Research Article GU J Sci 37(2): 912-922 (2024) DOI: 10.35378/gujs.967261

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Characterizations of Filter Convergent in Terms of Ideal

Shyamapada MODAK[1, *], Kulchhum KHATUN[1], Jiarul HOQUE[2]

[1]*Department of Mathematics, University of Gour Banga, 732103, Malda, India*
[2]*Department of Science and Humanities, Gangarampur Government Polytechnic, 733124, Gangarampur, India*

Highlights

- This paper focuses on filter convergent via ideal.
- An equivalent characterization of maximal ideal has been obtained.
- Preservations under homeomorphism of different generalized local functions were studied.

Article Info

Received: 08 July 2021
Accepted: 24 Sep 2023

Keywords

Net
Filter
Ultrafilter
Ideal
Local Function

Abstract

In this paper, convergences of a filter and a net have been characterized through ideal on topological spaces. Furthermore, we characterized the local function in an ideal topological space in terms of convergence of filter. Using Zorn's Lemma, we have found a maximal element in the collection of all proper ideals on a nonempty set which is called maximal ideal. We provide a convenient characterization of maximal ideals. We also consider simple properties of the image of an ideal, a net and various local functions under a homeomorphism.

1. INTRODUCTION

An ideal is a nonempty collection $\mathcal{I} \subseteq 2^Z$ where Z is a nonempty set that satisfies hereditary property and finite additivity property, where 2^Z is the collection of all subsets of Z. Notion of ideal on a topological space has been incorporated by Kuratowski [1]. There are so many branches for which the study of ideals is going on so far. One of the branches is the common representation of limit points. For representation of limit points, the ideals $\mathcal{I} = \{\emptyset\}$, $\mathcal{I} = 2^Z$, $\mathcal{I} = \mathcal{I}_f$ (collection of all finite subsets of Z), and $\mathcal{I} = \mathcal{I}_c$ (collection of all countable subsets of Z) play vital role. For generalization of limit points, we consider an ideal $\mathcal{I}$ on a topological space (Z,η), and consider the local function $(\cdot)^* : 2^Z \to 2^Z$ which is defined in [1] as: for $X \subseteq Z$, $X^*(\mathcal{I},\eta) = \{z \in Z : U \cap X \notin \mathcal{I}, \text{ for all } U \in \eta(z)\}$, where $\eta(z) = \{U \in \eta : z \in U\}$. In particular, $X^*(\{\emptyset\},\eta) = Cl(X)$ ('Cl' denotes the closure operator), $X^*(\mathcal{I}_f,\eta)$ is the collection of all ω-accumulation points of X, $X^*(\mathcal{I}_c,\eta)$ is the collection of all condensation points of X, and $X^*(2^Z,\eta) = \emptyset$. Recall that for a given subset Q of the topological space (Z,η), a point $z \in Z$ is called a ω-accumulation (resp. condensation) point of Q if for every $U \in \eta(z)$, $U \cap Q$ is infinite (resp. uncountable). In view of the last case, we consider only proper ideals (recall that an ideal $\mathcal{I}$ on a nonempty set Z is a proper ideal if $Z \notin \mathcal{I}$) throughout this paper. A filter $\mathcal{F}$ on a

*Corresponding author, e-mail: spmodak2000@yahoo.co.in

nonempty set Z is a nonempty aggregate of subsets of Z having the properties: (i) $\varnothing \notin \mathcal{F}$; (ii) $K \subseteq L$ and $K \in \mathcal{F}$ implies $L \in \mathcal{F}$; (iii) K, $L \in \mathcal{F}$ implies $K \cap L \in \mathcal{F}$. If $\mathcal{I}$ be a proper ideal on Z, then $\mathcal{F}_{\mathcal{I}} = \{A \subseteq Z : Z \setminus A \in \mathcal{I}\}$ is a filter on Z. This type of filter is known as associated filter or dual filter, and such filters have been studied in [2-5]. Conversely, for a filter $\mathcal{F}$ on Z, $\mathcal{I}_{\mathcal{F}} = \{E \subseteq Z : Z \setminus E \in \mathcal{F}\}$ defines a proper ideal on Z. This ideal is called associated ideal or dual ideal. In this respect, filter containing the empty set has been discussed in [6].

In this paper, we shall discuss the conversion between net and ideal as well as filter and ideal. Due to this conversation, we can characterize the convergence of a net and a filter via ideal. This conversion will be helpful to characterize the local function in terms of net and filter. The study of maximal ideal like the study of maximal filter is also a part of the above conversation. Through this paper, we also consider homeomorphic image of various local functions.

2. FILTERS AND NETS

In this section, we shall study net [7,8], filter [7,8], and their convergence in terms of ideal. Characterization of local function in terms of filter is also a part of this section.

At first, we shall discuss about the notion of directed set [8] from literature. A pair $(D, \geq)$ of a nonempty set D and a binary relation $\geq$ on D is a directed set having three properties: (i) For all $m, n, p \in D, m \geq n$ and $n \geq p$ imply $m \geq p$; (ii) For all $n \in D,\ n \geq n$; (iii) For all $m, n \in D$, there exists $p \in D$ such that $p \geq m$ and $p \geq n$.

Lemma 2.1. Let $\mathcal{J}$ be a proper ideal on a nonempty set Z and $D(\mathcal{J}) := \{t \times F : F \in \mathcal{F}_{\mathcal{J}} \text{ and } t \in F\}$. For $t \times H$, $s \times L \in D(\mathcal{J})$, we define

$$t \times H \geq s \times L \text{ iff } H \subseteq L.$$

Then $(D(\mathcal{J}), \geq)$ is a directed set.

Proof. We shall show only directive property. Suppose $t \times H$, $s \times L \in D(\mathcal{J})$. Thus $H \cap L \in \mathcal{F}_{\mathcal{J}}$ and hence is nonempty. Pick $z \in H \cap L$. Clearly then $z \times (H \cap L) \in D(\mathcal{J})$ and $z \times (H \cap L) \geq t \times H$ and $z \times (H \cap L) \geq s \times L$.

Note that if we have a proper ideal $\mathcal{J}$ on a nonempty set Z, then we always get a net $S : D(\mathcal{J}) \to Z$ in Z by the rule $S(t \times H) = t$.

On the other hand, if we start with a net $S : D \to Z$ in Z, where $(D, \geq)$ is a directed set, then one can obtain a proper ideal on Z as follows:

Theorem 2.2. Let $S : D \to Z$ be a net, and for every $m \in D$, let $B_m := \{S(n) : n \in D \text{ and } n \geq m\}$. Then $\mathcal{I}_S = \{A : A \subseteq Z \setminus B_m \text{ for some } m \in D\}$ is a proper ideal on Z.

Proof. It obvious that $\varnothing \in \mathcal{I}_S$. Now, let $M \subseteq N$ and $N \in \mathcal{I}_S$. Then $N \subseteq Z \setminus B_m$ for some $m \in D$. Hence $M \subseteq Z \setminus B_m$. Consequently, $M \in \mathcal{I}_S$. Finally, let M, $N \in \mathcal{I}_S$. Then $M \subseteq Z \setminus B_m$ and $N \subseteq Z \setminus B_n$ for some m, $n \in D$. As m, $n \in D$, there exists $p \in D$ such that $p \geq m$ and $p \geq n$. Now, $M \cup N \subseteq (Z \setminus B_m) \cup (Z \setminus B_n) = Z \setminus (B_m \cap B_n) \subseteq Z \setminus B_p$. Hence $M \cup N \in \mathcal{I}_S$. Since for every $m \in D$, B_m is nonempty, we have $Z \notin \mathcal{I}_S$. Hence $\mathcal{I}_S$ is a proper ideal on Z.

Theorem 2.3. Let (Z,η) be a topological space. A net $S: D \to Z$ converges to z if and only if for each $U_z \in \eta(z)$, $Z \setminus U_z \in \mathcal{I}_S$.

Proof. Suppose $S: D \to Z$ converges to z. Then for each $U_z \in \eta(z)$, $\exists$ $p \in D$ such that $S(n) \in U_z$ for all $n \geq p$. By Theorem 2.2, $B_p \subseteq U_z$. Thus $Z \setminus U_z \subseteq Z \setminus B_p$ and hence $Z \setminus U_z \in \mathcal{I}_S$.

Conversely, let for each $U_z \in \eta(z)$, $Z \setminus U_z \in \mathcal{I}_S$. Then for some $m \in D$, $Z \setminus U_z \subseteq Z \setminus B_m$. Thus $\{S(n): n \in D \text{ and } n \geq m\} = B_m \subseteq U_z$ and hence $S(n) \in U_z$ for all $n \geq m$. Therefore S converges to z.

Proposition 2.4. [8] Let (Z,η) be a topological space, and $A \subseteq Z$, $z \in Z$. Then $z \in Cl(A)$ if and only if there exists a net in A which converges to z.

Theorem 2.5. Let (Z,η) be a topological space, and $A \subseteq Z$. Then $z_o \in Cl(A)$ if and only if there is a proper ideal $\mathcal{I}$ on A such that for every $U_{z_o} \in \eta(z_o)$, $A \setminus U_{z_o} \in \mathcal{I}$.

Proof. Suppose that $z_o \in Cl(A)$. Then for all $U \in \eta(z_o)$, $U \cap A \neq \varnothing$. Consider $D = \{(x,U) \in A \times \eta(z_o) : x \in U \cap A\}$. Define $\geq$ on D by $(x,U) \geq (y,V)$ if $U \subseteq V$. Then $(D, \geq)$ is a directed set. Obviously, $S: D \to A$ defined by $S(x,U) = x$ is a net in A. Now, for every $(y,V) \in D$, consider $B_{(y,V)} = \{S(x,U) : (x,U) \in D \text{ and } (x,U) \geq (y,V)\}$. Then, by Theorem 2.2, $\mathcal{I}_S = \{M : M \subseteq A \setminus B_{(y,V)} \text{ for some } (y,V) \in D\}$ is a proper ideal on A. Furthermore, for $(x,U) \geq (y,V)$, $S(x,U) = x \in U \subseteq V$. Thus S converges to z_o, and hence by Theorem 2.3, for each $U_{z_o} \in \eta(z_o)$, $A \setminus U_{z_o} \in \mathcal{I}_S$.

Converse part is obvious from Lemma 2.1 and Proposition 2.4.

Theorem 2.6. For a topological space (Z,η), a filter $\mathcal{F}$ on Z converges to $z_o \in Z$ if and only if for every $U_{z_o} \in \eta(z_o)$, $Z \setminus U_{z_o} \in \mathcal{I}_{\mathcal{F}}$.

Proof. Suppose that $\mathcal{F}$ converges to z_o. Then $\eta(z_0) \subseteq \mathcal{F}$. Thus, for every $U_{z_o} \in \eta(z_0)$, $Z \setminus U_{z_o} \in \mathcal{I}_{\mathcal{F}}$.

Conversely, suppose that for every $U_{z_o} \in \eta(z_o)$, $Z \setminus U_{z_o} \in \mathcal{I}_{\mathcal{F}}$. Since the associated filter of the ideal $\mathcal{I}_{\mathcal{F}}$ is $\mathcal{F}$, then $U_{z_o} = Z \setminus (Z \setminus U_{z_o}) \in \mathcal{F}$. Thus, $\eta(z_o) \subseteq \mathcal{F}$ and consequently, $\mathcal{F}$ converges to z_o.

Proposition 2.7. [8] A topological space is Hausdorff iff no filter can converge to more than one point.

Proposition 2.8. [8] Let Z be a nonempty set and $\mathcal{S} \subseteq 2^Z$. Then there exists a filter on Z having $\mathcal{S}$ as a sub-base if and only if $\mathcal{S}$ has the finite intersection property.

Theorem 2.9. A topological space (Z,η) is Hausdorff iff there is no proper ideal $\mathcal{I}$ on Z having the property: for all $U_y \in \eta(y)$ and for all $V_z \in \eta(z)$ with $y \neq z$, $Z \setminus U_y \in \mathcal{I}$, $Z \setminus V_z \in \mathcal{I}$.

Proof. Firstly, suppose that (Z,η) is a Hausdorff space. Assume that $\mathcal{I}$ be a proper ideal on Z having the property: for all $U_y \in \eta(y)$ and for all $V_z \in \eta(z)$ with $y \neq z$, $Z \setminus U_y \in \mathcal{I}$, $Z \setminus V_z \in \mathcal{I}$. Then for all $U_y \in \eta(y)$, $U_y \in \mathcal{F}_{\mathcal{I}}$ and for all $V_z \in \eta(z)$, $V_z \in \mathcal{F}_{\mathcal{I}}$. Thus, $\eta(y) \subseteq \mathcal{F}_{\mathcal{I}}$ and $\eta(z) \subseteq \mathcal{F}_{\mathcal{I}}$. This shows that $\mathcal{F}_{\mathcal{I}}$ converges to both y and z which contradicts Proposition 2.7.

Conversely, suppose that the condition holds. Assume that (Z,η) is not a Hausdorff space. Then there exists y, $z \in Z$ with $y \neq z$ such that for all $U_y \in \eta(y)$ and for all $V_z \in \eta(z)$, $U_y \cap V_z \neq \emptyset$. Then $\eta(y) \cup \eta(z)$ has finite intersection property and by Proposition 2.8, there exists a filter $\mathcal{F}$ on Z containing $\eta(y) \cup \eta(z)$. Consequently, $\mathcal{I}_{\mathcal{F}}$ is a proper ideal on Z such that for all $U_y \in \eta(y)$ and for all $V_z \in \eta(z)$ with $y \neq z$, $Z \setminus U_y \in \mathcal{I}_{\mathcal{F}}$, $Z \setminus V_z \in \mathcal{I}_{\mathcal{F}}$. This is a contradiction.

Theorem 2.10. Let (Z,η) be a topological space, and $\mathcal{I}$ be an ideal on Z. If $x \in A^*\big(\mathcal{I},\eta\big)$, then there is a filter $\mathcal{M}$ on Z such that $A \in \mathcal{M}$ and $\mathcal{M}$ converges to x.

Proof. Given that $x \in A^*\big(\mathcal{I},\eta\big)$. Then, for all $W \in \eta(x)$, $W \cap A \notin \mathcal{I}$ and hence $W \cap A \neq \emptyset$. Put $\mathcal{M} = \big\{B \supseteq W \cap A : W \in \eta(x)\big\}$. It is obvious that $\mathcal{M}$ is a filter on Z. Now, $A \supseteq W \cap A$ and hence $A \in \mathcal{M}$. Furthermore, for $W \in \eta(x)$, $W \supseteq W \cap A$, and hence $W \in \mathcal{M}$. Therefore, $\mathcal{M}$ converges to x.

The proof of the Theorem 2.10 can also be done by the fact that $A^*(\mathcal{I},\eta) \subseteq Cl(A)$.

For converse of the Theorem 2.10, we have following:

Theorem 2.11. Let (Z,η) be a topological space, and $A \subseteq Z$. If a filter $\mathcal{F}$ on Z contains A and $\mathcal{F}$ converges to $x \in Z$, then $x \in A^*\big(\mathcal{I}_{\mathcal{F}},\eta\big)$.

Proof. Given that $\eta(x) \subseteq \mathcal{F}$ and $A \in \mathcal{F}$. Then, for each $W \in \eta(x)$, $W \cap A \in \mathcal{F}$ and so $X \setminus (W \cap A) \in \mathcal{I}_{\mathcal{F}}$. Therefore, $W \cap A \notin \mathcal{I}_{\mathcal{F}}$ (because if $W \cap A \in \mathcal{I}_{\mathcal{F}}$, then $(W \cap A) \cup \big(X \setminus (W \cap A)\big) \in \mathcal{I}_{\mathcal{F}}$ implies $X \in \mathcal{I}_{\mathcal{F}}$ and hence $\emptyset \in \mathcal{F}$, a contradiction). Thus, for every $W \in \eta(x)$, $W \cap A \notin \mathcal{I}_{\mathcal{F}}$ and hence $x \in A^*\big(\mathcal{I}_{\mathcal{F}},\eta\big)$.

Hence, we conclude that one can define the convergence of a filter in terms of associated ideal.

Let $\mathcal{N}$ and $\mathcal{H}$ be two filters on a set Z. Then $\mathcal{H}$ is called a sub-filter of $\mathcal{N}$ if $\mathcal{N} \subseteq \mathcal{H}$. If $\mathcal{H}$ is a sub-filter of $\mathcal{N}$ on Z, then it is obvious that $\mathcal{I}_{\mathcal{N}} \subseteq \mathcal{I}_{\mathcal{H}}$.

Remark 2.12. Let (Z,η) be a topological space and $\mathcal{F}$, $\mathcal{H}$ be two filters on Z. If $\mathcal{H}$ be a sub-filter of $\mathcal{F}$ and $\mathcal{F}$ converges to $z \in Z$, then for each $U_z \in \eta(z)$, $Z \setminus U_z \in \mathcal{I}_{\mathcal{H}}$.

3. MAXIMAL IDEALS

In this section, we are considering Zorn's Lemma [9] for discussing the maximal ideal. The study of maximal ideal is likely to be similar with ultrafilter [8].

Theorem 3.1. Let Z be a nonempty set. Then the collection Θ_Z of all proper ideals on Z forms a partial ordered set with respect to $\subseteq$.

Proof. The proof is straightforward.

Theorem 3.2. Consider the partial ordered set $(\Theta_Z, \subseteq)$ of Theorem 3.1, and let $\{\mathcal{J}_j : j \in \Omega\}$ be a chain in $(\Theta_Z, \subseteq)$, where Ω is an index set. Then $\mathcal{J} = \bigcup_{j\in\Omega} \mathcal{J}_j$ is an upper bound of the chain $\{\mathcal{J}_j : j \in \Omega\}$.

Proof. Since $\emptyset \in \mathcal{J}_j$ for all $j \in \Omega$, it follows that $\emptyset \in \mathcal{J}$. Let $S \in \mathcal{J}$ and $T \subseteq S$. Then there exists $k \in \Omega$ such that $S \in \mathcal{J}_k$, and hence $T \in \mathcal{J}_k$. This implies that $T \in \mathcal{J}$. Now let S, $T \in \mathcal{J}$. Then there exist k, $l \in \Omega$ such that $S \in \mathcal{J}_k$ and $T \in \mathcal{J}_l$. Since $\{\mathcal{J}_j : j \in \Omega\}$ is chain, either $\mathcal{J}_k \subseteq \mathcal{J}_l$ or $\mathcal{J}_l \subseteq \mathcal{J}_k$. Suppose $\mathcal{J}_l \subseteq \mathcal{J}_k$. Then S, $T \in \mathcal{J}_k$ and hence $S \cup T \in \mathcal{J}_k$. Therefore, $S \cup T \in \mathcal{J}$. Since no member of Θ_Z contains Z, we have $Z \notin \mathcal{J} = \bigcup_{j\in\Omega} \mathcal{J}_j$. Hence $\mathcal{J}$ is a proper ideal on Z, and as a result, $J \in \Theta_Z$. By construction, $\mathcal{J}$ is an upper bound of $\{\mathcal{J}_j : j \in \Omega\}$.

In view of the Zorn's Lemma, we conclude that $(\Theta_Z, \subseteq)$ has a maximal element, and we call it maximal ideal. Here, we give a nice characterization of maximal ideal and the characterization is a modification of a result of the ultrafilter form [8].

Theorem 3.3. Let Z be a nonempty set. Then for $\mathcal{I} \in \Theta_Z$, the following arguments are equivalent:

1. $\mathcal{I}$ is a maximal ideal;
2. for any $A \subseteq Z$, either $A \in \mathcal{I}$ or $Z \setminus A \in \mathcal{I}$;
3. for any A, $B \subseteq Z$, $A \cap B \in \mathcal{I}$ if and only if either $A \in \mathcal{I}$ or $B \in \mathcal{I}$.

Proof. $1 \Rightarrow 2$: If $A \notin \mathcal{I}$, then A is not contained in any member of $\mathcal{I}$ and hence A intersects every member of $\mathcal{F}_{\mathcal{I}}$. Thus, $\{A\} \cup \mathcal{F}_{\mathcal{I}}$ has finite intersection property, and so it generates a filter (see Proposition 2.8), say $\mathcal{G}$. Then, $\mathcal{F}_{\mathcal{I}} \subseteq \mathcal{G}$ and $A \in \mathcal{G}$. Now, it is quite obvious that $\mathcal{I} \subseteq \mathcal{I}_{\mathcal{G}}$. Since I is a maximal ideal, we have $\mathcal{I}_{\mathcal{G}} = \mathcal{I}$. Now, $A \in \mathcal{G}$ implies $Z \setminus A \in \mathcal{I}_{\mathcal{G}}$. Hence, $Z \setminus A \in \mathcal{I}$.

$2 \Rightarrow 1$: If possible suppose that $\mathcal{I}$ is not a maximal ideal on Z. Then there exists an ideal $\mathcal{I}_o \in \Theta_Z$ which contains $\mathcal{I}$ properly. Then $\mathcal{I}_o \setminus \mathcal{I} \neq \emptyset$. Pick $A \in \mathcal{I}_o \setminus \mathcal{I}$. Then $A \notin \mathcal{I}$ implies $Z \setminus A \in \mathcal{I}$, by assumption. Since $A \in \mathcal{I}_o$ and $Z \setminus A \in \mathcal{I}_o$, $Z = A \cup (Z \setminus A) \in \mathcal{I}_o$, a contradiction as $\mathcal{I}_o$ was a proper ideal. Hence, $\mathcal{I}$ is a maximal ideal on Z.

$2 \Rightarrow 3$: Firstly, consider that $A \cap B \in \mathcal{I}$ but neither $A \in \mathcal{I}$ nor $B \in \mathcal{I}$. Then, by assumption, $Z \setminus A \in \mathcal{I}$ and $Z \setminus B \in \mathcal{I}$ implying that $(Z \setminus A) \cup (Z \setminus B) \in \mathcal{I}$. Thus, $Z \setminus (A \cap B) \in \mathcal{I}$. Since $A \cap B \in \mathcal{I}$, $(A \cap B) \cup \big(Z \setminus (A \cap B)\big) \in \mathcal{I}$ showing that $Z \in \mathcal{I}$. This is a contradiction. Hence, either $A \in \mathcal{I}$ or

$B \in \mathcal{I}$. For converse, $A \cap B \subseteq A$ as well as B , and either $A \in \mathcal{I}$ or $B \in \mathcal{I}$, we have from definition of ideal that $A \cap B \in \mathcal{I}$.

$3 \Rightarrow 2$: Since for any $A \subseteq Z$, $A \cap (Z \setminus A) = \varnothing \in \mathcal{I}$, by assumption we have, either $A \in \mathcal{I}$ or $Z \setminus A \in \mathcal{I}$.

Corollary 3.4. Let Z be a nonempty set. Then for $\mathcal{J} \in \Theta_Z$, $\mathcal{J}$ is a maximal ideal if and only if $\mathcal{F}_{\mathcal{J}}$ is an ultrafilter (for ultrafilter see [2,7,8]).

4. HOMEOMORPHISMS

Before entering to the main study of this section, we recollect some necessary requirements from literature which will be helpful for our subsequent investigations.

Definition 4.1. A subset Q of a topological space (Z,η) is said to be

(i) semi-open [10] if $Q \subseteq Cl(Int(Q))$;
(ii) preopen [11,12] if $Q \subseteq Int(Cl(Q))$;
(iii) β -open [13] or semi-preopen [14] if $Q \subseteq Cl(Int(Cl(Q)))$;
(iv) b -open [15] if $Q \subseteq Int(Cl(Q)) \cup Cl(Int(Q))$
where '*Int*' stands for the interior operator in (Z,η).

We denote the collection of all semi-open (resp., pre-open, β -open and b -open) sets containing $z \in Z$ as $SO(Z,z)$ (resp., $PO(Z,z)$, $\beta O(Z,z)$ and $BO(Z,z)$). Complement of a semi-open set is addressed as semi-closed, and intersection of all semi-closed sets containing Q is called semi-closure of Q and is denoted as $sCl(Q)$.

Let (Z,η) be a topological space, and $\mathcal{I}$ be an ideal on Z . Utilizing the local function $(\cdot)^* : 2^Z \to 2^Z$, the set operators Ψ , $\veebar$, $\wedge$, $\overline{\wedge}$, ∇_1, $\nabla_2 : 2^Z \to 2^Z$ are respectively defined as: for $Q \subseteq Z$,

- $\Psi(Q)(\mathcal{I},\eta)($ simply $\Psi(Q)) = Z \setminus (Z \setminus Q)^*$ [16];
- $\veebar(Q)(\mathcal{I},\eta)($ simply $\veebar(Q)) = \Psi(Q) \setminus Q^*$ [5];
- $\wedge(Q)(\mathcal{I},\eta)($ simply $\wedge(Q)) = \Psi(Q) \setminus Q$ [5];
- $\overline{\wedge}(Q)(\mathcal{I},\eta)($ simply $\overline{\wedge}(Q)) = Q \setminus Q^*$ [5];
- $\nabla_1(Q)(\mathcal{I},\eta)($ simply $\nabla_1(Q)) = \veebar(Q) \cap \wedge(Q)$ [17];
- $\nabla_2(Q)(\mathcal{I},\eta)($ simply $\nabla_2(Q)) = \veebar(Q) \cap \overline{\wedge}(Q)$ [17].

For more operators, see [18] and [19].

Some researchers have generalized the concept of local function $(\cdot)^* : 2^Z \to 2^Z$ in aspect of different generalized open sets. We now enlist some generalized local functions below: for $X \subseteq Z$,

- $X^{*s}(\mathcal{I},\eta) = \{z \in Z : U_z \cap X \notin \mathcal{I} \text{ for every } U_z \in SO(Z,z)\}$ [20];

- $X^{*p}(\mathcal{I},\eta)=\{z\in Z: U_z\cap X\notin\mathcal{I}$ for every $U_z\in PO(Z,z)\}$ [20];
- $X^{*b}(\mathcal{I},\eta)=\{z\in Z: U_z\cap X\notin\mathcal{I}$ for every $U_z\in BO(Z,z)\}$ [20];
- $\gamma_c(X)(\mathcal{I},\eta)=\{z\in Z: Cl(U_z)\cap X\notin\mathcal{I}$ for every $U_z\in\eta(z)\}$ [21];
- $\gamma_{sc}(X)(\mathcal{I},\eta)=\{z\in Z: sCl(U_z)\cap X\notin\mathcal{I}$ for every $U_z\in\eta(z)\}$ [20].

Lemma 4.2. [4] Let $f:Z\to Y$ be a bijective function. For a proper ideal $\mathcal{I}$ on Z, $f(\mathcal{I})=\{f(I): I\in\mathcal{I}\}$ is a proper ideal on Y.

Lemma 4.3. [4] Let $f:X\to Y$ be a surjective function. For a proper ideal $\mathcal{J}$ on Y, $f^{-1}(\mathcal{J})=\{f^{-1}(J): J\in\mathcal{J}\}$ is a proper ideal on X.

Proposition 4.4. [22] Let (Z,η) and (Y,σ) be two topological spaces, and $\mathcal{I}$ be an ideal on Z. If $f:Z\to Y$ be a homeomorphism, then

1. $f[B^*(\mathcal{I},\eta)]=[f(B)]^*(f(\mathcal{I}),\sigma)$;
2. $f[\Psi(B)(\mathcal{I},\eta)]=\Psi[f(B)](f(\mathcal{I}),\sigma)$.

Theorem 4.5. Let (Z,η) and (Y,σ) be two topological spaces and $\mathcal{I}$ be an ideal on Z. If $f:Z\to Y$ be a homeomorphism, then for $X\subseteq Z$, the following properties hold:

1. $f[\nabla_1(X)(\mathcal{I},\eta)]=\nabla_1[f(X)](f(\mathcal{I}),\sigma)$;
2. $f[\nabla_2(X)(\mathcal{I},\eta)]=\nabla_2[f(X)](f(\mathcal{I}),\sigma)$;
3. $f[X^{*p}(\mathcal{I},\eta)]=[f(X)]^{*p}(f(\mathcal{I}),\sigma)$;
4. $f[X^{*s}(\mathcal{I},\eta)]=[f(X)]^{*s}(f(\mathcal{I}),\sigma)$;
5. $f[X^{*\beta}(\mathcal{I},\eta)]=[f(X)]^{*\beta}(f(\mathcal{I}),\sigma)$;
6. $f[X^{*b}(\mathcal{I},\eta)]=[f(X)]^{*b}(f(\mathcal{I}),\sigma)$;
7. $f[\gamma_c(X)(\mathcal{I},\eta)]=\gamma_c[f(X)](f(\mathcal{I}),\sigma)$;
8. $f[\gamma_{sc}(X)(\mathcal{I},\eta)]=\gamma_{sc}[f(X)](f(\mathcal{I}),\sigma)$.

Proof. 1. We have $f(\nabla_1(X)(\mathcal{I},\eta))=f((\veebar(X)\cap\wedge(X))(\mathcal{I},\eta))=f(\veebar(X)(\mathcal{I},\eta))\cap f(\wedge(X)(\mathcal{I},\eta))=f((\Psi(X)\setminus X^*)(\mathcal{I},\eta))\cap f((\Psi(X)\setminus X)(\mathcal{I},\eta))=[f(\Psi(X)(\mathcal{I},\eta))\setminus f(X^*(\mathcal{I},\eta))]\cap[f(\Psi(X)(\mathcal{I},\eta))\setminus f(X)]=[\Psi(f(X))(f(\mathcal{I}),\sigma)\setminus (f(X))^*(f(\mathcal{I}),\sigma)]\cap[\Psi(f(X))(f(\mathcal{I}),\sigma)\setminus f(X)]$ (by Proposition 4.4) $=[(\Psi(f(X))\setminus (f(X))^*)(f(\mathcal{I}),\sigma)]\cap[(\Psi(f(X))\setminus f(X))(f(\mathcal{I}),\sigma)]=(\veebar(f(X))(f(\mathcal{I}),\sigma))\cap(\wedge(f(X))(f(\mathcal{I}),\sigma))=(\veebar(f(X))\cap\wedge(f(X)))(f(\mathcal{I}),\sigma)=\nabla_1(f(X))(f(\mathcal{I}),\sigma)$.

2. Similar to 1.

3. Assume that $x \in Z$ with $f(x) \notin f\left[X^{*p}(\mathcal{I},\eta)\right]$. This implies that $x \notin X^{*p}(\mathcal{I},\eta)$. Thus, there exists $U_x \in PO(X,x)$ such that $U_x \cap X \in \mathcal{I}$ and hence $f(U_x \cap X) \in f(\mathcal{I})$. Therefore, $f(U_x) \cap f(X) \in f(\mathcal{I})$. We now show that $f(U_x) \in PO(Y, f(x))$ i.e., $f(U_x) \subseteq Int(Cl(f(U_x)))$. Since $U_x \subseteq Int(Cl(U_x))$, $f(U_x) \subseteq f(Int(Cl(U_x))) = Int(f(Cl(U_x))) = Int(Cl(f(U_x)))$. Thus, $f(U_x) \in PO(Y, f(x))$, and hence $f(x) \notin [f(X)]^{*p}(f(\mathcal{I}),\sigma)$. Therefore, $f\left[X^{*p}(\mathcal{I},\eta)\right] \supseteq [f(X)]^{*p}(f(\mathcal{I}),\sigma)$.

For reverse inclusion, assume that $t \in Z$ with $f(t) \notin [f(X)]^{*p}(f(\mathcal{I}),\sigma)$. Then there exists $U_{f(t)} \in PO(Y, f(t))$ such that $U_{f(t)} \cap f(X) \in f(\mathcal{I})$. Thus, $f^{-1}(U_{f(t)} \cap f(X)) = f^{-1}(U_{f(t)}) \cap X \in \mathcal{I}$. Moreover, $f^{-1}(U_{f(t)}) \in PO(Z,t)$. Hence, $t \notin X^{*p}(\mathcal{I},\eta)$ implies $f(t) \notin f\left[X^{*p}(\mathcal{I},\eta)\right]$. Therefore, $f\left[X^{*p}(\mathcal{I},\eta)\right] \subseteq [f(X)]^{*p}(f(\mathcal{I}),\sigma)$.

Proofs of rests are similar to that of 3.

Proposition 4.6. [22] Let (Z,η) and (T,τ) be two topological spaces, and $\mathcal{J}$ be an ideal on T. If $f: Z \to T$ be a homeomorphism, then for $B \subseteq T$,

1. $f^{-1}\left[B^{*}(\mathcal{J},\tau)\right] = [f^{-1}(B)]^{*}(f^{-1}(\mathcal{J}),\eta)$;
2. $f^{-1}\left[\Psi(B)(\mathcal{J},\tau)\right] = \Psi[f^{-1}(B)](f^{-1}(\mathcal{J}),\eta)$.

Theorem 4.7. Let (Z,η) and (T,τ) be two topological spaces, and $\mathcal{J}$ be an ideal on T. If $f: Z \to T$ be a homeomorphism, then for $B \subseteq T$, the following properties hold:

1. $f^{-1}\left[\nabla_1(B)(\mathcal{J},\tau)\right] = \nabla_1[f^{-1}(B)](f^{-1}(\mathcal{J}),\eta)$;
2. $f^{-1}\left[\nabla_2(B)(\mathcal{J},\tau)\right] = \nabla_2[f^{-1}(B)](f^{-1}(\mathcal{J}),\eta)$;
3. $f^{-1}\left[B^{*p}(\mathcal{J},\tau)\right] = [f^{-1}(B)]^{*p}(f^{-1}(\mathcal{J}),\eta)$;
4. $f^{-1}\left[B^{*s}(\mathcal{J},\tau)\right] = [f^{-1}(B)]^{*s}(f^{-1}(\mathcal{J}),\eta)$;
5. $f^{-1}\left[B^{*\beta}(\mathcal{J},\tau)\right] = [f^{-1}(B)]^{*\beta}(f^{-1}(\mathcal{J}),\eta)$;
6. $f^{-1}\left[B^{*b}(\mathcal{J},\tau)\right] = [f^{-1}(B)]^{*b}(f^{-1}(\mathcal{J}),\eta)$;
7. $f^{-1}\left[\gamma_c(B)(\mathcal{J},\tau)\right] = \gamma_c[f^{-1}(B)](f^{-1}(\mathcal{J}),\eta)$;
8. $f^{-1}\left[\gamma_{sc}(B)(\mathcal{J},\tau)\right] = \gamma_{sc}[f^{-1}(B)](f^{-1}(\mathcal{J}),\eta)$.

Proof. Proof is straightforward and thus omitted.

Theorem 4.8. Let (Z,η) and (T,τ) be two topological spaces, and $S: D \to Z$ be a net. If $f: Z \to T$ be a homeomorphism, then

1. $f(\mathcal{I}_S)$ is an ideal on T;
2. for each $U_z \in \eta(z)$, $Z \setminus U_z \in \mathcal{I}_S$ iff $T \setminus f(U_z) \in f(\mathcal{I}_S)$.

Proof. 1. Obvious from Lemma 4.2.

2. Forward part is obvious from the definition of $f(\mathcal{I}_S)$, and converse part is followed from Lemma 4.3.

Theorem 4.9. Let (Z,η) and (T,τ) be two topological spaces, and $\mathcal{F}$ be a filter on Z. If $f: Z \to T$ be a homeomorphism, then

1. $f(\mathcal{I}_{\mathcal{F}})$ is an ideal on T;
2. for each $U_z \in \eta(z)$, $Z \setminus U_z \in \mathcal{I}_{\mathcal{F}}$ iff $T \setminus f(U_z) \in f(\mathcal{I}_F)$.

Proof. 1. Obvious from Lemma 4.2.

2. Forward part is obvious from the definition of $f(\mathcal{I}_{\mathcal{F}})$ and converse part is followed from Lemma 4.3.

Theorem 4.10. Let Z and Y be two non-empty sets, and Θ_Z be the collection of all proper ideals on Z. Let $f: Z \to Y$ be a bijective function. Then for the partial order set $(\Theta_Z, \subseteq)$, $(f(\Theta_Z), \subseteq)$ is a partial ordered set, where $f(\Theta_Z) = \{f(\mathcal{I}) : \mathcal{I} \in \Theta_Z\}$.

Proof. Obvious from Lemma 4.2 and Theorem 3.1.

Theorem 4.11. Let Z and T be two non-empty sets, and $f: Z \to T$ be a bijective function. If $\{\mathcal{J}_j : j \in \Omega\}$ be a chain in $(\Theta_Z, \subseteq)$, then

1. $\{f(\mathcal{J}_j) : j \in \Omega\}$ is a chain in $(f(\Theta_Z), \subseteq)$;
2. $f\left(\bigcup_{j\in\Omega} \mathcal{J}_j\right) = \bigcup_{j\in\Omega} f(\mathcal{J}_j)$ is an upper bound of the chain $\{f(\mathcal{J}_j) : j \in \Omega\}$;
3. $(f(\Theta_Z), \subseteq)$ has a maximal element.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

ACKNOWLEDGEMENTS

The second author gives a special thanks to CSIR, India as she received fellowship (File No.: 09/1151/(0009)/2020-EMR-I) during this work from CSIR.

REFERENCES

[1] Kuratowski, K., Topology Vol. I, Academic Press, New York, (1966).

[2] Husain, T., Topology and Maps, First Edition, Plenum Press, New York and London, (1977).

[3] Lahiri, B. K., Das, P., “I and I^*-convergence in topological spaces”, Mathematica Bohemica, 130(2): 153–160, (2005).

[4] Modak, S., Hoque, J., Selim, Sk., “Homeomorphic image of some kernels”, Çankaya University Journal of Science and Engineering, 17(1): 052-062, (2020).

[5] Noiri, T., Selim, Sk., Modak, S., “Convergence of the associated filters via set-operators”, Çankaya University Journal of Science and Engineering, 17(2): 101-107, (2020).

[6] Baran, T. M., Erciyes, A., “Local T_3 constant filter convergence spaces”, Gazi University Journal of Science, 33(2): 446-454, (2020).

[7] Bourbaki, N., “Elements of Mathematics General Topology”, Chapters 1-4, Springer, New York, (1989).

[8] Joshi, K. D., “Introduction to general topology”, Revised ed., Wiley Estern Limited, New Delhi, (1984).

[9] Simmons, G. F., “Introduction to Topology and Modern Analysis”, Robert E. Krieger Publishing Company, Florida, (1963).

[10] Levine, N., “Semi-open sets and semi-continuity in topological spaces”, The American Mathematical Monthly, 70(1): 36–41, (1963).

[11] Mashhour, A. S., El-Monsef, M. E. Abd., El-Deeb, S. N., “On precontinous and weak precontinous mappings”, Proceedings of the Mathematical and Physical Society of Egypt, 53: 47-53, (1982).

[12] Ganster, M., “Preopen sets and resolvable spaces”, Kyungpook Mathematical Journal, 27(2): 135-143, (1987).

[13] El-Monsef, M. E. Abd., “β-open sets and β-continuous mappings”, Bulletin of the Faculty of Science, Assiut University, 12: 77-90, (1983).

[14] Andrijević, D., “Semi-preopen sets”, Matematički Vesnik, 38(93): 24-32, (1986).

[15] Andrijević, D., “On b-open sets”, Matematički Vesnik, 48: 59-64, (1996).

[16] Natkaniec, T., “On I-continuity and I-semicontinuity points”, Mathematica Slovaca, 36(3): 297-312, (1986).

[17] Selim, Sk., Noiri, T., Modak, S., “Operators in terms of * and Ψ”, Boletim da Sociedade Paranaense de Matemática, 41(3): 1-7, (2023).

[18] Al-Saadi, H., Al-Omari, A., “Some operators in ideal topological spaces”, Missouri Journal of Mathematical Sciences, 30(1): 59-71, (2018).

[19] Al-Omari, A., Noiri, T., “On operators in ideal minimal spaces”, Mathematica, 58(81), (1-2): 3-13, (2016).

[20] Selim, Sk., Islam, Md. M., Modak, S., “Common properties and approximations of local function and set operator Ψ”, Cumhuriyet Science Journal, 41(2): 360-368, (2020).

[21] Al-Omari, A., Noiri, T., “Local closure functions in ideal topological spaces”, Novi Sad Journal of Mathematics, 43(2): 139-149, (2013).

[22] Hoque, J., Modak, S., Acharjee, S., “Filter versus ideal on the topological spaces”, Advances in topology and their interdisciplinary applications, Springer, 183-195, (2023).