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Dual Covariant Derivative on Time Scales

Hatice Kuşak Samancı[1]

[1] *Department of Mathematics, Faculty of Science and Arts, Bitlis Eren University, Bitlis, Turkey*

Abstract

The covariant derivative is a kind of derivative along tangent vectors of a curve or a surface. The covariant derivative has many applications in physics, kinematics, robotics, machine engineering, and other scientific areas. Additionally, a dual vector or screw-vector in the dual space is an important tool widely used in kinematic and robotic studies to represent the space motion including the rotation and translation transformations. The aim of this paper is to introduce the dual covariant derivative on time scales defined as an arbitrary nonempty closed subset of the real numbers and to achieve unifying discrete and continuous forms. Consequently, some properties are analyzed.

Keywords: *Time Scales, Dual Space, Dual Covariant Derivative.*
2010 Mathematics Subject Classification: *53A04, 51B20.*

1. Introduction

W. K. Clifford (1873) first proposed the dual numbers which is defined by $\alpha = (\eta, \mu) = \eta + \varepsilon\mu$ combined with the real unit 1 and the non-zero dual unit ε, with $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. The dual numbers therefore constitute the elements of the set

$$\mathbb{D} = \left\{ \alpha = \eta + \varepsilon\mu \mid \eta, \mu \in \mathbb{R}, \text{ and } \varepsilon^2 = 0,\ \varepsilon \neq 0, \right\},$$

generated by 1 and ε. The addition operation in the set $\mathbb{D}$ is '+' and defined by $\alpha_1 + \alpha_2 = (\eta_1 + \eta_2, \mu_1 + \mu_2) = (\eta_1 + \eta_2) + \varepsilon(\mu_1 + \mu_2)$, while the multiplication operation in the set $\mathbb{D}$ is '.' and described by $\alpha_1.\alpha_2 = \eta_1.\eta_2 + \varepsilon(\eta_1.\mu_2 + \eta_2.\mu_1)$. The multiplication operation is commutative, associative and distributive over the addition. Clifford showed that since dual numbers do not have any inverse elements, they form an algebra and not a field. Therefore, the divisors of zero in the algebra of dual numbers are $\varepsilon\mu(\mu \in \mathbb{R})$. No $\varepsilon\mu$ numbers have an inverse in the algebra of the dual numbers. The conjugate of the dual number $\alpha = \eta + \varepsilon\mu$ is represented by $\bar{\alpha}$ and defined by $\bar{\alpha} = \eta - \varepsilon\mu$; hence, $\alpha.\bar{\alpha} = \eta^2$. The division of the dual number "$\alpha_1 = \eta_1 + \varepsilon\mu_1$" by the dual number "$\alpha_2 = \eta_2 + \varepsilon\mu_2$" becomes $\frac{\alpha_1}{\alpha_2} = \frac{\alpha_1.\overline{\alpha_2}}{\alpha_2.\overline{\alpha_2}} = \frac{\eta_1}{\eta_2} + \varepsilon\frac{\eta_2\mu_1 - \eta_1\mu_2}{{\eta_2}^2}$ where $\eta_2 \neq 0$. Hence, if $\eta_2^2 \neq 0$, the division $\frac{\alpha_1}{\alpha_2}$ becomes possible and unambiguous. The modulus of the dual number α is $|\alpha|$ and defined by $|\alpha|^2 = \alpha.\bar{\alpha} = \eta^2$. In other words, for a dual number "$\alpha = \eta + \varepsilon\mu$", the modulus $|\alpha|$ is replaced by η to allow the modulus of the dual number to be positive, zero, or negative. The dual plane is defined by the set of all dual number $\alpha \in \mathbb{D}$. The distance between two points of the dual plane as α and α_1 is denoted by $d(\alpha, \alpha_1)$ and satisfies the conditions $d(\alpha, \alpha_1) = |\alpha_1 - \alpha| = |\eta_1 - \eta|$ or $d^2(\alpha, \alpha_1) = (\alpha_1 - \alpha)(\overline{\alpha_1} - \overline{\alpha})$. In 1891, E. Study regarded using associative algebra as an ideal way to describe the group of motions of three-dimensional space [5,6,8,11,13]. Yaglom (1969) and Veldkamp (1976) studied on the dual numbers for getting more details on the other algebraic properties, see also in [15, 16]. The Taleshian (2009) studied on the dual covariant derivative in the dual space [14]. In addition, Messelmi (2013) developed a theory concerning the holomorphic dual functions using the dual-variable functions in the dual space [12]. The researcher also offered other properties that can be used in the analysis of the dual functions. On the other hand, the time scale calculus theory, which is of great importance and use to the unification of discrete and indiscrete analyses was developed by Hilger (1990) and Aulbach and Hilger (1990) at an earlier date [2]. The preliminaries for the timescale can be established by referring to [3,7]. The paper published by Bohner and Guseinov focused on the complex functions on the time scales [4]. By taking $\mathbb{T}_1$ and $\mathbb{T}_2$ as the time scales, $\mathbb{T}_1 + i\mathbb{T}_2$ was introduced a time scale complex plane. Then the classical Cauchy-Riemann equations on the time scales were derived and the complex-valued functions with a complex time scale variable were investigated. Aktan et al. introduced the directional nabla derivative on n-dimensional time scales [1]. Afterwards, Kuşak Samancı (2011) and (2018) introduced the concept of the delta nature connection in the other words the delta covariant derivative and the dual-valued functions on time scales and gave some definitions and theorems including the limit, derivative, partial differentiation and Cauchy-Riemann equation of the dual-variable functions on the time scales [9,10]. In our paper, we define for the first time the covariant derivation of dual-valued functions parametrized by the products of two time scales. We think that this study can lead to the emergence of new fields of study in geometry and physics.

Email addresses: **hkusak@beu.edu.tr (Hatice Kuşak Samancı)**

2. Preliminaries

2.1. Some Basics of Time Scale Calculus

Now, we will give some preliminaries about the time scale concept. Let $\mathbb{T}$ be an arbitrary timescale. The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ are denoted by $\sigma(t)=\inf\{s\in\mathbb{T}: s\rangle t, \forall t\in\mathbb{T}\}$ and $\rho(t)=\sup\{s\in\mathbb{T}: s\langle t\ ,\ \forall t\in\mathbb{T}\}$, respectively. If $\mathbb{T}$ have a left-scattered maximum t_1 and a right-scattered minimum t_2, then the sets will be $\mathbb{T}^{\kappa}=\mathbb{T}-\{t_1\}$, (otherwise $\mathbb{T}^{\kappa}=\mathbb{T}$) and $\mathbb{T}_{\kappa}=\mathbb{T}-\{t_2\}$ (otherwise $\mathbb{T}_{\kappa}=\mathbb{T}$), respectively. The delta-derivative of the function$f$ on the time scale is introduced by $f^{\Delta}(t)=\lim\limits_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}$ for $t\in\mathbb{T}^{\kappa}$. Suppose that the functions $f,g:\mathbb{T}\to\mathbb{R}$ are delta differentiable at the time scale variable $t\in\mathbb{T}^{\kappa}$. Then the equations satisfy following equations:

1) $(\alpha f+\beta g)^{\Delta}(t)=\alpha f^{\Delta}(t)+\beta g^{\Delta}(t)$,
2) $(fg)^{\Delta}(t)=g(t)f^{\Delta}(t)+f(\sigma(t))g^{\Delta}(t)=g^{\Delta}(t)f(t)+f^{\Delta}(t)g(\sigma(t))$,
3) $\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{g(t)f^{\Delta}(t)-g^{\Delta}(t)f(t)}{g(\sigma(t))g(t)},\ \ g\neq 0$,

for $\alpha,\beta\in\mathbb{R}$ [2,3]. On the other hand, we will give some basic concepts of the dual variable functions. Let the set $\mathbb{D}$ be a dual plane and Ω be an open subset of the dual plane $\mathbb{D}$. Additionally, the set O is called the generator of **Ω,** if there exists a subset $O\subset\mathbb{R}$ such that $\Omega=O\times\mathbb{R}$. Therefore, a dual-variable function f which is a mapping from a subset $\Omega\subset\mathbb{D}$ to $\mathbb{D}$. The dual-variable function $f:\Omega\to\mathbb{D}$, $z_0=x_0+\varepsilon y_0\to f(z_0)$ is called a homogeneous dual function if the function $f(real(z))\in\mathbb{R}$. The -variable function f is continuous at $z_0$if $\lim\limits_{z\to z_0}f(z)=f(z_0)$. Moreover, the function is continuous in $\Omega\subset\mathbb{D}$ if it is continuous at every point of **Ω.** The function f is called to be a differentiable function at $z_0=x_0+\varepsilon y_0$, since the limit $\frac{df}{dz}(z_0)=\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ is satisfied. Since the function f is a differentiable function for all points in a neighborhood of the point z_0 then the dual-variable function f is called holomorphic at z_0. Additionally, if it is holomorphic at every point of Ω, then the dual-variable function f is holomorphic in $\Omega\subset\mathbb{D}$. Assume that the dual-variable functions f and g are differentiable at the dual point $z\in\mathbb{D}$ for $c\in\mathbb{D}$, $n\in$ Z. Then following equations

1) $\frac{d(f+cg)}{dz}=\frac{df}{dz}+c\frac{dg}{dz}$
2) $\frac{d(f\cdot g)}{dz}=\frac{df}{dz}g+f\frac{dg}{dz}$
3) $\frac{d\left(\frac{f}{g}\right)}{dz}=\frac{\frac{df}{dz}g-f\frac{dg}{dz}}{g^2},\ \ g\neq 0$
4) $\frac{d(hog)}{dz}=\frac{dh}{dz}(g)\frac{dg}{dz}$

are satisfied using the differentiation of the dual-variable functions. On the other hand, the dual-variable function f in $\Omega\subset\mathbb{D}$ can be written with its real and dual parts as $f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$. Moreover, the function f is called holomorphic in $\Omega\subset\mathbb{D}$ if and only if the differentiation of f is provided $\frac{df}{dz}=\frac{\partial f}{\partial x}=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial x}\varepsilon$. Let the dual variable function can be written as$f=\varphi+\varepsilon\psi$ and assume that the partial derivatives of f exist. Then f is holomorphic in $\Omega\subset\mathbb{D}$ if and only if its partial differentiations hold $\varepsilon\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}$, or $\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y}$ and $\frac{\partial\varphi}{\partial y}=0$. The other properties of the dual-variable functions can be obtained from the reference [12]. Now, we will give some information concerning the dual covariant derivative, see in [14]. Assume that $\zeta=(\zeta_1,...,\zeta_n)$ and $\xi=(\xi_1,...,\xi_n)$be two vector fields at the point Q in $\mathbb{R}^n$ with C^{∞}. Therefore, each ξ_i is a C^{∞} real valued function on the domain of ξ which includes Q. As we know that the covariant derivative of ξ in the direction ζ is $\nabla_{\zeta}\xi|_Q=(\zeta_Q[\xi_1],...,\zeta_Q[\xi_n])$ where ∇ is denoted by the covariant derivative operator. Now, suppose that $\overline{\zeta}=\left(\overline{\zeta}_1,...,\overline{\zeta}_n\right)$ and $\overline{\xi}=(\overline{\xi}_1,...,\overline{\xi}_n)$ be the dual vectors at Q in $\mathbb{D}^n$ where each ξ_i is a C^{∞} dual-valued function on the domain of $\overline{\xi}$ which contains the point Q. The dual covariant derivative of $\overline{\xi}$ in the direction $\overline{\zeta}$ is defined by $\bar{\nabla}_{\overline{\zeta}}\overline{\xi}|_P=\left(\overline{\zeta}_P[\overline{\xi}_1],...,\overline{\zeta}_P[\overline{\xi}_n]\right)$ where $\bar{\nabla}$ is the dual covariant derivative operator. Furthermore, the dual gradient operator $\bar{\nabla}=\nabla+\varepsilon\nabla^*$ is a kind of a dual covariant derivative where ∇ and ∇^* denote the gradient derivatives. Let $\overline{\zeta}$ and $\overline{\xi}$are two dual vector fields at Q in $\mathbb{D}^n$ and let $\overline{\xi}=(\overline{\xi}_1,...,\overline{\xi}_n)$ and $\overline{\xi}_i$ is a C^{∞} dual function on the domain of $\bar{Y}$ which includes Q. The dual covariant derivative of $\overline{\xi}$ in the direction $\overline{\zeta}$ is the dual vector such that

$$
\begin{aligned}
&\bar{\nabla}_{\overline{\zeta}}\overline{\xi}|_Q=(\nabla+\varepsilon\nabla^*)_{\overline{\zeta}}\overline{\xi}\\
&=\left(\left\langle\overline{\zeta},(\nabla+\varepsilon\nabla^*)\overline{\xi}_1\right\rangle,...,\left\langle\bar{X},(\nabla+\varepsilon\nabla^*)\overline{\xi}_n\right\rangle\right)|_Q\\
&=\left(\sum_{i=1}^{n}\overline{\zeta}_i(\nabla\overline{\xi}_1)+\varepsilon\left(\overline{\zeta}_i\left(\nabla\overline{\xi}_1^*+\nabla^*\overline{\xi}_1\right)+\nabla\overline{\xi}_1(\overline{\zeta}_i^*)\right)\right)|_Q\\
&,...,\\
&\left(\sum_{i=1}^{n}\overline{\zeta}_i(\nabla\overline{\xi}_n)+\varepsilon\left(\overline{\zeta}_i\left(\nabla\overline{\xi}_n^*+\nabla^*\overline{\xi}_n\right)+\nabla\overline{\xi}_n(\overline{\zeta}_i^*)\right)\right)|_Q
\end{aligned}
$$

see in [14]. Moreover, in for any two points $A,B\in E$, and $s:E\to V$ a screw is a vector field which admits some $s\in V$ $s(A)-s(B)=s(A-B)$ [11]. The function $F(\zeta)=F(\zeta+\varepsilon\zeta^*)=f(\zeta,\zeta^*)+\varepsilon g(\zeta,\zeta^*)$ is a dual-variable function where$f(\zeta,\zeta^*)$ and $g(\zeta,\zeta^*)$ are real functions of the two real variables ζ and ζ^*. Then the dual function $F(\zeta)$ can be written by $F(\zeta)=\frac{\partial f}{\partial\zeta}+\varepsilon\left(\zeta^*\frac{\partial^2 f}{\partial\zeta^2}+\frac{\partial f^*}{\partial\zeta}\right)$ and the derivative of dual variable function can be calculated by $\frac{\partial F(\zeta)}{\partial\zeta}=\frac{\partial f}{\partial\zeta}+\varepsilon\left(\zeta^*\frac{\partial^2 f}{\partial\zeta^2}+\frac{\partial f^*}{\partial\zeta}\right)$ [5].

3. Main Results

In this section we will give a definition of the dual covariant derivative on the time scales for the first time. Assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are two arbitrary time scales. The addition set

$\mathbb{T}_1+\varepsilon\mathbb{T}_2:\{z=x+\varepsilon y : x\in\mathbb{T}_1, y\in\mathbb{T}_2\},$

is called the timescale dual plane where $\varepsilon\neq 0$ and $\varepsilon^2=0$ is the dual unit. Using the addition set the dual variable function $f:\mathbb{T}_1+\varepsilon\mathbb{T}_2\to\mathbb{D}$ is defined by
$f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$ for any dual variable $z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$,
where the real and dual part of the dual variable function f are $\varphi:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ and $\psi:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$, respectively. If the forward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$ are σ_1 and σ_2, respectively, the equations $z^{\sigma_1}=\sigma_1(t)+\varepsilon y$ and $z^{\sigma_2}=x+\varepsilon\sigma_2(y)$ are provided for the dual-variable $z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$, Similarly, if ρ_1 and ρ_2 are the backward jump operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, the equations $z^{\rho_1}=\rho_1(t)+\varepsilon y$ and $z^{\rho_2}=x+\varepsilon\rho_2(y)$ are satisfied for $z=x+\varepsilon y\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$.

Theorem 3.1. *Suppose f and g are differentiable at the dual number $z\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$ on the time scales, then the following conditions are satisfied for the time scale dual number $z\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$.*

1) $(f+cg)^\Delta(z)=f^\Delta(z)+cg^\Delta(z),$
2) $(fg)^\Delta(z)=f^\Delta(z)g(z)+f(\sigma(z))g^\Delta(z)=f(z)g^\Delta(z)+f^\Delta(z)g(\sigma(z)),$
3) $\left(\frac{f}{g}\right)^\Delta(z)=\frac{f^\Delta(z)g(z)-f(z)g^\Delta(z)}{g(z)g(\sigma(z))}\quad g\neq 0\ \ for\ \ c\in\mathbb{D},\ n\in\mathrm{Z}.$

Proof. From the properties of the delta differentiation, we get the following results ☐

1) $(f+cg)^\Delta(z)=\lim\limits_{z\to z_0}\frac{(f+cg)(\sigma(z))-(f+cg)(z_0)}{\sigma(z)-z_0}$
$=\lim\limits_{z\to z_0}\frac{[f(\sigma(z))-f(z_0)]+c.[g(\sigma(z))-g(z_0)]}{\sigma(z)-z_0}$
$=\lim\limits_{z\to z_0}\frac{[f(\sigma(z))-f(z_0)]}{\sigma(z)-z_0}+\lim\limits_{z\to z_0}\frac{[g(\sigma(z))-g(z_0)]}{\sigma(z)-z_0}$
$=f^\Delta(z)+cg^\Delta(z)$
2) $(f\cdot g)^\Delta(z)=\lim\limits_{z_0\to z}\frac{f\cdot g(\sigma(z))-f\cdot g(z_0)}{\sigma(z)-z_0}$
$=\lim\limits_{z_0\to z}\frac{f\cdot g(\sigma(z))-f\cdot g(z_0)}{\sigma(z)-z_0}+\lim\limits_{z_0\to z}\frac{f(\sigma(z)).g(z_0)-f(\sigma(z)).g(z_0)}{\sigma(z)-z_0}$
$=f(\sigma(z)).\lim\limits_{z_0\to z}\frac{g(\sigma(z))-g(z_0)}{\sigma(z)-z_0}+g(z).\lim\limits_{z_0\to z}\frac{f(\sigma(z))-f(z_0)}{\sigma(z)-z_0}$
$=f(\sigma(z)).g^\Delta(z)+g(z).f^\Delta(z)$
3) *The proof is obvious.*

Definition 3.2. *We say that a dual-valued function $f:\mathbb{T}_1+\varepsilon\mathbb{T}_2\to\mathbb{D}$ is a delta differentiable function at point $z_0=x_0+\varepsilon y_0\in\mathbb{T}_1^\kappa+\varepsilon\mathbb{T}_2^\kappa$ if there exist a dual number $A=A_1+\varepsilon A_2$ such that*

$f(z_0)-f(z)=A(z_0-z)+\alpha(z_0-z)$
$f(z_0^{\sigma_1})-f(z)=A(z_0^{\sigma_1}-z)+\beta(z_0^{\sigma_1}-z)$
$f(z_0^{\sigma_2})-f(z)=A(z_0^{\sigma_2}-z)+\gamma(z_0^{\sigma_2}-z)$

for all $z\in U_\delta(z_0)$ where $U_\delta(z_0)$ is a $\delta-$ neighborhood of z_0 in $\mathbb{T}_1+\varepsilon\mathbb{T}_2$. The coefficients $\alpha=\alpha(z_0,z)$, $\beta=\beta(z_0,z)$ and $\gamma=\gamma(z_0,z)$ are equal to zero at $z_0=z$, i.e. $\lim\limits_{z\to z_0}\alpha(z_0,z)=\lim\limits_{z\to z_0}\beta(z_0,z)=\lim\limits_{z\to z_0}\gamma(z_0,z)=0$ where they defined for $z\in U_\delta(z_0)$. Then the number A is called the delta differentiation of the dual-variable function f at z_0 and is denoted by $f^\Delta(z_0)$.

Theorem 3.3. *Assume that the function $f:\mathbb{T}_1+\varepsilon\mathbb{T}_2\to\mathbb{D}$, $f(z)=\varphi(x,y)+\varepsilon\psi(x,y)$ is delta differentiable and the functions $\varphi(x,y)$ and $\psi(x,y)$ are completely delta differentiable at the dual number $z_0=x_0+\varepsilon y_0\in\mathbb{T}_1^\kappa+\varepsilon\mathbb{T}_2^\kappa$. The dual-variable function is satisfied the Cauchy-Riemann equations $\frac{\Delta\varphi}{\Delta_1 x}=\frac{\Delta\psi}{\Delta_2 y}$ and $\frac{\Delta\varphi}{\Delta_2 y}=0$ at the dual number $z_0=x_0+\varepsilon y_0$. Therefore, the derivative formula $f^\Delta(z_0)$ can be denoted by $f^\Delta(z_0)=\frac{\Delta\varphi}{\Delta_1 x}+\varepsilon\frac{\Delta\psi}{\Delta_1 x}$.*

Lemma 3.4. *Suppose that f be a dual function in $\Omega\subset\mathbb{D}$, which can be denoted in terms of its real and dual parts as $f=\varphi+\varepsilon\psi$ and the partial derivatives of f exist on product of two time scales. Then*

1. *the dual variable-function f is holomorphic in $\Omega\subset\mathbb{D}$ necessary and sufficient condition its partial derivatives on the time scales satisfy $\varepsilon\frac{\partial f}{\Delta_1 x}=\frac{\partial f}{\Delta_2 y}$ on timescale.*
2. *the dual variable function f is holomorphic in the subset $\Omega\subset\mathbb{D}$ necessary and sufficient condition the below formula provides $\frac{\partial\varphi}{\Delta_1 x}=\frac{\partial\psi}{\Delta_2 y}$, $\frac{\partial\varphi}{\Delta_2 y}=0$ on the time scales.*

Theorem 3.5. *The function f is holomorphic in the open subset $\Omega\subset\mathbb{D}$, if and only if there exist a pair of real functions φ and κ, such that $\varphi\in C^{\sigma_1}(P_x(\Omega))$. $\frac{d\varphi}{\Delta_1 x}$ is delta1-differentiable in $P_x(\Omega)$ and κ is delta1-differentiable in $P_x(\Omega)$, where P_x is the first projection, so that the dual variable-function f is shown explicitly*

$$f(z)=\varphi(x)+(\frac{d\varphi}{\Delta_1 x}y+\kappa(x))\varepsilon,\quad \forall z\in\Omega. \tag{3.1}$$

Remark 3.6. *If, in particular, f is an homogeneous function, (3.1) gives $\kappa\equiv 0$. Thus $f(z)=\varphi(x)+\varepsilon\frac{d\varphi}{\Delta_1 x}y$.*

Definition 3.7. *Let the real functions $\varphi:[a,b]\to\mathbb{T}_1$ and $\psi:[a,b]\to\mathbb{T}_2$ be continuous functions. A dual-variable function $z=\lambda(t)=\varphi(t)+\varepsilon\psi(t)$, for $t\in[a,b]\in\mathbb{T}$ is defined a dual curve on the timescale plane $\mathbb{T}_1+\varepsilon\mathbb{T}_2$.*

If the dual-variable function $f(z)$ is constant on the set $\mathbb{T}_1+\varepsilon\mathbb{T}_2$, then the real function $\varphi(x,y)$ is constant and $\psi(x,y)=0$. Moreover, the Cauchy-Riemann equations of the dual-variable functions $\frac{\partial\varphi}{\Delta_1 x}=0=\frac{\partial\psi}{\Delta_2 x}$ and $\frac{\partial\varphi}{\Delta_2 y}=0$.

If the dual-variable function is given by $f(z)=z=x+\varepsilon y=\varphi(x,y)+\varepsilon\psi(x,y)$ on $\mathbb{T}_1+\varepsilon\mathbb{T}_2$, then the functions $\varphi(x,y)=x$ and $\psi(x,y)=y$ functions satisfy the Cauchy-Riemann equations of the dual-variable functions $\frac{\partial\varphi}{\Delta_1 x}=1=\frac{\partial\psi}{\Delta_2 y}$ and $\frac{\partial\varphi}{\Delta_2 y}=0$. Since $f^{\Delta}(z_0)=\frac{\partial\varphi}{\Delta_1 x}+\varepsilon\frac{\partial\psi}{\Delta_1 x}=1+\varepsilon 0$, the derivation of $f(z)=z$ becomes $f^{\Delta}(z_0)=1$ for $z_0\in\mathbb{T}_1+\varepsilon\mathbb{T}_2$.

Consider the dual-variable function $f(z)=x^2+\varepsilon 2xy$ on the time scales $\mathbb{T}_1+\varepsilon\mathbb{T}_2$. Therefore, the real functions $\varphi(x,y)=x^2$ and $\psi(x,y)=2xy$ provide the equations $\frac{\partial\varphi}{\Delta_1 x}=x+\sigma_1(x), \frac{\partial\psi}{\Delta_2 y}=2x$ and $\frac{\partial\varphi}{\Delta_2 y}=0$. Because of the Cauchy-Riemann equations of the dual-variable functions, the conditions $x+\sigma_1(x)=2x$ and $\frac{\partial\varphi}{\Delta_2 y}=0$ are satisfied. Now we will give a new definition about the dual delta-covariant derivative on the time scales.

Let $\varepsilon\neq 0$, $\varepsilon^2=0$ be the dual unit. Then, n-dimensional dual time scale is defined by $\Lambda^n=(\mathbb{T}_{11}+\varepsilon\mathbb{T}_{12},...,\mathbb{T}_{n1}+\varepsilon\mathbb{T}_{n2})$ where

$$\mathbb{T}_{i1}+\varepsilon\mathbb{T}_{i2}=\{Z_i=x_i+\varepsilon y_i | x_i\in\mathbb{T}_{i1}, y\in\mathbb{T}_{i2},\ i=1,\dots,n\}.$$

Any function $F:\Lambda^n\to\mathbb{D}^n$ can be represented in the form of $F=f+\varepsilon f^*$ where $f:\mathbb{T}_{11}\times\ldots\times\mathbb{T}_{n1}\to\mathbb{R}$ is the real part of F and $f^*:\mathbb{T}_{12}\times\ldots\times\mathbb{T}_{n2}\to\mathbb{R}$ is the dual part of F. Furthermore, σ_{i1} and σ_{i2} are the forward jump operators, and also Δ_{i1} and Δ_{i2} are the delta derivatives for $\mathbb{T}_{i1}$ and $\mathbb{T}_{i2}$, respectively.

Definition 3.8. *Assume that $Z=(Z_1,\dots,Z_n)$ be a dual vector in $\mathbb{D}^n$ where $Z_i=x_i+\varepsilon x_i^*$ $i=1,2,\dots,n$. The dual gradient operator with respect to the dual vector $Z=(Z_1,\dots,Z_n)$ can be defined by $\nabla=(\frac{\partial}{\Delta_1 Z_1},\dots,\frac{\partial}{\Delta_1 Z_n})$. Since, $F=f+\varepsilon f^*$ is a dual-variable function in $\mathbb{D}^n$, then the gradient operator of Fis described by $\nabla F=(\frac{\partial F}{\Delta_1 Z_1},\dots,\frac{\partial F}{\Delta_1 Z_n})$. If we substitute the derivative of dual functions in above equation, then we get*

$$\nabla F=\left(\frac{\partial f}{\Delta_{11}x_1}+\varepsilon\left(x_1^*\frac{\partial^2 f}{\Delta_{11}x_1^2}+\frac{\partial f^*}{\Delta_{11}x_1}\right),\dots,\frac{\partial f}{\Delta_{n1}x_n}+\varepsilon\left(x_n^*\frac{\partial^2 f}{\Delta_{n1}x_n^2}+\frac{\partial f^*}{\Delta_{n1}x_n}\right)\right).$$

Definition 3.9. *Let $Z=(Z_1,\dots,Z_n)$ be a dual vector in $\mathbb{D}^n$ where $Z_i=x_i+\varepsilon y_i$ and $F=(F_1,\dots,F_n)$ be a dual-variable function in $\mathbb{D}^n$ where $F_i=f_i+\varepsilon f_i^*$ for $i=1,2,\dots,n$.*

The dual delta-covariant derivative of $F=(F_1,\dots,F_n)$ with respect to the dual vector $Z=(Z_1,\dots,Z_n)$ is defined by

$$\frac{\partial F}{\Delta Z}(P)=\nabla_{Z^{\Delta}}F=\sum_{i=1}^{n}\frac{\partial F_i}{\Delta_i Z}\frac{\partial}{\Delta_i Z_i}(P)=(\langle\nabla F_1,Z_P\rangle,\dots,\langle\nabla F_n,Z_P\rangle)_P$$

where $\nabla F_i=\frac{\partial f_i}{\Delta_{11}x_1}+\varepsilon\left(x_1^*\frac{\partial^2 f_i}{\Delta_{11}x_1^2}+\frac{\partial f_i{}^*}{\Delta_{11}x_1}\right),\dots,\frac{\partial f_i}{\Delta_{n1}x_n}+\varepsilon\left(x_n^*\frac{\partial^2 f_i}{\Delta_{n1}x_n^2}+\frac{\partial f_i{}^*}{\Delta_{n1}x_n}\right)$, for $i=1,2,\dots,n$. Some special cases of the time scale as following:

1. *If we take all time scales as $\mathbb{T}_{i1}=\mathbb{T}_1$ and $\mathbb{T}_{i2}=\mathbb{T}_2$ for $i=1,2,\dots,n$, then the dual delta-covariant derivative will be*

$$\frac{\partial F}{\Delta Z}(P)=\begin{pmatrix}\left\langle\begin{pmatrix}\frac{\partial f_1}{\Delta_1 x_1}+\varepsilon\left(x_1^*\frac{\partial^2 f_1}{\Delta_1 x_1^2}+\frac{\partial f_1{}^*}{\Delta_1 x_1}\right)\\ ,\dots,\\ \frac{\partial f_1}{\Delta_1 x_n}+\varepsilon\left(x_n^*\frac{\partial^2 f_1}{\Delta_1 x_n^2}+\frac{\partial f_1{}^*}{\Delta_1 x_n}\right)_P\end{pmatrix},Z_P\right\rangle,\\ \vdots\\ \left\langle\begin{pmatrix}\frac{\partial f_n}{\Delta_1 x_1}+\varepsilon\left(x_1^*\frac{\partial^2 f_n}{\Delta_1 x_1^2}+\frac{\partial f_n{}^*}{\Delta_1 x_1}\right)\\ ,\dots,\\ \frac{\partial f_n}{\Delta_1 x_n}+\varepsilon\left(x_n^*\frac{\partial^2 f_n}{\Delta_1 x_n^2}+\frac{\partial f_n{}^*}{\Delta_1 x_n}\right)_P\end{pmatrix},Z_P\right\rangle\end{pmatrix}_P$$

2. *If we take all $\mathbb{T}_{i1}=\mathbb{T}_{i2}=\mathbb{T}$ for $i=1,2,\dots,n$, then the dual delta-covariant derivative will be*

$$\frac{\partial F}{\Delta Z}(P)=\begin{pmatrix}\left\langle\begin{pmatrix}\frac{\partial f_1}{\Delta x_1}+\varepsilon\left(x_1^*\frac{\partial^2 f_1}{\Delta x_1^2}+\frac{\partial f_1{}^*}{\Delta x_1}\right)\\ ,\dots,\\ \frac{\partial f_1}{\Delta x_n}+\varepsilon\left(x_n^*\frac{\partial^2 f_1}{\Delta x_n^2}+\frac{\partial f_1{}^*}{\Delta x_n}\right)_P\end{pmatrix},Z_P\right\rangle,\\ \vdots\\ \left\langle\begin{pmatrix}\frac{\partial f_n}{\Delta x_1}+\varepsilon\left(x_1^*\frac{\partial^2 f_n}{\Delta x_1^2}+\frac{\partial f_n{}^*}{\Delta x_1}\right)\\ ,\dots,\\ \frac{\partial f_n}{\Delta x_n}+\varepsilon\left(x_n^*\frac{\partial^2 f_n}{\Delta x_n^2}+\frac{\partial f_n{}^*}{\Delta x_n}\right)_P\end{pmatrix},Z_P\right\rangle\end{pmatrix}_P$$

3. If we take all $\mathbb{T}_{i1} = \mathbb{T}_{i2} = \mathbb{R}$ for $i = 1,2,\ldots,n$, then the dual delta-covariant derivative will be

$$\frac{\partial F}{\Delta Z}(P) = \begin{pmatrix} \left\langle \begin{pmatrix} \frac{\partial f_1}{\partial x_1} + \varepsilon\left(x_1^*\frac{\partial^2 f_1}{\partial x_1^2} + \frac{\partial f_1{}^*}{\partial x_1}\right) \\ ,\ldots, \\ \frac{\partial f_1}{\partial x_n} + \varepsilon\left(x_n^*\frac{\partial^2 f_1}{\partial x_n^2} + \frac{\partial f_1{}^*}{\partial x_n}\right)_P \end{pmatrix}, Z_P \right\rangle, \\ \vdots \\ \left\langle \begin{pmatrix} \frac{\partial f_n}{\partial x_1} + \varepsilon\left(x_1^*\frac{\partial^2 f_n}{\partial x_1^2} + \frac{\partial f_n{}^*}{\partial x_1}\right) \\ ,\ldots, \\ \frac{\partial f_n}{\partial x_n} + \varepsilon\left(x_n^*\frac{\partial^2 f_n}{\partial x_n^2} + \frac{\partial f_n{}^*}{\partial x_n}\right)_P \end{pmatrix}, Z_P \right\rangle \end{pmatrix}_P$$

Theorem 3.10. *Suppose F and G are differentiable at $Z \in \mathbb{D}^n$ and that $c \in \mathbb{D}$, $n \in Z$. The following equalities are satisfied.*

1) $\frac{\partial(F+cG)}{\Delta Z}(P) = \frac{\partial(F)}{\Delta Z}(P) + c\frac{\partial(G)}{\Delta Z}(P)$

2) $\frac{\partial(FG)}{\Delta Z}(P) = \frac{\partial(F)}{\Delta Z}G(P) + F(\sigma(Z))(P)\frac{\partial(G)}{\Delta Z}(P) = F(P)\frac{\partial(G)}{\Delta Z} + \frac{\partial(F)}{\Delta Z}(P)G(\sigma(Z))(P)$

3) $\frac{\partial\left(\frac{F}{G}\right)}{\Delta Z}(P) = \frac{\frac{\partial(F)}{\Delta Z}(P)G(Z) - F(Z)\frac{\partial(G)}{\Delta Z}(P)}{G(Z)G(\sigma(Z))}, \ G \neq 0$

Proof. The proof is obvious from the delta derivative properties. □

4. Numeric Example

Let $F = (F_1,\ldots,F_n) = (f_1 + \varepsilon f_1^*, f_2 + \varepsilon f_2^*)$ be a dual function with the components $f_1 = 3x_1 + x_2,\ f_1^* = x_1 - 2x_2,\ f_2 = x_1^2 + x_2^2,\ f_2^* = 2x_1^2 - x_2^2$. If the dual point is given by $Z = (Z_1, Z_2) = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*) = (2+3\varepsilon, 4+5\varepsilon)$, then we will compute the dual covariant derivative $\frac{\partial F}{\Delta Z}(P) = \nabla_{Z^\Delta}F$.

$$\begin{aligned}
\frac{\partial F}{\Delta Z}(P) &= \nabla_{Z^\Delta}F = (\langle \nabla F_1, Z_P\rangle, \langle \nabla F_2, Z_P\rangle)_P \\
&= \begin{pmatrix} \left\langle \begin{pmatrix} \frac{\partial f_1}{\partial x_1} + \varepsilon\left(x_1^*\frac{\partial^2 f_1}{\partial x_1^2} + \frac{\partial f_1{}^*}{\partial x_1}\right), \\ \frac{\partial f_1}{\partial x_2} + \varepsilon\left(x_2^*\frac{\partial^2 f_1}{\partial x_2^2} + \frac{\partial f_1{}^*}{\partial x_2}\right) \end{pmatrix}, Z_P \right\rangle, \\ \left\langle \begin{pmatrix} \frac{\partial f_2}{\partial x_1} + \varepsilon\left(x_1^*\frac{\partial^2 f_2}{\partial x_1^2} + \frac{\partial f_2{}^*}{\partial x_1}\right), \\ \frac{\partial f_2}{\partial x_2} + \varepsilon\left(x_2^*\frac{\partial^2 f_2}{\partial x_2^2} + \frac{\partial f_2{}^*}{\partial x_2}\right) \end{pmatrix}, Z_P \right\rangle \end{pmatrix} \\
&= \begin{pmatrix} \left\langle \begin{pmatrix} 3 + \varepsilon(x_1^* + 0 + 1), \\ 1 + \varepsilon(x_2^*.0 + (-2)) \end{pmatrix}, Z_P \right\rangle, \\ \left\langle \begin{pmatrix} (2x_1 + \varepsilon(x_1^* + 2 + 4x_1), \\ 2x_2 + \varepsilon(2x_2^* - 2x_2)) \end{pmatrix}, Z_P \right\rangle \end{pmatrix} \\
&= (\langle(3+\varepsilon, 1-2\varepsilon), Z_P\rangle, \langle(4+14\varepsilon, 8+2\varepsilon), Z_P\rangle) \\
&= \begin{pmatrix} \langle(3+\varepsilon, 1-2\varepsilon), (2+\varepsilon 3, 4+5\varepsilon)\rangle, \\ \langle(4+14\varepsilon, 8+2\varepsilon), (2+\varepsilon 3, 4+5\varepsilon)\rangle \end{pmatrix} \\
&= (10+8\varepsilon, 40+88\varepsilon).
\end{aligned}$$

On the other hand we will compute this example on the time scales. Suppose that all time scales are equal and the derivative on time scale is denoted by Δ.

$$\begin{aligned}
\frac{\partial F}{\Delta Z}(P) &= \nabla_{Z^\Delta}F = (\langle \nabla F_1, Z_P\rangle, \ldots, \langle \nabla F_n, Z_P\rangle)_P \\
&= \begin{pmatrix} \left\langle \begin{pmatrix} \frac{\partial f_1}{\Delta x_1} + \varepsilon\left(x_1^*\frac{\partial^2 f_1}{\Delta x_1^2} + \frac{\partial f_1{}^*}{\Delta x_1}\right), \\ \frac{\partial f_1}{\Delta x_2} + \varepsilon\left(x_2^*\frac{\partial^2 f_1}{\Delta x_2^2} + \frac{\partial f_1{}^*}{\Delta x_2}\right) \end{pmatrix}, Z_P \right\rangle, \\ \left\langle \begin{pmatrix} \frac{\partial f_2}{\Delta x_1} + \varepsilon\left(x_1^*\frac{\partial^2 f_2}{\Delta x_1^2} + \frac{\partial f_2{}^*}{\Delta x_1}\right), \\ \frac{\partial f_2}{\Delta x_n} + \varepsilon\left(x_2^*\frac{\partial^2 f_2}{\Delta x_2^2} + \frac{\partial f_2{}^*}{\Delta x_2}\right) \end{pmatrix}, Z_P \right\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle(3+\varepsilon, 1-2\varepsilon), Z_P\rangle, \\ \left\langle \begin{matrix} (\sigma(x_1)+2) + \varepsilon \begin{pmatrix} 3\left(\frac{\partial(\sigma(x_1))}{\Delta x_1} + 1\right) \\ +2(\sigma(x_1)+2) \end{pmatrix}, \\ (\sigma(x_2)+4) + \varepsilon \begin{pmatrix} 5\left(\frac{\partial(\sigma(x_2))}{\Delta x_2} + 1\right) \\ -(\sigma(x_2)+4) \end{pmatrix}, Z_P \end{matrix} \right\rangle \end{pmatrix}.
\end{aligned}$$

If we take the time scale as $T = \mathbb{R}$ then $\sigma(t) = t$. Therefore the result will be

$$\frac{\partial F}{\Delta Z}(P)=\left(\begin{array}{l}\langle(3+\varepsilon,1-2\varepsilon),Z_P\rangle,\\ \left\langle\left(\begin{array}{l}(x_1+2)+\varepsilon\left(\begin{array}{l}3\left(\frac{\partial(x_1)}{\Delta x_1}+1\right)\\+2(x_1+2)\end{array}\right),\\ (x_2+4)+\varepsilon\left(\begin{array}{l}5\left(\frac{\partial(x_2)}{\Delta x_2}+1\right)\\-(x_2+4)\end{array}\right)\end{array}\right),Z_P\right\rangle\end{array}\right)$$
$$=(\langle(3+\varepsilon,1-2\varepsilon),Z_P\rangle,\langle(4+14\varepsilon,8+2\varepsilon),Z_P\rangle)$$
$$=\left(\begin{array}{l}\langle(3+\varepsilon,1-2\varepsilon),(2+\varepsilon 3,4+5\varepsilon)\rangle,\\ \langle(4+14\varepsilon,8+2\varepsilon),(2+\varepsilon 3,4+5\varepsilon)\rangle\end{array}\right)$$
$$=(10+8\varepsilon,40+88\varepsilon).$$

5. Conclusion

In current paper, we have researched the covariant derivative of dual-variable functions on time scales. In the literature, up to now, there has been no any study about this concept. This research is a guideline for future work.

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