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# ON GENERALIZATIONS OF LOCALLY ARTINIAN SUPPLEMENTED MODULES

BURCU NİŞANCI TÜRKMEN

*0000-0001-7900-0529*

Abstract. The aim of this paper is to investigate generalizations of locally artinian supplemented modules in module theory, namely locally artinian radical supplemented modules and strongly locally artinian radical supplemented modules. We have obtained elementary features of them. Also, we have characterized strongly locally artinian radical supplemented modules by left perfect rings. Finally, we have proved that the reduced part of a strongly locally artinian radical supplemented $R$-module has the same property over a Dedekind domain $R$.

## 1. Introduction

Throughout this paper, the ring $R$ will denote an associative ring with identity element and modules will be left unital. We will use the notation $U \ll M$ to stress that $U$ is a *small* submodule of $M$. $Rad(M)$ will indicate radical of $M$ which is sum of all small submodules of $M$, and $Soc(M)$ will indicate socle of $M$ which is sum of all semisimple submodules of $M$. A non-zero module $M$ is called *hollow* if every proper submodule of $M$ is small in $M$, and $M$ is called *local* if the sum of all proper submodules of $M$ is also a proper submodule of $M$. A module $M$ is called *semilocal* if $\frac{M}{Rad(M)}$ is semisimple. $M$ is called *locally artinian* if every finitely generated submodule of $M$ is artinian [8, 31]. A submodule $V$ of $M$ is called *a supplement* of $U$ in $M$ if $M = U + V$ and $U \cap V \ll V$. The module $M$ is called *supplemented* if every submodule of $M$ has a supplement in $M$. A submodule $U$ of $M$ has ample supplements in $M$ if every submodule $V$ of $M$ with $M = U + V$ contains a supplement $V^{'}$ of $U$ in $M$. The module $M$ is called *amply supplemented* if every submodule of $M$ has ample supplements in $M$ [8]. Moreover, it is called $\oplus$-supplemented if every submodule of $M$ has a supplement in the form of a direct summand of $M$. Clearly, the $\oplus$-supplemented modules are supplemented.

In [10], Zöschinger introduced a notion of modules with radical which has supplements and called them *radical supplemented*. In the same paper and in [12],

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the structure of radical supplemented modules is determined. Motivated by this, Büyükaşık and Türkmen call a module $M$ *strongly radical supplemented* (or, briefly, a srs-module) if every submodule containing the radical has a supplement [2]. In [4], it is introduced another notion of $\oplus$-radical supplemented modules. A module $M$ is called $\oplus$-radical supplemented if $Rad(M)$ has a supplement which is a direct summand of $M$. In this paper, a module $M$ is called *strongly $\oplus$-radical supplemented* provided that every submodule containing the radical has a supplement which is a direct summand of $M$.

In [9], a generalization of concept of socle as a $Soc_s(M) = \sum\{U \ll M \,|\, U$ is simple$\}$ is defined. Here $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$. In [3], a module $M$ is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule $U$ of $M$ is called an *ss-supplement* of $U$ in $M$ if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. The module $M$ is called *ss-supplemented* if every submodule of $M$ has an $ss$-supplement in $M$. A submodule $U$ of $M$ has ample $ss$-supplements in $M$ if every submodule $V$ of $M$ such that $M = U + V$ contains an $ss$-supplement $V'$ of $U$ in $M$. The module $M$ is called *amply ss-supplemented* if every submodule of $M$ has ample $ss$-supplements in $M$. In [6], strongly local and (amply) $ss$-supplemented modules are generalized as $RLA$-local and (amply) locally artinian supplemented modules, respectively. A local module $M$ is called *$RLA$-local* if $Rad(M)$ is a locally artinian submodule of $M$. A module $M$ is called *locally artinian supplemented* if every submodule $U$ of $M$ has a locally artinian supplement in $M$, that is, $V$ is a supplement of $U$ in $M$ such that $U \cap V$ is locally artinian. $M$ is called *amply locally artinian supplemented* if every submodule $U$ of $M$ has ample locally artinian supplements in $M$. Here a submodule $U$ of $M$ has ample locally artinian supplements in $M$ if every submodule $V$ of $M$ such that $M = U + V$ contains a locally artinian supplement $V'$ of $U$ in $M$.

Motivated by this, we define locally artinian radical supplemented modules as a generalization of locally artinian supplemented modules and also define the concept of strongly locally artinian radical supplemented modules which is contained in the concept of locally artinian radical supplemented modules. In Section 2, it is shown that a module $M$ with small radical is strongly locally artinian radical supplemented if and only if $M$ is strongly radical supplemented and $Rad(M)$ is locally artinian if and only if $M$ is locally artinian supplemented. It is also shown that every factor module of a strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented. It is proved that any finite sum of strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented. It is also proved that $R$ is a left perfect ring and $Rad(M)$ is locally artinian if and only if every $R$-module is a strongly locally artinian radical supplemented module. Finally, it is obtained that over a Dedekind domain $R$, an $R$-module $M$ is strongly locally artinian radical supplemented if and only if the reduced part $N$ of $M$ is strongly locally artinian radical supplemented.

## 2. Strongly Locally Artinian Radical Supplemented Modules

**Definition 1.** Let $M$ be a module. Then $M$ is called a *locally artinian radical supplemented module* if $Rad(M)$ has a locally artinian supplement in $M$. A module $M$ is called *strongly locally artinian radical supplemented* if every submodule which contains $Rad(M)$ in $M$ has a locally artinian supplement in $M$.

**Proposition 1.** Let $M$ be a module with $Rad(M) = 0$. Then $M$ is a locally artinian radical supplemented module.

*Proof.* Since $M$ is a locally artinian supplement of $Rad(M)$ in $M$, the proof is clear. □

Recall that a module $M$ is called *radical* if $Rad(M) = M$.

**Proposition 2.** Let $M$ be a radical module. Then $M$ is strongly locally artinian radical supplemented.

*Proof.* Let $U$ be a submodule with $Rad(M) \subseteq U$. Since $Rad(M) = M$, $U = M$. So 0 is a locally artinian supplement of $U$ in $M$. Therefore $M$ is strongly locally artinian radical supplemented. □

Recall that $P(M)$ is the sum of all radical submodule of a module $M$ and $P(M)$ is a largest radical submodule of $M$. So, note that $Rad(P(M)) = P(M)$.

**Proposition 3.** $P(M)$ is a strongly locally artinian radical supplemented module for every module $M$.

*Proof.* Since $Rad(P(M)) = P(M)$, the proof follows from Proposition 2. □

It is clear that every locally artinian supplemented modules are locally artinian radical supplemented. Definition 1, notice that every strongly locally artinian supplemented module is locally artinian radical supplemented. The following example shows that the converse of these situations are not always true.

Recall that an integral domain $R$ is a *Dedekind domain* if every non-zero ideal of $R$ is invertible.

**Example 1.** $(i)$ Let $M =_{\mathbb{Z}} \mathbb{Z}$. Since $Rad(\mathbb{Z}) = 0$, $M$ is locally artinian radical supplemented by Proposition 1. But $M$ is not a locally artinian supplemented module.

$(ii)$ Let $R$ be a local Dedekind domain and $K$ be a quotient field of $R$. Since $Rad(K) = K$, $K$ is strongly locally artinian radical supplemented by Proposition 2. It follows from [6, Example 2.7] that $K$ is not locally artinian supplemented.

**Proposition 4.** Let $M$ be a module with small radical. Then $M$ is locally artinian radical supplemented if and only if $Rad(M)$ is a locally artinian submodule of $M$.

*Proof.* ($\Rightarrow$) Since $M$ is locally artinian radical supplemented, there exists a submodule $N$ of $M$ such that $M = Rad(M) + N$, $Rad(M) \cap N \ll N$ and $Rad(M) \cap N$ is locally artinian. Since $Rad(M) \ll M$, then $N = M$. So $Rad(M) \cap M = Rad(M)$ is locally artinian.

($\Leftarrow$) By the hypothesis, $M$ is a locally artinian supplement of $Rad(M)$ in $M$, as desired. □

**Corollary 1.** Let $M$ be a finitely generated module. Then $M$ is locally artinian radical supplemented if and only if $Rad(M)$ is a locally artinian submodule of $M$.

*Proof.* Since $M$ is finitely generated, $M$ has a small radical. So the proof follows from Proposition 4. □

**Example 2.** (see [6, Example 2.2]) Consider $\mathbb{Z}$-module $M = \mathbb{Z}_8$. Since $Rad(M) = \langle \overline{2} \rangle \ll M$ and $M$ is locally artinian, $M$ is an RLA-local module. It follows from [6, Theorem 2.11] that $M$ is locally artinian supplemented. Then $Rad(M)$ is locally artinian by [8, 31.2.(1)(i)]. So $M$ is locally artinian radical supplemented by Proposition 4. In addition, as $M$ is locally artinian supplemented, $M$ is strongly locally artinian radical supplemented. But $Rad(M)$ has not an ss-supplement in $M$.

Recall that a ring $R$ is called *a left max ring* if every non-zero $R$-module has a maximal submodule.

**Corollary 2.** Let $R$ be a left max ring and $M$ be an $R$-module. Then $M$ is locally artinian radical supplemented if and only if $Rad(M)$ is a locally artinian submodule of $M$.

*Proof.* By the hypothesis, there exists a submodule $N$ of $M$ such that $M = Rad(M) + N$. It follows that $Rad(\frac{M}{N}) = \frac{M}{N}$. Since $R$ is a left max ring, $\frac{M}{N} = 0$. So $M = N$. Thus $Rad(M) \ll M$. The proof follows from Proposition 4. □

**Proposition 5.** Every factor module of a strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented.

*Proof.* Let $M$ be a strongly locally artinian radical supplemented module with $N \subseteq K \subseteq M$ and $Rad(\frac{M}{N}) \subseteq \frac{K}{N}$. Let $\pi : M \longrightarrow \frac{M}{N}$ be a canonical projection. Then $\pi(Rad(M)) = \frac{Rad(M)+N}{N} \subseteq Rad(\frac{M}{N}) \subseteq \frac{K}{N}$. So $Rad(M) \subseteq K$. By the hypothesis, there exists a submodule $T$ of $M$ such that $M = K + T$, $K \cap T \ll T$ and $K \cap T$ is locally artinian. Then $\frac{M}{N} = \frac{K}{N} + \frac{(T+N)}{N}$, $\frac{K}{N} \cap \frac{(T+N)}{N} \ll \frac{(T+N)}{N}$. By [8, 31.2 (1)(i)], $\frac{K}{N} \cap \frac{(T+N)}{N}$ is locally artinian. Therefore $\frac{M}{N}$ is strongly locally artinian radical supplemented. □

**Corollary 3.** Every homomorphic image of a strongly locally artinian radical supplemented module is strongly locally artinian radical supplemented.

**Proposition 6.** Let $M$ be a module and $N \subseteq M$. If $N$ is a strongly locally artinian radical supplemented module and $Rad(\frac{M}{N}) = \frac{M}{N}$, then $M$ is a strongly locally artinian radical supplemented module.

*Proof.* Let $U$ be a submodule of $M$ with $Rad(M) \subseteq U$. Since $Rad(\frac{M}{N}) = \frac{M}{N}$, $M = Rad(M) + N$. So $M = U + N$. Then $Rad(N) \subseteq Rad(M) \subseteq U$ and $Rad(N) \subseteq N$. Note that $Rad(N) \subseteq U \cap N$. Since $N$ is strongly locally artinian radical supplemented, $N = (U \cap N) + K$, $(U \cap N) \cap K = U \cap K \ll K$ and $U \cap K$ is locally artinian for some submodule $K$ of $M$. Then we have $M = Rad(M) + (U \cap N) + K = U + (U \cap N) + K = U + K$. Thus $M$ is strongly locally artinian radical supplemented. □

**Lemma 1.** *Let $M$ be a module, $M_1$ and $K$ be submodules of $M$ and $Rad(M) \subseteq K$. If $M_1$ is a strongly locally artinian radical supplemented and $M_1 + K$ has a locally artinian radical supplement in $M$, then $K$ has a locally artinian supplement in $M$.*

*Proof.* Let $N$ be a locally artinian supplement of $M_1 + K$ in $M$ and $T$ be a locally artinian supplement of $(N + K) \cap M_1$ in $M_1$. Then we have $M = N + K + T$, $(M_1 + K) \cap N$ is locally artinian. Also we have $(N + K) \cap T \ll T$ and $(N + K) \cap T$ is locally artinian. Since $(M_1 + K) \cap N$ is locally artinian, $N \cap (K + T)$ is locally artinian by [8, 31.2(1)(i)]. It follows from $(M_1 + K) \cap N \ll N$ and $(N + K) \cap T \ll T$ that $K \cap (N+T) \subseteq N \cap (K+T) + T \cap (K+N) \subseteq N \cap (K+M_1) + T \cap (K+N) \ll N+T$. So $T \cap (K + N)$ is locally artinian by [8, 31.2(2)], as required. □

**Proposition 7.** Let $M = M_1 + M_2$ be a module with submodules $M_1, M_2 \subseteq M$. If $M_1$ and $M_2$ are strongly locally artinian radical supplemented, then $M$ is strongly locally artinian radical supplemented.

*Proof.* Let $K$ be a module with $Rad(M) \subseteq K$. Since $M_1 + M_2 + K$ has a locally artinian radical supplement 0 in $M$, $M_1 + K$ has a locally artinian supplement in $M$ by Lemma 1. Applying again Lemma 1, we obtain that $M$ is strongly locally artinian supplemented. □

**Corollary 4.** Every finite sum of strongly locally artinian radical supplemented modules is a strongly locally artinian radical supplemented module.

**Proposition 8.** Let $M$ be a module with $Rad(M) \ll M$. Then $M$ is strongly locally artinian radical supplemented if and only if $M$ is locally artinian supplemented.

*Proof.* ($\Rightarrow$) Let $N$ be a submodule of $M$. Then $Rad(M) \subseteq Rad(M) + N$. By the hypothesis, $Rad(M) + N$ has a locally artinian supplement $K$ in $M$. So, $M = Rad(M) + N + K$, $(Rad(M) + N) \cap K \ll K$ and $(Rad(M) + N) \cap K$ is locally artinian. Since $Rad(M) \ll M$, then $M = N + K$. It is clear that $N \cap K \ll K$. By [8, 31.1(i)] $N \cap K$ is locally artinian. Thus $M$ is locally artinian supplemented.

($\Leftarrow$) It is clear. □

Recall from a module $M$ is called *coatomic* if every proper submodule of $M$ is contained in a maximal submodule of $M$, equivalently, for a submodule $N$ of $M$, whenever $Rad(\frac{M}{N}) = \frac{M}{N}$, then $M = N$. Since every coatomic module has small radical, the following corollary is obtained clearly.

**Corollary 5.** Let $M$ be a coatomic module. Then $M$ is locally artinian supplemented if and only if $M$ is strongly locally artinian radical supplemented.

**Corollary 6.** Let $M$ be a module with $Rad(M) \ll M$. Then the following statements are equivalent.

(1) $M$ is locally artinian supplemented;
(2) $M$ is supplemented and $M$ is locally artinian radical supplemented;
(3) $M$ is strongly radical supplemented and $Rad(M)$ is locally artinian;
(4) $M$ is strongly locally artinian radical supplemented.

*Proof.* (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Clear by Proposition 4.

(3) $\Rightarrow$ (4) Let $K$ be a module with $Rad(M) \subseteq K$. Since $M$ is strongly radical supplemented, there exists a submodule $L$ of $M$ such that $M = K + L$, $K \cap L \ll L$. Then $K \cap L \subseteq Rad(L) \subseteq Rad(M)$. It follows from [8, 31.2(1)(i)] that $K \cap L$ is locally artinian, as desired.

(4) $\Rightarrow$ (1) Since $M$ is strongly locally artinian radical supplemented, $M$ is locally artinian radical supplemented. The proof follows from Proposition 8. □

It follows from [8, 43.9] that a ring $R$ is *left perfect* if and only if $R$ is semilocal and $Rad(R)$ is right T-nilpotent if and only if every $R$-module has a projective cover, that is, for any $R$-module $M$, there exists a projective module $P$ and an epimorphism $f : P \longrightarrow M$ with small kernel.

**Theorem 1.** *Let $R$ be a ring. Then the following statements are equivalent.*

(1) *$R$ is a left perfect ring and $Rad(R)$ is locally artinian;*
(2) *every free $R$-module is strongly locally artinian radical supplemented;*
(3) *every $R$-module is strongly locally artinian radical supplemented.*

*Proof.* (1) $\Rightarrow$ (2) Let $F$ be free $R$-module $R^{(I)}$ for some index set $I$. It follows from [8, 31.2(2)and 43.9] that $Rad(F) = Rad(R)^{(I)}$ is locally artinian and $F$ is supplemented. Since $Rad(F) \ll F$, $F$ is locally artinian supplemented by [6, Theorem 2.9]. We obtain that $F$ is strongly locally artinian radical supplemented by Proposition 8.

(2) $\Rightarrow$ (3) Since every $R$-module is a homomorphic image of a free $R$-module, the proof is obvious by Proposition 5.

(3) $\Rightarrow$ (1) Clear by Proposition 8 and [8, 43.9]. □

Recall that $P(M)$ is the divisible part of $M$ for an $R$-module $M$ over a Dedekind domain $R$. According to [1, Lemma 4.4], $P(M)$ is (divisible) injective, and so there exists a submodule $N$ of $M$ such that $M = P(M) \oplus N$. Here, $N$ is called *the reduced part* of $M$. Note that $P(M) \subseteq Rad(M)$. By Proposition 3, $P(M)$ is strongly locally artinian radical supplemented. Using these facts, we obtain the following result.

**Proposition 9.** Let $R$ be a Dedekind domain and $M$ be an $R$-module. Then $M$ is strongly locally artinian radical supplemented if and only if the reduced part $N$ of $M$ is strongly locally artinian radical supplemented.

*Proof.* ($\Rightarrow$) Since $N$ is a homomorphic image of $M$, $N$ is strongly locally artinian radical supplemented by Proposition 5.

($\Leftarrow$) Clear by Proposition 7. □

## 3. Conclusion

In this paper, we obtain new classes of modules from locally artinian supplemented modules. To obtain these class of modules, we have associated with radical of the module and every submodule that contains radical of the module. Also, we study on the algebraic structure of these modules. We characterize strongly locally artinian radical supplemented modules over a left perfect ring.

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(Burcu Nişancı Türkmen) Amasya university, Mathematics department, 05100, Amasya, Turkey

*Email address*, Burcu Nişancı Türkmen: burcu.turkmen@amasya.edu.tr