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Some Characterizations of Spherical Indicatrix Curves Generated by Flc Frame

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Abstract. In this study, the spherical indicatrices of Flc frame vectors were defined on unit sphere. The arc length parameters and the Frenet vectors of these indicatrix curves were calculated, as well. Last, we have provided the geodesic curvatures according to both Euclidean space E^3 and unit sphere S^2.

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1. Introduction

In classical differential geometry, the curves are characterized by a moving frame of which the most referred one is the Frenet frame. However, to settle a frame on a given curve like Frenet, it is required to make a choice of at least C^3 class non-degenerate curve. In 1975, L.Bishop defined another frame known to be alternative parallel frame that can be settled on the points at which even the curvatures vanish [1]. As Bishop pointed out in his study that ”there is more than one way to frame a curve”, recently Dede has introduced a new moving frame relatively having some advantages but under the constraint that the curve to be framed should be polynomial. They named it as the Flc (Frenet-like curve) frame. Calculations to be made for Flc frame is much easier compared to Frenet frame and it has an analytic form which is not the case for Bishop frame. In addition to these, Flc frame is less singular than Frenet frame and has much less undesirable rotations around the tangent of the curve [3]. Spherical indicatrices, on the other hand, are special curves drawn by the unit vectors centered at a unit sphere. In 3 dimensional Euclidean space, the characteristics of such curves like arc-lengths and geodesic curvatures are given in [6]. These are examined for Minkowski space in [2]. The spherical indicatrix curves according to Bishop frame and Dual Bishop frame were investigated in [12] and [5], respectively. The spherical indicatrices were charecterized on slant helix as a special curve in [7]. There are also studies spherical indicatrices curves of some special curves in [10, 11].

In this study, we first defined the spherical indicatrix curves of each vector components of Flc frame on unit sphere. Then for each spherical indicatrices, we calculated the Frenet frame and defined the arc lengths and the geodesic curvatures for both E^3 and S^2.

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2. Preliminaries

In this section, we remind some basic concepts that will be used through out the paper. Let $\alpha = \alpha(s)$ be a regular space curve and non-degenerate condition $\alpha^{'}(s) \wedge \alpha^{''}(s) \neq 0$. Then, three orthogonal vector fields which called Frenet frame are defined as [9]

$$T(s) = \frac{\alpha^{'}(s)}{\|\alpha^{'}(s)\|}, \quad B(s) = \frac{\alpha^{'}(s) \wedge \alpha^{''}(s)}{\|\alpha^{'}(s) \wedge \alpha^{''}(s)\|}, \quad N(s) = B(s) \wedge T(s), \tag{2.1}$$

where T is the tangent, N is the principal normal and B is the binormal vector field. The Frenet formulas are given by

$$T' = \kappa v N, \quad N' = -\kappa v T + \tau v B, \quad B' = -\tau v N \ , \quad \left\|\alpha'\right\| = v, \tag{2.2}$$

where the curvature κ and torsion τ of the curve are

$$\kappa = \frac{\left\|\alpha^{'}(s) \wedge \alpha^{''}(s)\right\|}{\|\alpha^{'}(s)\|^3}, \qquad \tau = \frac{\left\langle \alpha^{'}(s) \wedge \alpha^{''}(s), \alpha^{'''}(s)\right\rangle}{\|\alpha^{'}(s) \wedge \alpha^{''}(s)\|^2}.$$

We define the n^{th} degree polynomial with parameter t as

$$P(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t^1 + a_0, \qquad a_n \neq 0,$$

where $n \in N_0$, $a_i \in \mathbb{R}$, $(0 \leq i \leq n)$ [8].

Now let us define a curve such that, $\alpha : [a, b] \to E^n, \quad \alpha(t) = (\alpha_1(t), \alpha_2(t), ..., \alpha_n(t))$. If each α_is are polynomial for $1 \leq i \leq n$, then $a_t \in \mathbb{R}[t]$ is defined to be an n-dimensional polynomial curve [4]. The degree of such a polynomial curve as $\alpha(t)$ is given by

$$\deg \alpha(t) = max \{\deg(\alpha_1(t)), \deg(\alpha_2(t)), ..., \deg(\alpha_n(t))\} \ [8].$$

Regarding to these, let us define $\alpha = \alpha(s)$ as a polynomial space curve. The definition of the Flc frame for α introduced by Dede in [3] is as following

$$T(s) = \frac{\alpha^{'}(s)}{\|\alpha^{'}(s)\|}, \quad D_1(s) = \frac{\alpha^{'}(s) \wedge \alpha^{(n)}(s)}{\left\|\alpha^{'}(s) \wedge \alpha^{(n)}(s)\right\|}, \quad D_2(s) = D_1(s) \wedge T(s), \tag{2.3}$$

where the prime $'$ indicates the differentiation with respect to s and $^{(n)}$ stands for the n^{th} derivative. The new vectors D_1 and D_2 are called as binormal-like vector and normal-like vector, respectively. The curvatures of the Flc-frame d_1, d_2 and d_3 are given by

$$d_1 = \frac{\langle T', D_2 \rangle}{v}, \quad d_2 = \frac{\langle T', D_1 \rangle}{v}, \quad d_3 = \frac{\langle D_2', D_1 \rangle}{v},$$

where $\|\alpha'\| = v$. The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} T' \\ D_2' \\ D_1' \end{bmatrix} = v \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}. \tag{2.4}$$

The Darboux vector D_W of the Flc-Frame can be obtained as in the following form

$$D_W = v(d_3 T - d_2 D_2 + d_1 D_1). \tag{2.5}$$

Thus, the instantaneous angular speed of the Flc-frame is calculated as follows

$$\|D_W\| = v\sqrt{{d_1}^2 + {d_2}^2 + {d_3}^2}.$$

The instantaneous rate of change for each of these vectors T, D_2 and D_1 has two components, namely from (2.5), we write

- $D_W \wedge T = \ \ d_2 D_1 + d_1 D_2$, thus T changes instantaneous rate d_2 and d_1 in the direction of D_2 and D_1, respectively
- $D_W \wedge D_2 = \ \ d_3 D_1 - d_1 T$, thus D_2 changes instantaneous rate d_3 and d_1 in the direction of D_2 and T, respectively.
- $D_W \wedge D_1 = -d_3 D_2 - d_2 T$, thus D_1 changes instantaneous rate d_3 and d_2 in the direction of D_2 and T, respectively.

For a surface, M in E^3, let us denote the shape operator with S, the unit normal vector field with ξ and the Riemann connection with D, then for $\forall X, Y \in \chi(M)$, the Gauss equation is defined by

$$\bar{D}_X Y = D_X Y + \langle S(X), Y\rangle \xi. \tag{2.6}$$

Here, $\bar{D}$ is derivative operator in Gauss sense.
The geodesic curvatures of α according to E^3 and S^2 is given by

$$k_g = \|D_T T\|, \tag{2.7}$$

$$\mu_g = \left\|\bar{D}_T T\right\|, \tag{2.8}$$

respectively [6].

3. Some Characterizations of Spherical Indicatrix Curves Generated by Flc Frame

In this section, we define the spherical indicatrix curves of Flc frame on the unit sphere. We also provide the Frenet elements of each indicatrices and by using Gauss equation we calculate geodesic curvatures according to both E^3 and S^2.

Definition 3.1. Let α be a polynomial curve in E^3 and $\{T, D_2, D_1\}$ denote its Flc frame. The curve traced out by the vector T centered at unit sphere is called $T-$ or tangent indicatrix curve according to Flc frame and is defined as

$$\alpha_T(s) = T(s).$$

Theorem 3.2. *The Frenet vectors T_T, N_T, B_T of $T-$ indicatrix curve according to Flc frame are given as follows:*

$$\begin{aligned}
T_T &= \frac{d_1}{\sqrt{{d_1}^2+{d_2}^2}}D_2 + \frac{d_2}{\sqrt{{d_1}^2+{d_2}^2}}D_1,\\
N_T &= \frac{x_2d_2 - x_3d_1}{\sqrt{{d_1}^2+{d_2}^2}}T - \frac{x_1d_2}{\sqrt{{d_1}^2+{d_2}^2}}D_2 + \frac{x_1d_1}{\sqrt{{d_1}^2+{d_2}^2}}D_1,\\
B_T &= \frac{(v^3d_3(d_1^2+d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')T - (v^3d_2(d_1^2+d_2^2))D_2 + (v^3d_1(d_1^2+d_2^2))D_1}{\sqrt{(v^3d_3(d_1^2+d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')^2 + v^6(d_1^2+d_2^2)^3}},
\end{aligned} \tag{3.1}$$

where x_1, x_2, x_3 are coefficients of the following form

$$x_1 = \frac{v^3d_3(d_1^2+d_2^2) + vd_1(vd_2)' - vd_2(vd_1)'}{\sqrt{(v^3d_3(d_1^2+d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')^2 + v^6(d_1^2+d_2^2)^3}},$$

$$x_2 = -\frac{v^3d_2(d_1^2+d_2^2)}{\sqrt{(v^3d_3(d_1^2+d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')^2 + v^6(d_1^2+d_2^2)^3}},,$$

$$x_3 = \frac{v^3d_1(d_1^2+d_2^2)}{\sqrt{(v^3d_3(d_1^2+d_2^2) + vd_1(vd_2)' - vd_2(vd_1)')^2 + v^6(d_1^2+d_2^2)^3}}.$$

Proof. By referring the relation (2.4), the first and second derivative of $\alpha_T(s) = T(s)$ can be easily given as

$$\begin{aligned}
{\alpha_T}'(s) &= v(d_1D_2 + d_2D_1),\\
{\alpha_T}''(s) &= -v^2(d_1^2+d_2^2)T + ((vd_1)' - v^2d_2d_3)D_2 + ((vd_2)' + v^2d_1d_3)D_1.
\end{aligned}$$

Now, recalling the definitions given in (2.1) and (2.2), we calculate the required components as

$$\left\|\alpha_T'(s)\right\| = v\sqrt{{d_1}^2+{d_2}^2},$$

$$\alpha_T'(s)\wedge\alpha_T''(s) = (v^3d_3(d_1^2+d_2^2)+vd_1(vd_2)'-vd_2(vd_1)')T-(v^3d_2(d_1^2+d_2^2))D_2+(v^3d_1(d_1^2+d_2^2))D_1,$$

$$\left\|\alpha_T'(s)\wedge\alpha_T''(s)\right\| = \sqrt{(v^3d_3(d_1^2+d_2^2)+vd_1(vd_2)'-vd_2(vd_1)')^2+v^6(d_1^2+d_2^2)^3}.$$

Substituting these relations in place we complete the proof. □

Theorem 3.3. *The geodesic curvatures of T- indicatrix curve according to E^3 and S^2 are given as following:*

$$k_g^T = \frac{\sqrt{\left({d_1}^2+{d_2}^2\right)(d_1d_2'-d_2d_1')^2+2d_3\left(d_1d_2'-d_2d_1'\right)\left({d_1}^2+{d_2}^2\right)^2+{d_3}^2\left({d_1}^2+{d_2}^2\right)^3+\left({d_1}^2+{d_2}^2\right)^4}}{\left({d_1}^2+{d_2}^2\right)^2},$$

$$\mu_g^T = \frac{\sqrt{\left({d_1}^2+{d_2}^2\right)(d_1d_2'-d_2d_1')^2+2d_3\left(d_1d_2'-d_2d_1'\right)\left({d_1}^2+{d_2}^2\right)^2+{d_3}^2\left({d_1}^2+{d_2}^2\right)^3}}{\left({d_1}^2+{d_2}^2\right)^2},$$

respectively.

Proof. From (3.1), if we take the derivative of T_T and consider the relation (2.4), then we get

$$D_{T_T}T_T = \frac{-\left(d_2\left(d_1d_2'-d_2d_1'\right)+d_2d_3\left({d_1}^2+{d_2}^2\right)\right)D_2+\left(d_1\left(d_1d_2'-d_2d_1'\right)+d_1d_3\left({d_1}^2+{d_2}^2\right)\right)D_1}{\left({d_1}^2+{d_2}^2\right)^2}-T.$$

By using the relation given in (2.7), k_g^T can be easily obtained. On the other hand, when considered the Gauss equation given in (2.6), we get the following relations

$$\begin{aligned}\bar{D}_{T_T}T_T &= D_{T_T}T_T+\langle S\left(T_T\right),T_T\rangle T\\ &= \frac{-\left(d_2\left(d_1d_2'-d_2d_1'\right)+d_2d_3\left({d_1}^2+{d_2}^2\right)\right)D_2+\left(d_1\left(d_1d_2'-d_2d_1'\right)+d_1d_3\left({d_1}^2+{d_2}^2\right)\right)D_1}{\left({d_1}^2+{d_2}^2\right)^2}\end{aligned}$$

to be used to calculate μ_g^T by the definition given in (2.8), which completes the proof. □

Remark 3.4. Since, Frenet and Flc frame have a common tangent vector field which means the images of tangent indicatrix for both are the same on a unit sphere, one would argue the necessity of calculations. However, we are concerned to the Flc frame, and express those characterizations of T-indicatrix according to the Flc invariants.

Definition 3.5. Let α be a polynomial curve in E^3 and $\{T, D_2, D_1\}$ denote the Flc frame. The curve traced out by the vector D_2 centered at unit sphere is called D_2- or normal like indicatrix curve and is defined as

$$\alpha_{D_2}(s) = D_2(s).$$

Theorem 3.6. *The Frenet vectors $T_{D_2}, N_{D_2}, B_{D_2}$ of D_2- indicatrix curve according to Flc frame are given as follows:*

$$T_{D_2} = \frac{-d_1}{\sqrt{{d_1}^2+{d_3}^2}}T + \frac{d_3}{\sqrt{{d_1}^2+{d_3}^2}}D_1,$$

$$N_{D_2} = \frac{u_2d_3}{\sqrt{{d_1}^2+{d_3}^2}}T - \frac{u_1d_3+u_3d_1}{\sqrt{{d_1}^2+{d_3}^2}}D_2 + \frac{u_2d_1}{\sqrt{{d_1}^2+{d_3}^2}}D_1, \tag{3.2}$$

$$B_{D_2} = \frac{(v^3d_3(d_1^2+d_3^2))T + (-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))D_2 + (v^3d_1(d_1^2+d_3^2))D_1}{\sqrt{(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))^2+v^6(d_1^2+d_3^2)^3}}$$

where the coefficients u_1, u_2 and u_3 are of the following forms:

$$u_1 = \frac{v^3d_3(d_1^2+d_3^2)}{\sqrt{(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))^2+v^6(d_1^2+d_3^2)^3}},$$

$$u_2 = \frac{-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2)}{\sqrt{(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))^2+v^6(d_1^2+d_3^2)^3}},$$

$$u_3 = \frac{v^3d_1(d_1^2+d_3^2)}{\sqrt{(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))^2+v^6(d_1^2+d_3^2)^3}},$$

respectively.

Proof. By taking the first and second derivative of the expression $\alpha_{D_2}(s) = D_2(s)$ and using (2.4) we get

$$\alpha_{D_2}{}'(s) = v(-d_1T + d_3D_1),$$

$$\alpha_{D_2}{}''(s) = -((vd_1)' + d_2d_3v^2)T - v^2(d_1^2+d_3^2)D_2 + ((vd_3)' - d_1d_2v^2)D_1.$$

Similarly, by calculating the necessary relations as following

$$\left\|\alpha_{D_2}{}'(s)\right\| = v\sqrt{{d_1}^2+{d_3}^2},$$

$$\alpha_{D_2}{}'(s)\wedge\alpha_{D_2}{}''(s) = (v^3d_3(d_1^2+d_3^2))T + (-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))D_2 + (v^3d_1(d_1^2+d_3^2))D_1,$$

$$\left\|\alpha_{D_2}{}'(s)\wedge\alpha_{D_2}{}''(s)\right\| = \sqrt{(-vd_3((vd_1)'+d_2d_3v^2)+vd_1((vd_3)'-d_1d_2v^2))^2+v^6(d_1^2+d_3^2)^3}$$

to substitute in (2.1), we complete the proof. □

Theorem 3.7. *The geodesic curvatures of D_2- indicatrix curve according to E^3 and S^2 are given as following:*

$$k_g^{D_2} = \frac{\sqrt{\left({d_1}^2+{d_3}^2\right)(d_1{d_3}'-d_3{d_1}')^2 - 2d_2\left(d_1{d_3}'-d_3{d_1}'\right)\left({d_1}^2+{d_3}^2\right)^2 + {d_2}^2\left({d_1}^2+{d_3}^2\right)^3 + \left({d_1}^2+{d_3}^2\right)^4}}{\left({d_1}^2+{d_3}^2\right)^2},$$

$$\mu_g^{D_2} = \frac{\sqrt{\left({d_1}^2+{d_3}^2\right)(d_1{d_3}'-d_3{d_1}')^2 - 2d_2\left(d_1{d_3}'-d_3{d_1}'\right)\left({d_1}^2+{d_3}^2\right)^2 + {d_2}^2\left({d_1}^2+{d_3}^2\right)^3}}{\left({d_1}^2+{d_3}^2\right)^2},$$

respectively.

Proof. By taking the derivative of T_{D_2} given in (3.2), and considering again (2.4), we have

$$D_{T_{D_2}}T_{D_2} = \frac{\left(d_3\left(d_1d_3' - d_3d_1'\right) - d_2d_3\left({d_1}^2 + {d_3}^2\right)\right)T + \left(d_1\left(d_1d_3' - d_3d_1'\right) - d_1d_2\left({d_1}^2 + {d_3}^2\right)\right)D_1}{\left({d_1}^2 + {d_3}^2\right)^2} - D_2.$$

From the definition, (2.7), the geodesic curvature according to E^3, $k_g^{D_2}$ can be easily calculated. By the Gauss relation (2.6), we write

$$\bar{D}_{T_{D_2}}T_{D_2} = D_{T_{D_2}}T_{D_2} + \langle S\left(T_{D_2}\right), T_{D_2}\rangle D_2$$

$$= \frac{\left(d_3\left(d_1d_3' - d_3d_1'\right) - d_2d_3\left({d_1}^2 + {d_3}^2\right)\right)T + \left(d_1\left(d_1d_3' - d_3d_1'\right) - d_1d_2\left({d_1}^2 + {d_3}^2\right)\right)D_1}{\left({d_1}^2 + {d_3}^2\right)^2}.$$

Now, by referring the definition in (2.8) the geodesic curvature according to S^2, $\mu_g^{D_2}$ can be obtained, which completes the proof. □

Definition 3.8. Let α be a polynomial curve in E^3 and $\{T, D_2, D_1\}$ denote its Flc frame. The curve traced out by the vector D_1 centered at unit sphere is called D_1- or binormal like indicatrix curve and is defined as

$$\alpha_{D_1}(s) = D_1(s).$$

Theorem 3.9. *The Frenet vectors $T_{D_1}, N_{D_1}, B_{D_1}$ of D_1- indicatrix curve according to Flc frame are given as follows:*

$$T_{D_1} = -\frac{d_2}{\sqrt{{d_2}^2 + {d_3}^2}}T - \frac{d_3}{\sqrt{{d_2}^2 + {d_3}^2}}D_2,$$

$$N_{D_1} = \frac{m_3d_3}{\sqrt{{d_2}^2 + {d_3}^2}}T - \frac{m_3d_2}{\sqrt{{d_2}^2 + {d_3}^2}}D_2 + \frac{m_2d_2 - m_1d_3}{\sqrt{{d_2}^2 + {d_3}^2}}D_1, \tag{3.3}$$

$$B_{D_1} = \frac{(v^3d_3(d_2^2 + d_3^2))T - (v^3d_2(d_2^2 + d_3^2))D_2 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2d_1(d_2^2 + d_3^2))D_1}{\sqrt{v^6(d_2^2 + d_3^2)^3 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2d_2(d_2^2 + d_3^2))^2}},$$

where the coefficients m_1, m_2 and m_3 are of the following form

$$m_1 = \frac{v^3d_3(d_2^2 + d_3^2)}{\sqrt{v^6(d_2^2 + d_3^2)^3 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2d_2(d_2^2 + d_3^2))^2}},$$

$$m_2 = -\frac{v^3d_2(d_2^2 + d_3^2)}{\sqrt{v^6(d_2^2 + d_3^2)^3 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2d_2(d_2^2 + d_3^2))^2}},$$

$$m_3 = \frac{vd_2((vd_3)' - vd_3(vd_2)' + v^2d_1(d_2^2 + d_3^2)}{\sqrt{v^6(d_2^2 + d_3^2)^3 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2d_2(d_2^2 + d_3^2))^2}}.$$

Proof. By taking the first and second derivative of the expression $\alpha_{D_1}(s) = D_1(s)$ and using (2.4) we get

$$\alpha_{D_1}'(s) = -v(d_2T + d_3D_2),$$

$$\alpha_{D_1}''(s) = -((vd_2)' - vd_1d_3)T - ((vd_3)' + vd_1d_2)D_2 - v^2(d_2^2 + d_3^2)D_1.$$

By following the same steps as before, we calculate the necessary relations as

$$\left\|\alpha_{D_1}{}'(s)\right\| = v\sqrt{{d_2}^2 + {d_3}^2},$$

$$\alpha_{D_1}{}'(s) \wedge \alpha_{D_1}{}''(s) == (v^3 d_3(d_2^2 + d_3^2))T - (v^3 d_2(d_2^2 + d_3^2))D_2 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2 d_1(d_2^2 + d_3^2))D_1.$$

$$\left\|\alpha_{D_1}{}'(s) \wedge \alpha_{D_1}{}''(s)\right\| = \sqrt{v^6(d_2^2 + d_3^2)^3 + (vd_2((vd_3)' - vd_3(vd_2)' + v^2 d_2(d_2^2 + d_3^2))^2}$$

to substitute in (2.1) and complete the proof. □

Theorem 3.10. *The geodesic curvatures of D_1- indicatrix curve according to E^3 and S^2 are given as following:*

$$k_g^{D_1} = \frac{\sqrt{\left({d_2}^2 + {d_3}^2\right)(d_2 d_3' - d_3 d_2')^2 + 2d_1\left(d_2 d_3' - d_3 d_2'\right)\left({d_2}^2 + {d_3}^2\right)^2 + {d_1}^2\left({d_2}^2 + {d_3}^2\right)^3 + \left({d_2}^2 + {d_3}^2\right)^4}}{\left({d_2}^2 + {d_3}^2\right)^2},$$

$$\mu_g^{D_1} = \frac{\sqrt{\left({d_2}^2 + {d_3}^2\right)(d_2 d_3' - d_3 d_2')^2 + 2d_1\left(d_2 d_3' - d_3 d_2'\right)\left({d_2}^2 + {d_3}^2\right)^2 + {d_1}^2\left({d_2}^2 + {d_3}^2\right)^3}}{\left({d_2}^2 + {d_3}^2\right)^2},$$

respectively.

Proof. By taking the derivative of T_{D_1} given in (3.3), and recalling once more (2.4), we have

$$D_{T_{D_1}} T_{D_1} = \frac{\left(d_3\left(d_2 d_3' - d_3 d_2'\right) + d_1 d_3\left({d_2}^2 + {d_3}^2\right)\right)T + \left(d_2\left(d_3 d_2' - d_2 d_3'\right) - d_1 d_2\left({d_2}^2 + {d_3}^2\right)\right)D_2}{\left({d_2}^2 + {d_3}^2\right)^2} - D_1.$$

From the definition in (2.7), $k_g^{D_1}$ is simply the norm of latter relation. By recalling the Gauss equation (2.6) again, we have

$$\bar{D}_{T_{D_1}} T_{D_1} = D_{T_{D_1}} T_{D_1} + \langle S\left(T_{D_1}\right), T_{D_1}\rangle D_1$$

$$= \frac{\left(d_3\left(d_2 d_3' - d_3 d_2'\right) + d_1 d_3\left({d_2}^2 + {d_3}^2\right)\right)T + \left(d_2\left(d_3 d_2' - d_2 d_3'\right) - d_1 d_2\left({d_2}^2 + {d_3}^2\right)\right)D_2}{\left({d_2}^2 + {d_3}^2\right)^2}$$

and by using (2.8) we obtain $\mu_g^{D_1}$ and complete the proof. □

4. Examples

Example 4.1. Let α be the 3^{rd} degree polynomial curve in E^3 given by the following parameterization

$$\alpha(s) = (s, \frac{s^2}{2}, \frac{s^3}{6}).$$

From the relations, (2.1), (2.3) and (2.5) the vector elements of Frenet and Flc-frame can be found as

$$T(s) = \left(\frac{2}{s^2+2}, \frac{2s}{s^2+2}, \frac{s^2}{s^2+2}\right), \qquad N(s) = \left(\frac{-2s}{s^2+2}, -\frac{s^2-2}{s^2+2}, \frac{2s}{s^2+2}\right),$$

$$B(s) = \left(\frac{s^2}{s^2+2}, -\frac{-2s}{s^2+2}, \frac{2}{s^2+2}\right), \quad D_2(s) = \left(\frac{-s^2}{(s^2+2)\sqrt{s^2+1}}, \frac{-s^3}{(s^2+2)\sqrt{s^2+1}}, \frac{2\sqrt{s^2+1}}{s^2+2}\right),$$

$$D_1(s) = \left(\frac{s}{\sqrt{s^2+1}}, \frac{-1}{\sqrt{s^2+1}}, 0\right).$$

The curvatures and the corresponding unit Darboux vectors (C and C_W) of $\alpha(s)$ according to both Frenet and Flc frame are given in respective order as

$$\kappa = \tau = \frac{4}{(s^2+2)^2},$$
$$d_1 = \frac{4s}{\sqrt{(s^2+1)(s^2+2)^2}}, \quad d_2 = -\frac{4}{\sqrt{(s^2+1)(s^2+2)^2}}, \quad d_3 = \frac{2s^2}{(s^2+1)(s^2+2)^2}.$$

The spherical indicatrix of tangent vector $T(s)$ of $\alpha(s)$ is given in Fig. 1

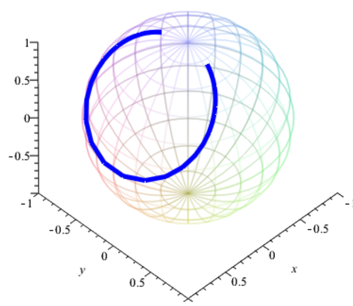

FIGURE 1. Tangent Spherical Image of $\alpha(s)$ (is same for both frames)

The normal and normal like indicatrix curves of $\alpha(s)$ are illustrated in Fig. 2

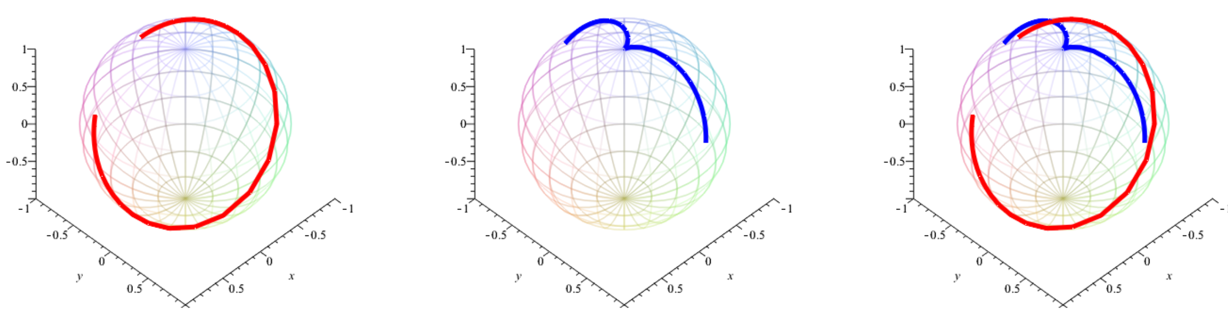

FIGURE 2. The $N-$ indicatrix (red) and D_2- indicatrix (blue) curves of $\alpha(s)$

The binormal and binormal like indicatrix curves of $\alpha(s)$ are illustrated in Fig. 3

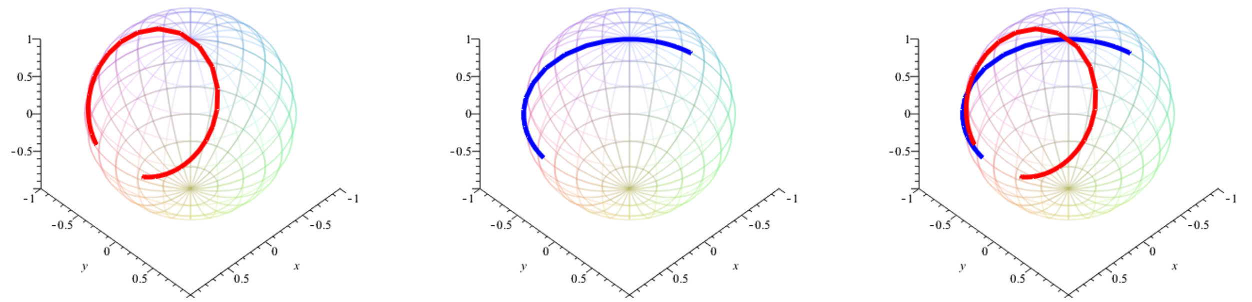

FIGURE 3. The $B-$ indicatrix (red) and D_1- indicatrix (blue) curves of $\alpha(s)$

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

AUTHORS CONTRIBUTIONS STATEMENT

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

REFERENCES

[1] Bishop, R.L., *There is more than one way to Frame a curve,* American Mathematical Monthly, **82**(3)(1975), 246–251.

[2] Capın, R., Minkowski Uzayında Küresel Gösterge Eğrileri, Yüksek Lisans Tezi, Gaziantep Üniversitesi Fen Bilimleri Enstitüsü, Gaziantep, 77p, 2016.

[3] Dede, M., *A new representation of tubular surfaces,* Houston J. Math. **45**(3)(2019), 707—720.

[4] Farouki, R.T., Han, C.Y., Manni, C., Sestini, A., *Characterization and construction of helical polynomial space curves,* Journal of Computational and Applied Mathematics **162**(2004), 365-–392.

[5] Gökyeşil, D., Dual Uzayda Bazı Eğrilerin Dual Bishop Çatısına Göre Karakterizasyonları, Yüksek Lisans Tezi, Manisa Celal Bayar Üniversitesi Fen Bilimleri Enstitüsü , Manisa, 65p, 2018.

[6] Hacısalihoğlu, H.H., Diferansiyel Geometri, Ankara Üniversitesi Fen Fakültesi, Ankara, 1993.

[7] Kula, L., Yayli, Y. *On slant helix and its spherical indicatrix,* Appl. Math. Comput., **169**(1)(2005), 600–607.

[8] Larson, R., Elemantary Linear Algebra, The Pennsylvania State University, Boston, 2012.

[9] O'Neill, B., Elementary Diferential Geometry. New York, Academic Press Inc., 1966.

[10] Şenyurt, S., *Natural lifts and the geodesic sprays for the spherical indicatrices of the Mannheim partner curves in* E^3*,* International Journal of the Physical Sciences, **7**(16)(2012), 2414–2421.

[11] Şenyurt, S., Özgüner, Z., *Bertrand eğri çiftinin küresel göstergelerinin geodezik eğrilikleri ve tabii liftleri,* Ordu Üniv. Bil. Tek. Derg., **3**(2)(2013), 58–81.

[12] Yılmaz, S., Özyılmaz, E., Turgut, M., *New spherical indicatrices and their characterizations,* An. Şt. Univ. Ovidius Constanta., **18**(2)(2010), 337–354.