2 + 2t\lambda_1'(s)$, $F = \lambda_1(s)$ and $G = 0$. Thus we say that $EG - F^2 = -(\lambda_1(s))^2$. Normal vector field of the $M^1(s,t)$ surface is given by

$$n(s,t) = xT(s) + yN(s) + zB(s). \tag{3.1.4}$$

Since $n \perp M_s^1(s,t)$ and $n \perp M_t^1(s,t)$, the vector field of the surface $M^1(s,t)$ is obtained as follows

$$n(s,t) = N(s) + t\tau(s)\lambda_1(s)B(s). \tag{3.1.5}$$

As can be seen from the above equation, the normal vector $n(s,t)$ is the spacelike vector. This implies M^1 is a timelike surface. After computations, we can easily obtain coefficients e, f and g of the second fundamental form of $M^1(s,t)$ as

$$e = 1 - t\tau'(s)\lambda_1(s) - 2t\lambda_1'(s) - t^2\tau^3(s)\lambda_1(s)^2, \tag{3.1.6}$$
$$f = -\tau(s)\lambda_1(s), \tag{3.1.7}$$
$$g = 0. \tag{3.1.8}$$

Thus, the Gaussian curvature K and mean curvature H of the surface $M^1(s,t)$ are

$$K(s,t) = \frac{eg - f^2}{EG - F^2} = \tau^2(s), \tag{3.1.9}$$
$$H(s,t) = \frac{Eg - 2Ff + Ge}{2(EG - F^2)} = -\tau(s) \tag{3.1.10}$$

respectively. □

Remark 3.1. The Gaussian and mean curvature of the surface do not depend on the choice of the differentiable function λ_1. They only depend on the torsion function of null Cartan curve α. The above theorem shows that the principal curvatures of timelike surface M^1 can be found as follows:

$$k_1(s,t) = k_2(s,t) = -\tau(s). \tag{3.1.11}$$

Definition 3.1. A point p on a timelike surface M is called quasi-umbilic if the shape operator of M is non-diagonalizable over C [3].

Corollary 3.1. *If null Cartan curve α is a space curve then every point of the surface M^1 is quasi-umbilic.*

Proof. Using above theorem, we obtain

$$H^2(s,t) - K(s,t) = \tau^2(s) - \tau^2(s) = 0 \tag{3.1.12}$$

for timelike surface M^1. This implies that the shape operator is non-diagonalizable over C. Thus, we get the proof. □

For details of the diagonalizability of the shape operator of a timelike surface, the readers are referred to [1].

Remark 3.2. Let $\alpha : I \to \mathbb{E}_1^3$ be a regular curve, then $\chi(s,t) = \alpha(s) + tb(s)$ is called a null scroll if

$$\langle \alpha'(s), \alpha'(s)\rangle = \langle b(s), b(s)\rangle = 0, \;\; \langle \alpha'(s), b(s)\rangle = 1. \tag{3.1.13}$$

Therefore, if the constant function $\lambda_1(s) = 1$ is taken for surface M^1, the surface is obtained as a null scroll.

3.2. Surface with principal normal generating direction

In this subsection, we will give the surface whose base curve is the null Cartan curve $\alpha : I \to \mathbb{E}_1^3$ and the generating line is the spacelike vector field with the same direction of the Cartan frame field N. The parametric representation of surface is obtained as follows

$$M^2(s,t) = \alpha(s) + t\lambda_2(s)N(s) \tag{3.2.1}$$

where $\lambda_2 : I \to \mathbb{R}$ is a differentiable function.

Theorem 3.2. *The Gaussian curvature K of the surface $M^2 = M^2(s,t)$ is*

$$K(s,t) = \varepsilon \frac{1}{4t^2\lambda_2(s)^2 (1 + t\lambda_2(s)\tau(s))^2} \tag{3.2.2}$$

such that $\varepsilon = \pm 1$.

Proof. The coefficients of first fundamental forms of the $M^2(s,t)$ are obtained as $E = -2t\lambda_2(s) - 2t^2\lambda_2^2(s)\tau(s) + t^2\lambda_2'(s)^2$, $F = t\lambda_2'(s)\lambda_2(s)$ and $G = \lambda_2^2(s)$. So we obtain that $EG - F^2 = -2t\lambda_2^3(s)(1 + t\lambda_2(s)\tau(s))$. Since $n \perp M_s^2(s,t)$ and $n \perp M_t^2(s,t)$, the vector field of the surface $M^2(s,t)$ is given as follows

$$n(s,t) = (1 + \lambda_2(s)t\tau(s))T(s) + t\lambda_2(s)B(s). \tag{3.2.3}$$

If we take the vector field $n(s,t)$ as a unit vector, then we get

$$n(s,t) = \frac{(1 + \lambda_2(s)t\tau)}{\sqrt{2t\lambda_2(s)(1 + \lambda_2(s)t\tau(s))}} T(s) + \frac{t\lambda_2(s)}{\sqrt{2t\lambda_2(s)(1 + \lambda_2(s)t\tau(s))}} B(s). \tag{3.2.4}$$

We obtain coefficients e, f and g of the second fundamental form of $M^2(s,t)$ as

$$e = \frac{-2t\lambda_2'(s) + t^2\lambda_2^2(s)\tau'(s)}{\sqrt{2t\lambda_2(s)(1 + \lambda_2(s)t\tau(s))}}, \tag{3.2.5}$$

$$f = \frac{-\lambda_2(s)}{\sqrt{2t\lambda_2(s)(1 + \lambda_2(s)t\tau(s))}}, \tag{3.2.6}$$

$$g = 0. \tag{3.2.7}$$

Thus, the Gaussian curvature K of the surface $M^2(s,t)$ is

$$K(s,t) = \varepsilon \frac{1}{4t^2\lambda_2(s)^2 (1 + t\lambda_2(s)\tau)^2} \tag{3.2.8}$$

respectively. □

Theorem 3.3. *The mean curvature H of $M^2 = M^2(s,t)$ is*

$$H(s,t) = -\varepsilon \frac{(\lambda_2(s)t\tau'(s))}{4(1 + \lambda_2(s)t\tau(s))\sqrt{2t\lambda_2(s)(1 + \lambda_2(s)t\tau(s))}} \tag{3.2.9}$$

respectively.

Proof. If the coefficients of first and second fundamental forms are written in the formula of the mean curvature, then this theorem is proved. □

Corollary 3.2. *If the torsion is constant, the surface $M^2(s,t)$ is minimal surface.*

Special Case: If we take $\lambda_2(s) = 1$ of the surface $M^2(s,t)$, then we get one parameter family of Bertrand curves of null Cartan curve α. The geometric properties of this surface will be discussed in the following.

Theorem 3.4. *Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by pseudo arclength parameter and $\{T, N, B\}$ be the pseudo orthonormal frame of the curve α. Suppose that $\widetilde{M}^2 = \widetilde{M}^2\,(s,t)$ is a curve flow such that*

$$\widetilde{M}^2\,(s,t) = \alpha\,(s) + tN(s). \tag{3.2.10}$$

If $t \in \left(\frac{-1}{\tau}, 0\right)$, then $\widetilde{M}^2$ is a spacelike regular surface with Gaussian curvature

$$K(s,t) = \frac{1}{4t^2\,(1+\tau\,(s)\,t)^2} \tag{3.2.11}$$

and the mean curvature

$$H(s,t) = 0 \tag{3.2.12}$$

where τ is torsion function of the null Cartan curve α.

Proof. We need to find the coefficients of the first and second fundamental forms of the surface $\widetilde{M}^2$. By using derivative formulas of the pseudo orthonormal frame of the curve α and the fact that $\alpha'(s) = T(s)$, we get

$$\frac{\partial}{\partial s}\widetilde{M}^2\,(s,t) = (1+\tau\,(s)\,t)\,T(s) - tB(s) \tag{3.2.13}$$

and

$$\frac{\partial}{\partial t}\widetilde{M}^2\,(s,t) = N(s). \tag{3.2.14}$$

Actually, we have

$$\left\langle \frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)\,, \frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)\right\rangle_L = -2t\,(1+\tau\,(s)\,t) > 0 \tag{3.2.15}$$

since $\kappa = 1$ and $\langle T, T\rangle_L = \langle B, B\rangle_L = 0$, $\langle T, B\rangle = 1$. According to the assumption we have $t \in \left(\frac{-1}{\tau}, 0\right)$. This means that the vector field $\frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)$ is a spacelike vector field. Let n denotes the unit normal vector of the surface $\widetilde{M}^2$. Since $\frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)$ and $\frac{\partial}{\partial t}\widetilde{M}^2\,(s,t)$ are two spacelike vector fields, the surface normal vector field should be timelike. We can easily obtain the unique timelike vector field n which is orthogonal to both $\frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)$ and $\frac{\partial}{\partial t}\widetilde{M}^2\,(s,t)$ as follows

$$n(s,t) = \frac{1}{\sqrt{-2t\,(1+\tau\,(s)\,t)}}\left[(1+\tau\,(s)\,t)\,T(s) + tB(s)\right]. \tag{3.2.16}$$

Now we are ready to find the coefficients of the first and second fundamental form of the spacelike surface $\widetilde{M}^2$. Let $\{E, F, G\}$ and $\{e, f, g\}$ denote the coefficients of the first and second fundamental form of the $\widetilde{M}^2$, respectively. Therefore we find

$$E = \left\langle \frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)\,, \frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)\right\rangle_L = -2t\,(1+\tau\,(s)\,t)\,, \tag{3.2.17}$$

$$F = \left\langle \frac{\partial}{\partial s}\widetilde{M}^2\,(s,t)\,, \frac{\partial}{\partial t}\widetilde{M}^2\,(s,t)\right\rangle_L = 0, \tag{3.2.18}$$

$$G = \left\langle \frac{\partial}{\partial t}\widetilde{M}^2\,(s,t)\,, \frac{\partial}{\partial t}\widetilde{M}^2\,(s,t)\right\rangle_L = 1 \tag{3.2.19}$$

and

$$e = \left\langle \frac{\partial^2}{\partial s^2}\widetilde{M}^2\,(s,t)\,, n(s,t)\right\rangle_L = 0, \tag{3.2.20}$$

$$f = \left\langle \frac{\partial^2}{\partial s \partial t}\widetilde{M}^2\,(s,t)\,, n(s,t)\right\rangle_L = \frac{-1}{\sqrt{-2t\,(1+\tau\,(s)\,t)}}, \tag{3.2.21}$$

$$g = \left\langle \frac{\partial^2}{\partial t^2}\widetilde{M}^2\,(s,t)\,, \frac{\partial^2}{\partial t^2}\widetilde{M}^2\,(s,t)\right\rangle_L = 0. \tag{3.2.22}$$

Finally, we find the Gaussian curvature of the spacelike surface

$$K(s,t) = -\frac{eg-f^2}{EG-F^2} = -\frac{-\left(\frac{-1}{\sqrt{-2t(1+\tau(s)t)}}\right)^2}{-2t\left(1+\tau\left(s\right)t\right)}$$
$$= \frac{1}{4t^2\left(1+\tau\left(s\right)t\right)^2} \tag{3.2.23}$$

and the mean curvature

$$H(s,t) = -\frac{1}{2}\left(\frac{eG-2fF+gE}{EG-F^2}\right) = 0. \tag{3.2.24}$$

□

Corollary 3.3. *The spacelike surface* $\widetilde{M}^2\left(s,t\right) = \alpha\left(s\right) + tN(s)$ *is a minimal surface where* $\alpha : I \to \mathbb{E}_1^3$ *is a null Cartan curve with the torsion function* $\tau\left(s\right)$ *and* $t \in \left(\frac{-1}{\tau}, 0\right)$.

Remark 3.3. The above theorem shows that the principal curvatures of spacelike surface $\widetilde{M}^2$ can be found as follows:

$$k_1(s,t) = -k_2(s,t) = \frac{1}{2t(1+t\tau(s))}. \tag{3.2.25}$$

Theorem 3.5. *Let* $\alpha : I \to \mathbb{E}_1^3$ *be a null Cartan curve given by pseudo arclength and* $\{T, N, B\}$ *be the pseudo orthonormal frame of the curve* α*. Suppose that* $\widetilde{M}^2 = \widetilde{M}^2\left(s,t\right)$ *is a curve flow such that* $\widetilde{M}^2\left(s,t\right) = \alpha\left(s\right) + tN(s)$*. If* $t \in \mathbb{R} - [\frac{-1}{\tau}, 0]$*, then* $\widetilde{M}^2$ *is a timelike regular surface with Gaussian curvature*

$$K(s,t) = -\frac{1}{4t^2\left(1+\tau\left(s\right)t\right)^2} \tag{3.2.26}$$

and the mean curvature

$$H(s,t) = 0 \tag{3.2.27}$$

where τ *is torsion function of the null Cartan curve* α*.*

Proof. With the use of the fact that $\alpha'(s) = T(s)$ and $N'(s) = \tau(s)T(s) - B(s)$, we get

$$\left\langle \frac{\partial}{\partial s}\widetilde{M}^2\left(s,t\right), \frac{\partial}{\partial s}\widetilde{M}^2\left(s,t\right)\right\rangle_L = -2t\left(1+\tau\left(s\right)t\right) < 0. \tag{3.2.28}$$

Therefore, $\frac{\partial}{\partial s}\widetilde{M}^2\left(s,t\right)$ is a timelike vector field. Consequently, the unit normal n of the surface is a spacelike vector field. Since n is orthogonal to both $\frac{\partial}{\partial s}\widetilde{M}^2$ and $\frac{\partial}{\partial t}\widetilde{M}^2$, we get

$$n(s,t) = \frac{1}{\sqrt{2t\left(1+\tau\left(s\right)t\right)}}\left[\left(1+\tau\left(s\right)t\right)T(s) + tB(s)\right]. \tag{3.2.29}$$

At the same time, the coefficients of the first and second fundamental form of the timelike surface $\widetilde{M}^2$ are obtain as follows

$$E = -2t\left(1+\tau\left(s\right)t\right), \quad F = 0, \quad G = 1 \tag{3.2.30}$$

and

$$e = 0, \quad f = \frac{-1}{\sqrt{2t\left(1+\tau\left(s\right)t\right)}}, \quad g = 0 \tag{3.2.31}$$

respectively. Finally, we find the Gaussian curvature of the timelike surface

$$K(s,t) = \frac{eg-f^2}{EG-F^2} = \frac{-\left(\frac{-1}{\sqrt{-2t(1+\tau(s)t)}}\right)^2}{-2t\left(1+\tau\left(s\right)t\right)}$$
$$= -\frac{1}{4t^2\left(1+\tau\left(s\right)t\right)^2} \tag{3.2.32}$$

and the mean curvature

$$H(s,t) = \frac{1}{2}\left(\frac{eG - 2fF + gE}{EG - F^2}\right) = 0. \tag{3.2.33}$$

□

Corollary 3.4. *The timelike surface $\widetilde{M}^2(s,t) = \alpha(s) + tN(s)$ is a minimal surface where $\alpha : I \to \mathbb{E}_1^3$ is a null Cartan curve with the torsion function $\tau(s)$ and $t \in \mathbb{R} - [\frac{-1}{\tau}, 0]$.*

Remark 3.4. The above theorem shows that the principal curvatures of timelike surface $\widetilde{M}^2$ can be found as follows:

$$k_1(s,t) = -k_2(s,t) = \frac{1}{2t(1 + t\tau(s))}. \tag{3.2.34}$$

Example 3.1. Let $\alpha : I \to \mathbb{E}_1^3$ be a null Cartan curve given by pseudo arclength parameter with following parametric expression

$$\alpha(s) = (s, \cos s, \sin s). \tag{3.2.35}$$

Then the Frenet frame is obtained

$$T(s) = (1, -\sin s, \cos s), \tag{3.2.36}$$

$$N(s) = (0, -\cos s, -\sin s), \tag{3.2.37}$$

$$B(s) = (-\frac{1}{2}, -\frac{1}{2}\sin s, \frac{1}{2}\cos s) \tag{3.2.38}$$

where $\kappa(s) = 1$ and $\tau(s) = -\frac{1}{2}$. Suppose that $\widetilde{M}^2 = \widetilde{M}^2(s,t)$ is a curve flow such that

$$\begin{aligned}\widetilde{M}^2(s,t) &= (s, \cos s, \sin s) + t(0, -\cos s, -\sin s)\\ &= (s, \cos s - t\cos s, \sin s - t\sin s).\end{aligned} \tag{3.2.39}$$

The figure of the surface is given below for $t \in (0, 2)$.

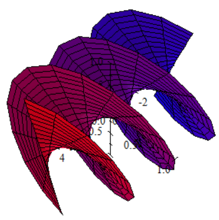

Figure 1. The surface is given for $t \in (0, 2)$.

If $t \in (0, 2)$, then $\widetilde{M}^2$ is a spacelike regular surface with Gaussian curvature

$$K(s,t) = \frac{1}{4t^2\left(1 - \frac{1}{2}t\right)^2} = \frac{1}{t^2(t-2)^2} \tag{3.2.40}$$

and the mean curvature

$$H(s,t) = 0. \tag{3.2.41}$$

Suppose that $\widetilde{M}^2 = \widetilde{M}^2(s,t)$ is a curve flow such that

$$\widetilde{M}^2(s,t) = \alpha(s) + tN(s). \tag{3.2.42}$$

The figure of the surface is given below for $t \in (-5, 0)$:

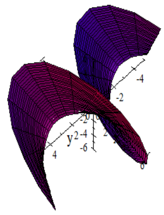

Figure 2. The surface is given for $t \in (-5, 0)$.

If $t \in \mathbb{R} - [0, 2]$, then $\widetilde{M}^2$ is a timelike regular surface with Gaussian curvature

$$K(s,t) = -\frac{1}{t^2 (t-2)^2} \tag{3.2.43}$$

and the mean curvature

$$H(s,t) = 0. \tag{3.2.44}$$

Remark 3.5. The points $\widetilde{M}^2(s, 0)$ and $\widetilde{M}^2(s, -\frac{1}{\tau})$ are the singular points of the surface

$$\widetilde{M}^2(s,t) = \alpha(s) + tN(s) \tag{3.2.45}$$

which is one parameter family of the Bertrand curves of null Cartan curve α.

3.3. *Surface with tangent generating direction*

In this subsection, we will investigate the surface whose base curve is the null Cartan curve $\alpha : I \to \mathbb{E}_1^3$ and the generating line is the null vector field with the same direction of the tangent frame field T. The surface give the following parametric representation

$$M^3(s,t) = \alpha(s) + t\lambda_3(s)T(s) \tag{3.3.1}$$

where $\lambda_3 : I \to \mathbb{R}$ is a differentiable function.

Theorem 3.6. *The Gaussian curvature K and mean curvature H of $M^3 = M^3(s,t)$ are*

$$K(s,t) = 0, \tag{3.3.2}$$

$$H(s,t) = \frac{-1}{2t\lambda_3(s)} \tag{3.3.3}$$

respectively.

Proof. The coefficients of first fundamental forms of the $M^3(s,t)$ as follows

$$E = t^2\lambda_3^2(s), \quad F = 0, \quad G = 0. \tag{3.3.4}$$

Since $EG - F^2 = 0$, the surface $M^3(s,t)$ is degenerate surface. Normal vector field of the $M^3(s,t)$ surface is given by

$$n(s,t) = xT(s) + yN(s) + zB(s). \tag{3.3.5}$$

Since $n \perp M_s^3(s,t)$ and $n \perp M_t^3(s,t)$, the vector field of the surface $M^3(s,t)$ is obtained as follows

$$n(s,t) = T(s). \tag{3.3.6}$$

The coefficients e, f and g of the second fundamental form of $M^3(s,t)$ obtained as

$$e = -t\lambda_3(s), \quad f = 0, \quad g = 0. \tag{3.3.7}$$

The Weingarten matrix of $M^3(s,t)$ is obtained as follows

$$\begin{bmatrix} \frac{-1}{t\lambda_3(s)} & 0 \\ \frac{1}{t\lambda_3^2(s)\left(1+t\lambda_3'(s)\right)} & 0 \end{bmatrix}. \tag{3.3.8}$$

So, the Gaussian curvature K and mean curvature H of the surface $M^3(s,t)$ are

$$K(s,t)=0, \quad H(s,t)=\frac{-1}{2t\lambda_3(s)} \tag{3.3.9}$$

respectively. □

4. Surfaces whose generating lines have unit direction and lie on Cartan planes

In this section, we will investigate the surfaces with null Cartan base curve whose generating lines lie on the planes spanned by $\{N,B\}$, $\{T,B\}$ and $\{T,N\}$, respectively. These are the last four cases which are given in the Table 1.

4.1. Surface with unit spacelike generating direction lie on span{N,B}

In this subsection, we will examine the surface with the null Cartan base curve $\alpha: I \to \mathbb{E}_1^3$ and whose generating line is non-constant unit spacelike vector field spanning by the Cartan frame fields N and B. The surface has the following parametric representation

$$M^4(s,t)=\alpha(s)+t(N(s)+\lambda_4(s)B(s)) \tag{4.1.1}$$

where $\lambda_4: I \to \mathbb{R}$ is a differentiable function.

Theorem 4.1. *The Gaussian curvature K of $M^4(s,t)$ is*

$$K(s,t)=\varepsilon\frac{\left(\lambda_4^2(s)\tau(s)+\tau(s)k+\lambda_4'(s)-1\right)^2}{\left(\lambda_4^2(s)+2k\right)\left(\left(t^2\tau^2(s)-1\right)\lambda_4^2(s)+2t\left(1+t\tau(s)\right)\left(\lambda_4'(s)-1\right)\right)} \tag{4.1.2}$$

such that $\tau \neq \frac{-1}{t}$ where

$$k(s,t)=\frac{t-t\lambda_4'(s)-t\lambda_4^2(s)\tau(s)}{1+t\tau(s)} \tag{4.1.3}$$

and $\varepsilon=\pm 1$.

Proof. The coefficients of first fundamental forms of the $M^4(s,t)$ as follows

$$E=t^2\lambda_4^2(s)\tau^2(s)+2\left(1+t\tau(s)\right)\left(t\lambda_4'(s)-t\right), \quad F=\lambda_4(s), \quad G=1. \tag{4.1.4}$$

and $EG-F^2=\left(t^2\tau^2(s)-1\right)\lambda_4^2(s)+2t\left(1+t\tau(s)\right)\left(\lambda_4'(s)-1\right)$. If $\tau=\frac{-1}{t}$, we obtained that $EG-F^2=0$. So the surface $M^4(s,t)$ is defined as degenerate surface. Therefore we assume that $\tau \neq \frac{-1}{t}$, when defining the surface. Normal vector field of the $M^4(s,t)$ surface is obtained as follows

$$n(s,t)=\frac{1}{\sqrt{\lambda_4^2(s)+2k}}(T(s)-\lambda(s)N(s)+kB(s). \tag{4.1.5}$$

where $k=\frac{t-t\lambda_4'(s)-t\lambda_4^2(s)\tau(s)}{1+t\tau(s)}$ and $\langle n,n\rangle=\varepsilon=\pm 1$. The coefficients e,f and g of the second fundamental form of $M^4(s,t)$ obtained as

$$e=\frac{1}{\sqrt{\lambda_4^2(s)+2k}}\begin{pmatrix} -\lambda_4(s)+t\lambda_4^2(s)\tau'(s)+t\tau(s)\lambda_4(s) \\ +t\lambda_4''(s)+k\left(t\lambda_4'(s)-t\lambda_4(s)-\tau^2(s)\right) \end{pmatrix}, \tag{4.1.6}$$

$$f=\frac{1}{\sqrt{\lambda_4^2(s)+2k}}\left(\lambda_4^2(s)\tau(s)+\tau(s)k+\lambda_4'(s)-1\right), \tag{4.1.7}$$

$$g=0. \tag{4.1.8}$$

So, the Gaussian curvature K of the surface $M^4(s,t)$ obtain as follows

$$K(s,t)=\varepsilon\frac{\left(\lambda_4^2(s)\,\tau(s)+\tau(s)k+\lambda_4'(s)-1\right)^2}{\left(\lambda_4^2(s)+2k\right)\left(\left(t^2\tau^2(s)-1\right)\lambda_4^2(s)+2t\left(1+t\tau(s)\right)\left(\lambda_4'(s)-1\right)\right)} \tag{4.1.9}$$

where $\varepsilon=\pm1$. □

Theorem 4.2. *The mean curvature H of $M^4(s,t)$ is obtained as follows:*

$$H(s,t)=\varepsilon\frac{\begin{array}{c}t\lambda_4''(s)-\lambda_4(s)-k\left(\tau^2(s)+t\lambda_4(s)-t\lambda_4'(s)\right)+t\tau(s)\lambda_4(s)+t\tau'(s)\lambda_4^2(s)\\-2\lambda_4(s)\left(\tau(s)\lambda_4^2(s)+\lambda_4'(s)+k\tau(s)-1\right)\end{array}}{\left(4t\left(\lambda_4'(s)-1\right)\left(t\tau(s)+1\right)-2t^2\tau(s)^2\lambda_4^2(s)\right)\sqrt{\lambda_4^2(s)+2k}} \tag{4.1.10}$$

where $\varepsilon=\pm1$.

Proof. The proof can be done similar to the previous theorem. □

4.2. Surface with unit generating direction which lies on span {T,B}

In this subsection, we will investigate the surface with the null Cartan base curve $\alpha: I\to\mathbb{E}_1^3$ and whose generating line is non-constant unit vector field spanning by the Cartan frame fields T and B. There is two possible subcases.

4.2.1. Surface with unit spacelike generating direction which lies on span {T,B} In this situation, the surface has the following parametric representation

$$\widetilde{M}^5(s,t)=\alpha(s)+t(\lambda_5(s)T(s)+\frac{1}{2\lambda_5(s)}B(s)) \tag{4.2.1}$$

where $\lambda_5: I\to\mathbb{R}$ is a differentiable function.

Theorem 4.3. *Gaussian curvature K and mean curvature H of the spacelike surface $\widetilde{M}_5^5=\widetilde{M}_5^5(s,t)$ are*

$$K(s,t)=\frac{1}{h^2(s,t)}\frac{\left(\lambda_5'(s)\left(-2\lambda_5^2(s)+2-\frac{1}{\lambda_5^2(s)}\right)+\frac{1}{t}\right)^2}{\left(t\lambda_5(s)-\frac{1}{2}\right)^2-\frac{\left(2\lambda_5'(s)t-1\right)^2-2}{4\lambda_5^2(s)}}, \tag{4.2.2}$$

$$H(s,t)=\frac{\frac{1}{h(s,t)}\left(\begin{array}{c}\frac{1}{2\lambda_5(s)}\left(-2\lambda_5^2(s)\lambda_5'(s)+\frac{1}{t}+2\lambda_5'(s)-\frac{\lambda_5'(s)}{\lambda_5^2(s)}\right)\\+\left(-2t\lambda_5^2(s)\lambda_5''(s)-2\lambda_5^3(s)t+2+4t\lambda_5'(s)+\frac{\lambda_5''(s)}{2\lambda_5^2(s)}\right)\end{array}\right)}{\left(t\lambda_5(s)-\frac{1}{2}\right)^2-\frac{\left(2\lambda_5'(s)t-1\right)^2-2}{4\lambda_5^2(s)}} \tag{4.2.3}$$

where

$$h^2(s,t)=-4\lambda_5^2(s)+4\frac{(2t\lambda_5'(s)+1)^2}{(t-2t\lambda_5(s))^2}. \tag{4.2.4}$$

Proof. We have found the coefficients of first fundamental forms of the $\widetilde{M}^5(s,t)$ as

$$E=\left(t\lambda_5(s)-\frac{1}{2}\right)^2-\frac{\lambda_5'(s)t-\lambda_5'(s)^2t^2}{\lambda_5^2(s)}, \tag{4.2.5}$$

$$F=\frac{1}{2\lambda_5(s)}, \tag{4.2.6}$$

$$G=1. \tag{4.2.7}$$

Therefore, we obtain

$$EG-F^2=\left(t\lambda_5(s)-\frac{1}{2}\right)^2-\frac{\left(2\lambda_5'(s)t-1\right)^2-2}{4\lambda_5^2(s)} \tag{4.2.8}$$

and

$$e = \frac{1}{h(s,t)}\left[-2t\lambda_5^2(s)\lambda_5''(s) - 2\lambda_5^3(s)t + 2 + 4t\lambda_5'(s) + \frac{\lambda_5''(s)}{2\lambda_5^2(s)}\right], \tag{4.2.9}$$

$$g = 0, \tag{4.2.10}$$

$$f = \frac{1}{h(s,t)}(-2\lambda_5^2(s)\lambda_5'(s) + \frac{1}{t} + 2\lambda_5'(s) - \frac{\lambda_5'(s)}{\lambda_5^2(s)}) \tag{4.2.11}$$

where $h^2(s,t) = -4\lambda_5^2(s) + 4\frac{(2t\lambda_5'(s)+1)^2}{(t-2t\lambda_5(s))^2}$. Using obtained values of the coefficients of first and second fundamental forms, we get the proof. □

4.2.2. Surface with unit timelike generating direction which lies on span {T,B} In this situation, the surface has the following parametric representation

$$\overline{M}_5(s,t) = \alpha(s) + t(\lambda_5(s)T(s) - \frac{1}{2\lambda_5(s)}B(s)) \tag{4.2.12}$$

where $\lambda_5 : I \to \mathbb{R}$ is a differentiable function.

Theorem 4.4. *Gaussian curvature K and mean curvature H of $\overline{M}^5 = \overline{M}^5(s,t)$ are*

$$K = -\frac{(\lambda_5^2(s)+\tau)^2(2t\lambda_5'(s)-1)^2(1+2t\lambda_5'(s)+2(\lambda_5'(s)t)^2+\lambda_5^4(s)t^4+2\lambda_5^2(s)t^2\tau+\tau^2 t)}{(t^2\lambda_5^2(s)+\tau t)(2t^2\lambda_5^2(s)+\tau t+2t\lambda_5'(s)-1)}, \tag{4.2.13}$$

$$H = -\frac{(-e\lambda_5(s)+2f)\lambda_5(s)}{2(1+2t\lambda_5'(s)+2(\lambda_5'(s)t)^2+\lambda_5^4(s)t^4+2\lambda_5^2(s)t^2\tau+\tau^2 t)} \tag{4.2.14}$$

where

$$e = \frac{1}{h(s,t)(t^2\lambda_5^2(s)+\tau t)}\left[\begin{array}{c}(t^2\lambda_5^2(s)+\tau t)(-\lambda_5(s)\tau t + 2t\lambda_5''(s) + \frac{\tau^2 t - 2t\lambda_5'(s)}{\lambda_5(s)}) \\ +(2t\lambda_5'(s)-1)(\lambda_5(s)+2\lambda_5'(s)\lambda_5(s)t+\tau' t - \frac{2\tau t\lambda_5'(s)}{\lambda_5(s)}\end{array}\right], \tag{4.2.15}$$

$$f = \frac{1}{h(s,t)(t^2\lambda_5^2(s)+\tau t)}(\lambda_5^2(s)+\tau)(2t\lambda_5'(s)-1), \tag{4.2.16}$$

$$h^2(s,t) = 2\lambda_5^2(s) + \lambda_5^2(s)\frac{(2t\lambda_5'(s)-1)^2}{(t^{2\lambda_5 2}(s)+\tau t)^2}. \tag{4.2.17}$$

Proof. We have found the coefficients of first fundamental forms of the $\overline{M}^5(s,t)$ as

$$E = \frac{2t\lambda_5'(s) + 2(\lambda_5'(s)t)^2 + (\lambda_5^2(s)t+\tau t)^2}{\lambda_5^2(s)}, \tag{4.2.18}$$

$$F = -\frac{1}{\lambda_5(s)}, \tag{4.2.19}$$

$$G = -1. \tag{4.2.20}$$

Therefore, we obtain

$$EG - F^2 = -\frac{1+2t\lambda_5'(s)+2(\lambda_5'(s)t)^2+\lambda_5^4(s)t^4+2\lambda_5^2(s)t^2\tau+\tau^2 t}{\lambda_5^2(s)}. \tag{4.2.21}$$

Normal vector field of the $\overline{M}^5(s,t)$ surface is given by

$$n(s,t) = xT(s) + yN(s) + zB(s). \tag{4.2.22}$$

Since $n \perp \overline{M}_s^5(s,t)$ and $n \perp \overline{M}_t^5(s,t)$, the vector field of the surface $\overline{M}^5(s,t)$ is obtained as follows

$$n(s,t) = \frac{1}{h(s,t)}\left[\lambda_5^2(s)T(s) + \frac{\lambda_5(s)(2t\lambda_5'(s)-1)}{t^2\lambda_5^2(s)+\tau t}N(s) + B(s)\right] \tag{4.2.23}$$

where

$$h^2(s,t) = 2\lambda_5^2(s) + \lambda_5^2(s)\frac{(2t\lambda_5'(s)-1)^2}{(t^2\lambda_5^2(s)+\tau t)^2}. \tag{4.2.24}$$

The normal vector $n(s,t)$ is spacelike vector field. After computations, we can easily obtain coefficients e, f and g of the second fundamental form of $\overline{M}^5(s,t)$ as

$$e = \frac{1}{h(s,t)(t^2\lambda^2(s)+\tau t)}\begin{bmatrix} (t^2\lambda^2(s)+\tau t)(-\lambda^2(s)\tau t + 2t\lambda''(s) + \frac{\tau^2 t - 2t\lambda'(s)}{\lambda(s)}) \\ +(2t\lambda'(s)-1)(\lambda(s)+2\lambda'(s)\lambda(s)t + \tau' t - \frac{2\tau t\lambda'(s)}{\lambda(s)} \end{bmatrix}, \tag{4.2.25}$$

$$f = \frac{1}{h(s,t)(t^2\lambda^2(s)+\tau t)}(\lambda^2(s)+\tau)(2t\lambda'(s)-1), \tag{4.2.26}$$

$$g = 0. \tag{4.2.27}$$

Thus, the Gaussian curvature K and mean curvature H of the surface $\overline{M}^5(s,t)$ are

$$K = -\frac{(\lambda_5^2(s)+\tau)^2(2t\lambda_5'(s)-1)^2(1+2t\lambda_5'(s)+2(\lambda_5'(s)t)^2+\lambda_5^4(s)t^4+2\lambda_5^2(s)t^2\tau+\tau^2 t)}{(t^2\lambda_5^2(s)+\tau t)(2t^2\lambda_5^2(s)+\tau t+2t\lambda_5'(s)-1)}, \tag{4.2.28}$$

$$H = -\frac{(-e\lambda_5(s)+2f)\lambda_5(s)}{2(1+2t\lambda_5'(s)+2(\lambda_5'(s)t)^2+\lambda_5^4(s)t^4+2\lambda_5^2(s)t^2\tau+\tau^2 t)} \tag{4.2.29}$$

respectively. □

4.3. Surface with unit spacelike generating direction lie on span {T,N}

In this subsection, we will examine the surface with the null Cartan base curve $\alpha : I \to \mathbb{E}_1^3$ and whose generating line is non-constant unit spacelike vector field spanning by the Cartan frame fields T and N. The surface has the following parametric representation

$$M^6(s,t) = \alpha(s) + t(\lambda_6(s)T(s) + N(s)) \tag{4.3.1}$$

where $\lambda_6 : I \to \mathbb{R}$ is a differentiable function.

Theorem 4.5. *Gaussian curvature K and mean curvature H of $M^6 = M^6(s,t)$ are*

$$K(s,t) = -\varepsilon\frac{1}{t^4 p^4(s,t)}, \tag{4.3.2}$$

$$H(s,t) = -\frac{1}{2t^2p^3(s,t)}\begin{bmatrix} -2\lambda_6(s) - 3\lambda_6'(s)\lambda_6(s)t - 2\lambda_6(s)\tau(s)t \\ +\lambda_6''(s)t + \lambda_6^3(s)t + \tau'(s)t \end{bmatrix} \tag{4.3.3}$$

where

$$p^2(s,t) = \frac{2}{t} + 2\lambda_6'(s) + 2\tau(s) - \lambda_6^2(s) \neq 0 \tag{4.3.4}$$

and $\varepsilon^2 = \pm 1$.

Proof. We have found the coefficients of first fundamental forms of the $M^6(s,t)$ as

$$E = -t^2(\frac{2}{t} + 2\lambda_6'(s) + 2\tau(s) - \lambda_6^2(s)), \tag{4.3.5}$$

$$F = 0, \tag{4.3.6}$$

$$G = 1. \tag{4.3.7}$$

Therefore, we obtain

$$EG - F^2 = -t^2(\frac{2}{t} + 2\lambda_6'(s) + 2\tau(s) - \lambda_6^2(s)). \tag{4.3.8}$$

Normal vector field of M^6 surface is given by

$$n(s,t) = xT(s) + yN(s) + zB(s). \tag{4.3.9}$$

Since $n \perp M_s^6(s,t)$ and $n \perp M_t^6(s,t)$, the vector field of the surface $M^6(s,t)$ is obtained as follows

$$n(s,t) = \varepsilon \frac{1}{p(s,t)}[(1+\lambda_6'(s)+t\tau(s)-t\lambda_6^2(s))T(s)-\lambda_6(s)N(s)+B(s)] \tag{4.3.10}$$

where

$$p^2(s,t) = \frac{2}{t}+2\lambda_6'(s)+2\tau(s)-\lambda_6^2(s) \tag{4.3.11}$$

and $\varepsilon^2 = \pm 1$. The normal vector $n(s,t)$ can be spacelike or timelike vector field depending on the value of $p^2(s,t)$. We can easily obtain coefficients e, f and g of the second fundamental form of $M(s,t)$ as

$$e = \varepsilon \frac{1}{p(s,t)} \begin{bmatrix} -2\lambda_6(s)-3\lambda_6'(s)\lambda_6(s)t-2\lambda_6(s)\tau(s)t \\ +\lambda_6''(s)t+\lambda_6^3(s)t+\tau'(s)t \end{bmatrix}, \tag{4.3.12}$$

$$f = -\varepsilon \frac{1}{tp(s,t)}, \tag{4.3.13}$$

$$g = 0. \tag{4.3.14}$$

Thus, the Gaussian curvature K and mean curvature H of the surface $M^6(s,t)$ are

$$K = -\varepsilon \frac{1}{t^4p^4(s,t)}, \tag{4.3.15}$$

$$H = -\frac{1}{2t^2p^3(s,t)} \begin{bmatrix} -2\lambda_6(s)-3\lambda_6'(s)\lambda_6(s)t-2\lambda_6(s)\tau(s)t \\ +\lambda_6''(s)t+\lambda_6^3(s)t+\tau'(s)t \end{bmatrix}, \tag{4.3.16}$$

respectively. □

Remark 4.1. If $\frac{2}{t}+2\lambda_6'(s)+2\tau(s)-2\lambda_6^2(s)=0$, we obtained that $EG-F^2=0$. So the surface M^6 is defined as degenerate surface. Therefore we assume that $\frac{2}{t}+2\lambda_6'(s)+2\tau(s)-2\lambda_6^2(s) \neq 0$ for above theorem.

Acknowledgments

The authors would like to thank Prof. Dr. E. Nešović for her encouragement and support.

References

[1] Akamine, S.: *Behavior of the Gaussian curvature of timelike minimal surfaces with singularities*. Hokkaido Mathematical Journal. 48 (3), 537-568 (2019). https://doi.org/10.14492/hokmj/1573722017
[2] Aslan, S., Bekar, M., Yaylı, Y.: *Ruled surfaces constructed by quaternions*. Journal of Geometry and Physics. 161, (2021). https://doi.org/10.1016/j.geomphys.2020.104048
[3] Clelland, J.N.: *Totally quasi-umbilic timelike surfaces in* $\mathbb{R}_1^2$. Asian J. Math. 16(1), 189-208 (2012).
[4] Erdoğdu, M., Yavuz, A.: *On Backlund Transformation and Motion of Null Cartan Curves*.International Journal of Geometric Methods in Modern Physics. 19 (1), (2022). https://doi.org/10.1142/S0219887822500141
[5] Honda, K., Inoguchi, J.I.: *Deformation of Cartan framed null curves preserving the torsion*. Differ. Geom. Dyn. Syst. 5 (1), 31-37 (2003).
[6] Kim, Y.H., Yoon, D.W.: *Classification of ruled surfaces in Minkowski 3-spaces*. Journal of Geometry and Physics. 49 (1), 89-100 (2004). https://doi.org/10.1016/S0393-0440(03)00084-6
[7] Kim, Y.H., Yoon, D.W.: *On non-developable ruled surfaces in Lorentz-Minkowski 3-spaces*. Taiwanese Journal of Mathematics. 11 (1), 197-214 (2007). https://doi.org/10.11650/twjm/1500404646
[8] Liu, H.: *Characterizations of ruled surfaces with lightlike ruling in Minkowski 3-space*. Results in Mathematics. 56 (1), (2009). https://doi.org/10.1007/s00025-009-0431-8
[9] Nolasco, B., Pacheco, R.: *Evolutes of plane curves and null curves in Minkowski 3-space*. Journal of Geometry. 108 (1), 195-214 (2017). https://doi.org/10.1007/s00022-016-0334-2
[10] Wang, Z., Pei, D.: *Singularities of ruled null surfaces of the principal normal indicatrix to a null Cartan curve in de Sitter 3-space*. Physics Letters B. 689 (2-3), 101-106 (2010). https://doi.org/10.1016/j.physletb.2010.04.050
[11] Yaylı, Y.: *On the motion of the Frenet vectors and spacelike ruled surfaces in the Minkowski 3-space*. Mathematical and Computational Applications. 5 (1), 49-55 (2000).
[12] Yüca, G.: *Kinematics applications of dual transformations*. Journal of Geometry and Physics. 163, (2021). https://doi.org/10.1016/j.geomphys.2021.104139

Affiliations

AYŞE YAVUZ
ADDRESS: Department of Mathematics and Science Education, Necmettin Erbakan University, Konya TURKEY
E-MAIL: ayasar@erbakan.edu.tr.
ORCID ID:0000-0002-0469-3786

MELEK ERDOĞDU
ADDRESS: Department of Mathematics and Computer Sciences, Necmettin Erbakan University, Konya, TURKEY
E-MAIL: merdogdu@erbakan.edu.tr
ORCID ID:0000-0001-9610-6229