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ON QUARTIC DIOPHANTINE EQUATIONS WITH TRIVIAL SOLUTIONS IN THE GAUSSIAN INTEGERS

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ABSTRACT. We show that the quartic Diophantine equations $ax^4 + by^4 = cz^2$ has only trivial solution in the Gaussian integers for some particular choices of a, b and c. Our strategy is by elliptic curves method. In fact, we exhibit two null-rank corresponding families of elliptic curves over Gaussian field. We also determine the torsion groups of both families.

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1. Introduction

The integer solutions of the Diophantine equation

$$ax^4 + by^4 = cz^2 \tag{1}$$

can be mostly found in classical book on Diophantine equations. Our interest is the solutions in the Gaussian integers $\mathbb{Z}[i]$. By a trivial solution of (1) we mean $x = y = z = 0$ or, $a = b = c$ and, in addition, one of the x, y is zero and the square of the other equals z.

The Diophantine equation $x^4 + y^4 = z^2$ was studied by Fermat who proved by infinite-descent method that there exist no nontrivial solution in $\mathbb{Z}$. Hilbert [3] extended this result by showing that the equation $x^4 + y^4 = z^2$ has only trivial solution in $\mathbb{Z}[i]$. From his proof, it follows that the equation $x^4 - y^4 = z^2$ has also trivial solution in $\mathbb{Z}[i]$.

Other authors also examined similar problems. Najman [5] found all nontrivial solutions of the equations $x^4 \pm y^4 = iz^2$ in $\mathbb{Z}[i]$. He also gave a new proof of Hilbert's results. Szabó [9] solved the eight equations $x^4 + my^4 = z^2$ in $\mathbb{Z}[i]$, where $m = \pm 2^n$, and $0 \leq n \leq 3$. He also considered the equations of the form (1) with only trivial solution in $\mathbb{Z}[i]$ and proved that the equations $x^4 - py^4 = z^2$ and $x^4 - p^3y^4 = z^2$ have only trivial solutions in $\mathbb{Z}[i]$, where p is a prime $p \equiv 3 \mod 8$. However, the equations $x^4 + py^4 = z^2$ and $x^4 + p^3y^4 = z^2$ have integer solutions $(1, 1, 2)$ and $(2, 1, 5)$, respectively, when, say, $p = 3$.

Izadi et al. [4] studied two family of Diophantine equations of type (1) over the Gaussian integers. More precisely, they considered the families of equations $y^4 \pm p^3x^4 = z^2$ with $p \equiv 3 \mod 8$ or $\mod 16$ and $y^4 \pm px^4 = z^2$ with $p \equiv 7$ or 11 mod 16 over the Gaussian integers and showed by elliptic curves method that in either cases there are only trivial solutions.

By a prime we shall mean a prime in $\mathbb{Z}$; we shall refer to primes in $\mathbb{Z}[i]$ as Gaussian primes. Let p, q be primes $p \equiv 3 \mod 8$, $q \equiv 1 \mod 8$, and the Legendre symbol $\left(\frac{p}{q}\right) \neq 1$. In this article, we find out another equation of type (1) with only trivial solution in $\mathbb{Z}[i]$, where $a = 1$, $b = \pm qp^2$ with p, q as above, and c is a power of $i, 1 + i$, or 2. The approach is also by elliptic curves method.

Theorem 1.1. *Let p, q be primes $p \equiv 3 \mod 8$, $q \equiv 1 \mod 8$, and $\left(\frac{p}{q}\right) \neq 1$. The Diophantine equations $x^4 \pm qp^2y^4 = \pm z^2$ and $x^4 \pm qp^2y^4 = \pm iz^2$ have only trivial solutions in $\mathbb{Z}[i]$.*

The element $\omega = 1 + i$ is a Gaussian prime satisfying $\omega^4 = -4$. Each of the Diophantine equations

$$x^4 + 4qp^2y^4 = z^2,\ x^4 - 4qp^2y^4 = z^2,\ -4x^4 + 4qp^2y^4 = z^2 \tag{2}$$

may be transformed to one of the equations

$$x^4 - qp^2y^4 = z^2,\ x^4 + qp^2y^4 = z^2.$$

It suffices to substitute y by ωy in the first two, and an extra substitution ωx for x in the third. Therefore, the equations (2) also have only trivial solutions in $\mathbb{Z}[i]$. The following corollary gives even more equations of type (1) with trivial solutions.

Corollary 1.2. *Let p, q be primes with $p \equiv 3 \mod 8$, $q \equiv 1 \mod 8$, and $\left(\frac{p}{q}\right) \neq 1$. The Diophantine equations $x^4 \pm qp^2y^4 = \pm 2^n z^2$ and $x^4 \pm qp^2y^4 = \pm i2^n z^2$ have only trivial solutions in $\mathbb{Z}[i]$ for any $n \in \mathbb{Z}^+$.*

As seen, the coefficient of z^2 in each of the equations in Theorem 1.1 or Corollary 1.2 is a power of $i, 1 + i$, or 2.

Remark 1.3. Since any solution in $\mathbb{Q}(i)$ gives a solution in $\mathbb{Z}[i]$, through this work we shall consider all solutions in $\mathbb{Z}[i]$.

Elliptic curves are used to sketch the proof of Theorem 1.1. Elliptic curves over the Gaussian field $\mathbb{Q}(i)$ are not so known. We are interested the elliptic curves of type $Y^2 = X^3+dX$ over $\mathbb{Q}(i)$. For the same family over $\mathbb{Q}$ we cite the comprehensive reference [7].

The so-called Selmer-Mordell conjecture says that the rank of elliptic curves $Y^2 = X^3 + pX$, with p prime, over $\mathbb{Q}$ is exactly 1. Working over $\mathbb{Q}(i)$, Bremner and Cassels [1] showed that this conjecture is true for the primes $p \equiv 5 \mod 8$ less than 1000. Other authors enlarged the upper bound but worked entirely over $\mathbb{Q}$.

Consider the two families of elliptic curves

$$E: \ Y^2 = X^3 \pm qp^2X, \tag{3}$$

where p, q are as in Theorem 1.1. We show that the Mordell-Weil rank of both families (3) over the Gaussian field $\mathbb{Q}(i)$ is zero.

Theorem 1.4. *For the primes $p \equiv 3 \mod 8$ and $q \equiv 1 \mod 8$ with $\left(\frac{p}{q}\right) \neq 1$, the ranks of the two families of elliptic curves*

$$Y^2 = X^3 \pm qp^2X$$

over $\mathbb{Q}(i)$ are zero.

We also determine the torsion groups of the families (3).

Theorem 1.5. *For the primes $p \equiv 3 \mod 8$ and $q \equiv 1 \mod 8$ with $\left(\frac{p}{q}\right) \neq 1$, the torsion groups of the two families of elliptic curves*

$$Y^2 = X^3 \pm qp^2X$$

over $\mathbb{Q}(i)$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

2. Preliminaries

Let E be an elliptic curve over the field $\mathbb{K}$ of characteristic not equals 2 or 3, and let $E(\mathbb{K})$ denote the $\mathbb{K}$-rational points of E over $\mathbb{K}$. The so-called Mordell-Weil theorem asserts that $E(\mathbb{K})$ is a finitely generated abelian group and hence can be represented as

$$E(\mathbb{K}) = E(\mathbb{K})_{\text{tors}} \oplus \mathbb{Z}^r, \ r \geq 0,$$

where $E(\mathbb{K})_{\text{tors}}$ denotes the torsion group of $E(\mathbb{K})$ and r is called the (algebraic) rank of E over $\mathbb{K}$. If $\mathbb{K}$ is quadratic field, there are 26 possible torsion groups, while in the case of the Gaussian quadratic field $\mathbb{K} = \mathbb{Q}(i)$, there are exactly 16 possible torsion groups, namely, the 15 groups from Mazur's theorem and the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (see also [6]). To determine the torsion subgroup of $E(\mathbb{K})$ of our families (3) over $\mathbb{K} = \mathbb{Q}(i)$, we need the extended Lutz-Nagell theorem [7].

Theorem 2.1 (Extended Lutz-Nagell theorem). *Consider the elliptic curve* $Y^2 = X^3 + aX + b$ *with* $a, b \in \mathbb{Z}[i]$ *and let* $(X, Y) \in E(\mathbb{Q}(i))$ *be a torsion point. Then,*

(1) $X, Y \in \mathbb{Z}[i]$;
(2) *either* $Y = 0$ *or* $Y^2 \mid 4a^3 + 27b^2$.

The plan of proving Theorems 1.4 and 1.5 is hanging on the 2-decent method for determining the rank of $E(\mathbb{Q})$. We describe briefly this method. For more details see [2,8]. Suppose that $E: \; Y^2 = X^3 + AX^2 + BX$ is an elliptic curve over $\mathbb{Q}$ and $\widehat{E}: \; Y^2 = X^3 - 2AX^2 + (A^2 - 4B)X$ is the curve isogenous to E. Let $\mathbb{Q}^*$ be the multiplicative group of nonzero rational numbers, and $\mathbb{Q}^{*2}$ denote the subgroup of squares of elements of $\mathbb{Q}^*$. Then, $\mathbb{Q}^*/\mathbb{Q}^{*2}$ is the multiplicative group of rational numbers modulo squares. The process of determining the rank of E requires that we look at both curves E and $\widehat{E}$. Define the 2-descent homomorphism $\alpha : E(\mathbb{Q}) \longrightarrow \mathbb{Q}^*/\mathbb{Q}^{*2}$ by

$$\alpha(P) = \begin{cases} 1 \mod \mathbb{Q}^{*2}, & \text{if } P = \mathcal{O}, \text{ the point at infinity,} \\ B \mod \mathbb{Q}^{*2}, & \text{if } P = (0,0), \\ X \mod \mathbb{Q}^{*2}, & \text{if } P = (X, Y) \text{ with } X \neq 0. \end{cases}$$

The 2-descent homomorphism $\widehat{\alpha} : \widehat{E}(\mathbb{Q}) \longrightarrow \mathbb{Q}^*/\mathbb{Q}^{*2}$ is similarly defined.

$$\widehat{\alpha}(\widehat{P}) = \begin{cases} 1 \mod \mathbb{Q}^{*2}, & \text{if } \widehat{P} = \mathcal{O}, \text{ the point at infinity,} \\ \widehat{B} \mod \mathbb{Q}^{*2}, & \text{if } \widehat{P} = (0,0), \\ X \mod \mathbb{Q}^{*2}, & \text{if } \widehat{P} = (X, Y) \text{ with } X \neq 0, \end{cases}$$

where $\widehat{B} = A^2 - 4B$.

Proposition 2.2. *With the above notations, let* r *denote the rank of* $E(\mathbb{Q})$. *Then,*

$$|\alpha(E(\mathbb{Q}))| \, |\widehat{\alpha}(\widehat{E}(\mathbb{Q}))| = 2^{r+2}.$$

A practical method to compute $|\alpha(E(\mathbb{Q}))|$ is followed by looking at the factorization of B. We make this more precisely in the following theorem [2].

Theorem 2.3. *The group* $\alpha(E(\mathbb{Q}))$ *is equal to the classes modulo squares of* 1, B, *and the positive and negative divisors* B_1 *of* B *such that the quartic equation*

$$V^2 = B_1U^4 + AU^2W^2 + (B/B_1)W^4 \tag{4}$$

has a solution (U, V, W) *with* U, V *and* W *pairwise coprime such that* $UW \neq 0$ *and*

$$\gcd(B/B_1, U) = \gcd(B_1, W) = 1,$$

and the point $P = (\frac{B_1U^2}{W^2}, \frac{B_1UV}{W^3})$ *is in* $E(\mathbb{Q})$ *such that* $\alpha(P) = B_1$.

In general, Theorem 2.3 gives us a method for determining the rank of $E(\mathbb{Q})$, provided that we are able to determine whether or not each of the curves generated by the divisors of B and $\widehat{B}$ in the definitions of α and $\widehat{\alpha}$ have solutions. It is also important to note that calculating the rank of an elliptic curve using the 2-descent method can be rather time consuming, depending on how many square-free divisors of B and $\widehat{B}$ there are.

The group of elliptic curves $E(\mathbb{K})$ over the quadratic fields $\mathbb{K} = \mathbb{Q}(\sqrt{m})$, with square-free integer m has interesting properties. The next result [7] shows that the rank of E over $\mathbb{K}$ is the sum of the ranks of E and it's m-twist E_m over $\mathbb{Q}$.

Theorem 2.4. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{m})$ be a quadratic field, where m is a square-free integer. Let $E : y^2 = x^3 + ax^2 + bx$ be an elliptic curve over $\mathbb{Q}$ and $E_m : y^2 = x^3 + max^2 + m^2bx$ be the m-twist of E. Then*

$$\operatorname{rank}(E(\mathbb{K})) = \operatorname{rank}(E(\mathbb{Q})) + \operatorname{rank}(E_m(\mathbb{Q})).$$

3. Proofs

Proof of Theorem 1.4. We prove the result for the family $E : Y^2 = X^3 + qp^2X$. The proof for the other family uses similar techniques. Appealing to Proposition 2.2 we need to prove

$$|\alpha(E(\mathbb{Q}))|\,|\widehat{\alpha}(\widehat{E}(\mathbb{Q}))| = 4.$$

In other words, by the definitions of the 2-descent homomorphisms α and $\widehat{\alpha}$, we should prove

$$|\alpha(E(\mathbb{Q}))| = 2 = |\widehat{\alpha}(\widehat{E}(\mathbb{Q}))|.$$

We do this in two steps. Note that all calculations and equations are modulo squares.

Step 1. We show $|\alpha(E(\mathbb{Q}))| = 2$. In this part, the quartic equation (4) in Theorem 2.3 is

$$V^2 = B_1U^4 + \frac{qp^2}{B_1}W^4, \tag{5}$$

Therefore, modulo squares,

$$B_1 \in \{\pm 1, \pm p, \pm q, \pm qp\}.$$

By the definition of α, 1 and q are in the image of α. We show that none of the B_1, except to 1 and q, is in $\operatorname{Im}(\alpha)$. If $B_1 = -1$, the equation (5) turns to $V^2 = -U^4 - qp^2W^4$, which is impossible. The case $B_1 = -p$ has the same discussion.

For $B_1 = p$, we get $(V/U^2)^2 \equiv p \mod q$. By the property of the Legendre symbol we get $\left(\frac{p}{q}\right) = 1$, contradicts the hypothesis.

Now, let $B_1 = -q$. If $-q \in \mathrm{Im}(\alpha)$ then, $q \cdot (-q) = -1 \in \mathrm{Im}(\alpha)$, which is failed in the case $B_1 = -1$. The cases $B_1 = \pm qp$ have similar reason with different failures.

Step 2. We prove $|\widehat{\alpha}(\widehat{E}(\mathbb{Q}))| = 2$. Here, the quartic equation (4) in Theorem 2.3 is

$$\widehat{V}^2 = \widehat{B_1}\widehat{U}^4 - \frac{4qp^2}{\widehat{B_1}}\widehat{W}^4,$$

Therefore, modulo squares,

$$\widehat{B_1} \in \{\pm 1, \pm 2, \pm p, \pm 2p, \pm q, \pm 2q, \pm qp, \pm 2qp\}.$$

By the definition of $\widehat{\alpha}$, two elements 1 and $-q$ are in the image of α. We show that the other elements of $\widehat{B_1}$ are not in $\mathrm{Im}(\widehat{\alpha})$. We distinguish some cases and try to get contradiction in each case.

3.0.1. $\widehat{B_1} = -1; \widehat{V}^2 = -\widehat{U}^4 + 4qp^2\widehat{W}^4$. Then, $(\widehat{V}/\widehat{U}^2)^2 \equiv -1 \mod p$. By the property of the Legendre symbol, we get $\left(\frac{-1}{p}\right) = 1$, which implies $p \equiv 1 \mod 4$, contradicts the hypothesis.

3.0.2. $\widehat{B_1} = 2; \widehat{V}^2 = 2\widehat{U}^4 - 2qp^2\widehat{W}^4$. Hence, $(\widehat{V}/\widehat{U}^2)^2 \equiv 2 \mod p$. It follows that $\left(\frac{2}{p}\right) = 1$, showing that $p \equiv 1$ or $7 \mod 8$, contradicts again the hypothesis.

3.0.3. $\widehat{B_1} = p; \widehat{V}^2 = p\widehat{U}^4 - 4qp\widehat{W}^4$. Here, we have $(\widehat{V}/\widehat{U}^2)^2 \equiv p \mod q$, which gives $\left(\frac{p}{q}\right) = 1$, a contradiction.

3.0.4. $\widehat{B_1} = -p; \widehat{V}^2 = -p\widehat{U}^4 + 4qp\widehat{W}^4$. Now, $(\widehat{V}/\widehat{U}^2)^2 \equiv -p \mod q$ from which, $\left(\frac{-p}{q}\right) = 1$. This implies $\left(\frac{-1}{q}\right) = -1$ from which we get $q \equiv 3 \mod 4$, which is impossible.

3.0.5. $\widehat{B_1} = 2p; \widehat{V}^2 = 2p\widehat{U}^4 - 2qp\widehat{W}^4$. We obtain $(\widehat{V}/\widehat{U}^2)^2 \equiv 2p \mod q$ and hence $\left(\frac{2p}{q}\right) = 1$. Thus, $\left(\frac{2}{q}\right) = -1$ which shows $q \equiv 3$ or $5 \mod 8$, again impossible. The cases $\widehat{B_1} = 2q, 2qp$ have similar reason.

3.0.6. $\widehat{B_1} = q$. If $q \in \mathrm{Im}(\widehat{\alpha})$ then, $q \cdot (-q) = -q^2 = -1 \in \mathrm{Im}(\widehat{\alpha})$ which is failed in the case $\widehat{B_1} = -1$. The cases $\widehat{B_1} = -2, -2q, -2p, \pm qp, -2qp$ are similarly failed with different failures.

We showed $\mathrm{rank}(E(\mathbb{Q})) = 0$. An apply of Theorem 2.4 completes the proof of Theorem 1.4 since $m = -1$ and the m-twist of E is itself. □

Proof of Theorem 1.5. Let (X, Y) be a torsion point. Then, by Theorem 2.1, $X, Y \in \mathbb{Z}[i]$ and, in addition, either $Y = 0$ or $Y^2 \mid 4q^3p^6$. The relation $Y = 0$ gives the 2-torsion point $(0,0)$. We claim that there is no other torsion point, that is, Y^2 never divides $4q^3p^6$. Let by the contrary that $Y^2 \mid 4q^3p^6$. Then, $Y^2 = \ell q^s p^t$, where

$$\ell \in \{\pm 1, \pm 2i, \pm 4\},\ s \in \{0,2\},\ t \in \{0,2,4,6\}. \tag{6}$$

Regardless the sign of ℓ, we try to get contradiction in all 24 cases. The opposite sign of ℓ has similar discussion. We take $\omega = (1+i)$.

The case $Y^2 = 1 = (-i)^2 = X^3 + qp^2X$ is clearly impossible. Let $Y^2 = 2i = \omega^2 = X^3 + qp^2X$. The only Gaussian prime factor of X is ω. Then, we have $\omega^2 = \omega^{3k} + qp^2\omega^k$ with $k \geq 1$, showing that $\cos(\frac{k\pi}{2}) = \frac{1}{2} - \frac{qp^2}{2^{k+1}}$, which is false. The case $Y^2 = 4 = \omega^4$ has similar discussion.

Now, let $Y^2 = q^2 = X^3 + qp^2X$. Then, $q \mid X$. Let k be the largest power of q in X. Then, $q^2 = q^{3k}X_0^3 + q^{k+1}p^2X_0$, $k \geq 1$, where $q \nmid X_0$. By canceling q^2 we get an impossible equation.

For $Y^2 = 2q^2i = \omega^2q^2 = X^3 + qp^2X$ we have again $q \mid X$. Assume that $\omega^2q^2 = q^{3k}X_0^3 + q^{k+1}p^2X_0$, $k \geq 1$, where $q \nmid X_0$. Here again k is taken the largest power of q in X. Then, by canceling the q^2, we get $\omega^2 = q^{3k-2}X_0^3 + q^{k-1}p^2X_0$. The only Gaussian prime factor of X_0 is ω. Now, we get an equation in which the powers of ω in both sides do not match. The case $Y^2 = 4q^2 = \omega^4q^2$ has similar discussion.

Finally, let $Y^2 = \ell q^s p^t = X^3 + qp^2X$ where, as in (6), $\ell \in \{\pm 1, \pm 2i, \pm 4\}$, $s = 0, 2$, and $t = 2, 4, 6$. Then, $p \mid X$. Assume $\ell q^s p^t = p^{3k}X_0^3 + qp^{k+2}X_0$, with $k \geq 1$ the largest power of q in X, where $p \nmid X_0$. By canceling the $p^{k'}$ with $k' = \min\{k+2, t\}$, we obtain an impossible equation for any ℓ and any s. □

We are now ready to prove the main Theorem 1.1.

Proof of Theorem 1.1. We prove the result for the equations

$$x^4 \pm qp^2y^4 = z^2, \tag{7}$$

$$x^4 \pm qp^2y^4 = iz^2, \tag{8}$$

with positive signs in the right hand sides. The negative cases have similar proofs. Assume that (x,y,z) is a nontrivial solution of the equation (7). Dividing both sides by y^4 and putting $x/y = u$ and $z/y^2 = v$ we get $u^4 \pm qp^2 = v^2$. Taking $X = u^2$, we have two equations

$$X = u^2,\ X^2 \pm qp^2 = v^2.$$

Now, multiplying these two equations and putting $Y = uv$ we obtain a torsion point $\neq (0,0)$ on the elliptic curves $Y^2 = X^3 \pm qp^2 X$ which is impossible by Theorem 1.5.

Now, we work on the equation (8). Here again divide both sides by y^4 and put $x/y = u$ and $z/y^2 = v$ to get $u^4 \pm qp^2 = iv^2$. This time taking $-iX = u^2$, we obtain two equations

$$-iX = u^2, \ -X^2 \pm qp^2 = iv^2.$$

By the similar manner as in the previous case, the existence of a torsion point $\neq (0,0)$ on the elliptic curves $Y^2 = X^3 \pm qp^2 X$ leads to a contradiction and this completes the proof. □

Proof of Corollary 1.2. In the equation $x^4 \pm qp^2 y^4 = \pm 2^n z^2$ taking $n = 2k$ and $n = 2k+1$ we get, respectively, the equations

$$x^4 - qp^2 y^4 = \pm(2^k z)^2, \ x^4 - qp^2 y^4 = \pm i(i\omega 2^k z)^2,$$

which is in the form of equations of Theorem 1.1.

In a similar way, in the equation $x^4 \pm qp^2 y^4 = \pm i 2^n z^2$ taking $n = 2k$ and $n = 2k+1$ we get, respectively, the equations

$$x^4 - qp^2 y^4 = \pm i(2^k z)^2, \ x^4 - qp^2 y^4 = \pm(\omega 2^k z)^2,$$

which is again in the form of equations of Theorem 1.1. □

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