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SOME FIXED POINT THEOREMS ON ORTHOGONAL METRIC SPACES VIA EXTENSIONS OF ORTHOGONAL CONTRACTIONS

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Abstract. Orthogonal metric space is a considerable generalization of a usual metric space obtained by establishing a perpendicular relation on a set. Very recently, the notions of orthogonality of the set and orthogonality of the metric space are described and notable fixed point theorems are given in orthogonal metric spaces. Some fixed point theorems for the generalizations of contraction principle via altering distance functions on orthogonal metric spaces are presented and proved in this paper. Furthermore, an example is presented to clarify these theorems.

1. Introduction and Preliminaries

The well-known theorem on the presence and uniqueness of a fixed point of exact self maps defined on certain metric spaces were stated by Stefan Banach [3] in 1992: Every self mapping h on a complete metric space (Ω, ρ) satisfying the condition

$$\rho(hx, hy) \leq \lambda\rho(x, y), \text{ for all } x, y \in \Omega, \lambda \in (0, 1) \tag{1}$$

has a unique fixed point.
This gracious theorem has been used to show the presence and uniqueness of the solution of differantial equation

$$y'(x) = F(x; y); y(x_0) = y_0 \tag{2}$$

where F is a continuously differantiable function.
Consequently, after the Banach Contraction Principle on complete metric space, many researchers have investigated for anymore fixed point results and reported

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new fixed point theorems intended by the use of two very influential directions, assembled or apart (See [2], [4], [5], [6], [7], [8], [9], [10], [11], [16], [17], [18]).
One of them is involved with the attempts to generalize the contractive conditions on the maps and thus, soften them; the other with to attempts to generalize the space on which these contractions are described.
Among the results in the first direction, Khan et al. [15] installed fixed point theorems in complete and compact metric spaces by using altering distance functions in 1984. Then, Alber and Guerre-Delabriere [1] presented anohter generalization of the contraction principle in Hilbert spaces in 1997. In 2001, the results of [1] were shown to be valuable in complete metric spaces by Rhoades [19]. On the other hand, among the results in the second direction, Gordji et al. [13] introduced the concepts orthogonality of the set and orthogonality of the metric space in 2017. In the mentioned paper, a generalization of Banach fixed point theorem is presented in this exciting defined construction and also, acquired results in the mentioned paper is implemented to indicate the presence of a solution of an ordinary differential equation. In this paper considerable fixed point theorems on orthogonal metric spaces via orthogonal contractions are introduced inspired by [13], [15] and [19]. Furthermore, an example is presented to illustrate these theorems.

The main difference between studies in orthogonal metric spaces and studies in general metric spaces is that instead of a contraction condition provided by any two elements of the space, it is sufficient to provide a contraction condition given only for orthogonally related elements. Another important point is that orthogonal complete metric spaces do not have to be complete metric spaces. So the results of this paper, not only generalize the analogous fixed point theorems but are relatively simpler and more natural than the related ones.

In the sequel, respectively, $\mathbb{Z}, \mathbb{R}, \mathbb{R}^+$ denote integers, real numbers and positive real numbers.

Definition 1. *([13]) Let Ω be a non-empty set and $\perp \subseteq \Omega \times \Omega$ be a binary relation. $(\Omega, \perp)$ is called orthogonal set if $\perp$ satisfies the following condition*

$$\exists k_0 \in \Omega; (\forall l \in \Omega, l \perp k_0) \vee (\forall l \in \Omega, k_0 \perp l). \tag{3}$$

And also this k_0 element is named orthogonal element.

Example 1. *([12]) Let $\Omega = \mathbb{Z}$ ($\mathbb{Z}$ is integer numbers) and define $a \perp b$ if there exists $\gamma \in \mathbb{Z}$ such that $a = \gamma b$. It is effortless to see that $0 \perp b$ for all $b \in \mathbb{Z}$. On account of this $(\Omega, \perp)$ is an orthogonal set.*

This k_0 element does not have to be unique. For example;

Example 2. *([12]) Let $\Omega = [0, \infty)$, define $k \perp l$ if $kl \in \{k, l\}$, then by setting $k_0 = 0$ or $k_0 = 1$, $(\Omega, \perp)$ is an orthogonal set.*

Definition 2. *([13]) A sequence $\{k_n\}$ is named orthogonal sequence if*

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{4}$$

In the same way, a Cauchy sequence $\{k_n\}$ *is named to be an orthogonally Cauchy sequence if*

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{5}$$

Definition 3. *([13]) Let* $(\Omega, \perp)$ *be an orthogonal set,* ρ *be a usual metric on* Ω*. Afterwards* $(\Omega, \perp, \rho)$ *is named an orthogonal metric space.*

Definition 4. *([13]) An orthogonal metric space* $(\Omega, \perp, \rho)$ *is named to be a complete orthogonal metric space if every orthogonally Cauchy sequence converges in* Ω*.*

Definition 5. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal metric space and a function* $h : \Omega \to \Omega$ *is named to be orthogonally continuous at* k *if for each orthogonal sequence* $\{k_n\}$ *converging to* k *implies* $hk_n \to hk$ *as* $n \to \infty$*. Also* h *is orthogonal continuous on* Ω *if* h *is orthogonal continuous in each* $k \in \Omega$*.*

Definition 6. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal metric space and* $\lambda \in \mathbb{R}$*,* $0 < \lambda < 1$*. A function* $h : \Omega \to \Omega$ *is named to be orthogonal contraction with Lipschitz constant* λ *if*

$$\rho(hk, hl) \leq \lambda\rho(k, l) \tag{6}$$

for all $k, l \in \Omega$ *whenever* $k \perp l$*.*

Definition 7. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal metric space and a function* $h : \Omega \to \Omega$ *is named orthogonal preserving if* $hk \perp hl$ *whenever* $k \perp l$*.*

Remark 1. *The authors of [12] gave an example which shows the orthogonal continuity and orthogonal contraction are weaker than the classic continuity and classic contraction in classic metric spaces.*

Theorem 1. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal complete metric space and* $0 < \lambda < 1$ *and let* $h : \Omega \to \Omega$ *be orthogonal continuous , orthogonal contraction (with Lipschitz constant* λ *) and orthogonal preserving. Afterwards* h *has a unique fixed point* $k^* \in \Omega$ *and* $\lim_{n\to\infty} h^n(k) = k^*$ *for all* $k \in \Omega$*.*

Definition 8. *([15]) Let* $\psi : [0, \infty) \to [0, \infty)$ *be a function which satisfies*
(i) $\psi(s)$ *is continuous and nondecreasing,*
(ii) $\psi(s) = 0 \iff s = 0$
properties. Then ψ *is named altering distance function. And* Ψ *is denoted as the set of altering distance functions* ψ*.*

And in [14], notable fixed point theorems on orthogonal metric spaces via altering distance functions are presented by Bilgili Gungor and Turkoglu.

2. Main Results

Firstly, in the following theorem, by giving a contraction condition that will generalize the previous works using alterne distance functions is presented and proven.

Theorem 2. *Let $(\Omega, \perp, \rho)$ be an orthogonal complete metric space, $h : \Omega \to \Omega$ be a self map, $\eta, \kappa \in \Psi$ and η is a sub-additive function. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\eta(\rho(hk, hl)) \leq \eta(\rho(k, l)) - \kappa(\rho(k, l)) \tag{7}$$

for all $k, l \in \Omega$ where $k \perp l$ and $k \neq l$. Under this circumstance, there exists a point $k^* \in \Omega$ so that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n k_0\}$ converges to this point. Also, $k^* \in \Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^* \in \Omega$.

Proof. Because of $(\Omega, \perp)$ is an orthogonal set,

$$\exists k_0 \in \Omega; (\forall l \in \Omega, l \perp k_0) \vee (\forall l \in \Omega, k_0 \perp l). \tag{8}$$

And from h is a self mapping on Ω, for any orthogonal element $k_0 \in \Omega$, $k_1 \in \Omega$ can be chosen as $k_1 = h(k_0)$. Thus,

$$\begin{aligned} & k_0 \perp hk_0 \vee hk_0 \perp k_0 \\ \Rightarrow\ & k_0 \perp k_1 \vee k_1 \perp k_0. \end{aligned} \tag{9}$$

Then, if it continues similarly

$$k_1 = hk_0, k_2 = hk_1 = h^2 k_0, ..., k_n = hk_{n-1} = h^n k_0 \tag{10}$$

so $\{h^n k_0\}$ is an iteration sequence.
If any $n \in \mathbb{N}, k_n = k_{n+1}$ then $k_n = hk_n$ and so h has a fixed point. Suppose that $k_n \neq k_{n+1}$ for all $n \in \mathbb{N}$.
Since h is orthogonal preserving, $\{h^n k_0\}$ is an orthogonal sequence and by using inequality (7)

$$\begin{aligned} \eta(\rho(k_{n+1}, k_n)) &= \eta(\rho(hk_n, hk_{n-1})) \\ &\leq \eta(\rho(k_n, k_{n-1})) - \kappa(\rho(k_n, k_{n-1})). \end{aligned} \tag{11}$$

Using the monotone property of $\eta \in \Psi$, $\{\rho(k_{n+1}, k_n)\}$ is a sequence of decreasing nonnegative real numbers. Thus there is a $\theta \geq 0$ and $\lim_{n\to\infty} \rho(k_{n+1}, k_n) = \theta$. It can be shown that $\theta = 0$. Assume, on the contrary, that $\theta > 0$. At that rate, by taking the limit $n \to \infty$ in inequality (11) and using η, κ are continuous functions,

$$\eta(\theta) \leq \eta(\theta) - \kappa(\theta) \tag{12}$$

is obtained. This is a contradiction. Therefore $\theta = 0$. Now it can be proved that $\{k_n\}$ is an orthogonal Cauchy sequence. If $\{k_n\}$ is not an orthogonal Cauchy sequence, there exists $\epsilon > 0$ and suitable subsequences $\{r(n)\}$ and $\{s(n)\}$ of $\mathbb{N}$ satisfying $r(n) > s(n) > n$ for which

$$\rho(x_{r(n)}, x_{s(n)}) \geq \epsilon \tag{13}$$

and where $r(n)$ is selected as the least integer satisfying (13), that is

$$\rho(k_{r(n)-1}, k_{s(n)}) < \epsilon. \tag{14}$$

By (13),(14) and triangular inequality of ρ, it can be easily derived that

$$\varepsilon \leq \rho(k_{r(n)}, k_{s(n)}) \leq \rho(k_{r(n)}, k_{r(n)-1}) + \rho(k_{r(n)-1}, k_{s(n)}) < \rho(k_{r(n)}, k_{r(n)-1}) + \epsilon. \tag{15}$$

Letting $n \to \infty$, by using $\lim_{n\to\infty} \rho(k_{n+1}, k_n) = \theta$

$$\lim_{n\to\infty} \rho(k_{r(n)}, k_{s(n)}) = \epsilon \tag{16}$$

is obtained. Also, for each $n \in \mathbb{N}$, by using the triangular inequality of ρ,

$$\begin{aligned} &\rho(k_{r(n)}, k_{s(n)}) - \rho(k_{r(n)}, k_{r(n)+1}) - \rho(k_{s(n)+1}, k_{s(n)}) \\ &\leq \rho(k_{r(n)+1}, k_{s(n)+1}) \\ &\leq \rho(k_{r(n)}, k_{r(n)+1}) + \rho(k_{r(n)}, k_{s(n)}) + \rho(k_{s(n)+1}, k_{s(n)}). \end{aligned} \tag{17}$$

Passing to the limit when $n \to \infty$ in the last inequality

$$\rho(k_{r(n)+1}, k_{s(n)+1}) = \epsilon. \tag{18}$$

By using the inequality (7),

$$\begin{aligned} \eta(\rho(k_{r(n)+1}, k_{s(n)+1})) &= \eta(\rho(hk_{r(n)}, hk_{s(n)})) \\ &\leq \eta(\rho(k_{r(n)}, k_{s(n)})) - \kappa(\rho(k_{r(n)}, k_{s(n)})). \end{aligned} \tag{19}$$

Passing to the limit when $n \to \infty$ in the last inequality

$$\eta(\epsilon) \leq \eta(\epsilon) - \kappa(\epsilon). \tag{20}$$

It is a contradiction. Therefore $\{k_n\}$ is a orthogonal Cauchy sequence. By the orthogonal completeness of Ω, there exists $k^* \in \Omega$ so that $\{k_n\} = \{h^n k_0\}$ converges to this point.

Now it can be shown that k^* is a fixed point of h when h is orthogonally continuous at $k^* \in \Omega$. Assume that h is orthogonally continuous at $k^* \in \Omega$. Thus,

$$k^* = \lim_{n\to\infty} k_{n+1} = \lim_{n\to\infty} hk_n = hk^*. \tag{21}$$

so $k^* \in \Omega$ is a fixed point of h.

Now the uniqueness of the fixed point can be shown. Suppose that there exist two distinct fixed points k^* and l^*.Then,

(i) If $k^* \perp l^* \vee l^* \perp k^*$, by using the inequality (7)

$$\begin{aligned} \eta(\rho(k^*, l^*)) &= \eta(\rho(hk^*, hl^*)) \\ &\leq \eta(\rho(k^*, l^*)) - \kappa(\rho(k^*, l^*)) \end{aligned} \tag{22}$$

This is a contradiction and $k^* \in \Omega$ is a unique fixed point of h.

(ii) If not, for the chosen orthogonal element $k_0 \in \Omega$,

$$[(k_0 \perp k^*) \wedge (k_0 \perp l^*)] \vee [(k^* \perp k_0) \wedge (l^* \perp k_0)] \tag{23}$$

and since h is orthogonal preserving,

$$[(hk_n \perp k^*) \wedge (hk_n \perp l^*)] \vee [(k^* \perp hk_n) \wedge (l^* \perp hk_n)] \tag{24}$$

is obtained. Now, by using the triangular inequality of ρ, ψ is nondecreasing sub-additive function and the inequality (7)

$$\begin{aligned} \eta(\rho(k^*,l^*)) &= \eta(\rho(hk^*,hl^*)) \\ &\leq \eta(\rho(hk^*,hk_{n+1})+\rho(hk_{n+1},hl^*)) \\ &\leq \eta(\rho(hk^*,h(hk_n)))+\eta(\rho(h(hk_n),hl^*)) \\ &\leq \eta(\rho(k^*,hk_n))-\kappa(\rho(k^*,hk_n))+\eta(\rho(hk_n,l^*))-\kappa(\rho(hk_n,l^*)). \end{aligned} \tag{25}$$

and taking limit $n\to\infty$, $k^*=l^*$. Thus, $k^*\in\Omega$ is a unique fixed point of h. □

If assumed to be $\kappa(s)=(1-\lambda)\eta(s)$, for all $s>0$ where $0<\lambda<1$ in Theorem 2, the following Corollary is obtained.

Corollary 1. *Let $(\Omega,\perp,\rho)$ be an orthogonal complete metric space, $\lambda\in\mathbb{R}, 0<\lambda<1$, $h:\Omega\to\Omega$ be a self map , $\eta\in\Psi$ be a sub-additive function. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\eta(\rho(hk,hl))\leq\lambda\eta(\rho(k,l)) \tag{26}$$

for all $k,l\in\Omega$ whenever $k\perp l$ and $k\neq l$. Under this circumstance, there exists a point $k^*\in\Omega$ such that for any orthogonal element $k_0\in\Omega$, the iteration sequence $\{h^nk_0\}$ converges to this point. Also, $k^*\in\Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^*\in\Omega$.

If assume $\eta(s)=s$, for all $s>0$ in Theorem 2, the following Corollary is gotten.

Corollary 2. *Let $(\Omega,\perp,\rho)$ be an orthogonal complete metric space, $h:\Omega\to\Omega$ be a self map, $\kappa\in\Psi$. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\rho(hk,hl)\leq\rho(k,l)-\kappa(\rho(k,l)) \tag{27}$$

for all $k,l\in\Omega$ where $k\perp l$ and $k\neq l$. Under this circumstance, there exists a point $k^*\in\Omega$ such that for any orthogonal element $k_0\in\Omega$, the iteration sequence $\{h^nk_0\}$ converges to this point. Also, $k^*\in\Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^*\in\Omega$.

If assume $\eta(s)=s$ and $\kappa(s)=(1-\lambda)\eta(s)$, for all $s>0$ where $0<\lambda<1$ in Theorem 2, the following Corollary which is the main result of [12] is obtained.

Corollary 3. *Let $(\Omega,\perp,\rho)$ be an orthogonal complete metric space, $\lambda\in\mathbb{R}, 0<\lambda<1$, $h:\Omega\to\Omega$ be a self map. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\rho(hk,hl)\leq\lambda\rho(k,l)) \tag{28}$$

for all $k,l\in\Omega$ where $k\perp l$ and $k\neq l$. Under this circumstance, there exists a point $k^*\in\Omega$ such that for any orthogonal element $k_0\in\Omega$, the iteration sequence $\{h^nk_0\}$ converges to this point. Also, $k^*\in\Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^*\in\Omega$.

Example 3. *Let $\Omega = [0,1)$ be a set and define $\rho : \Omega \times \Omega \to \Omega$ such that $\rho(k,l) = \mid k-l \mid$. Also, let the binary relation $\perp$ on Ω such that $k \perp l \iff kl \in \{k,l\}$. Then, $(\Omega, \perp)$ is an orthogonal set and ρ is a metric on Ω. So $(\Omega, \perp, \rho)$ is an orthogonal metric space. In this space, any orthogonal Cauchy sequence is convergent. Indeed, suppose that (k_n) is an arbitrary orthogonal Cauchy sequence in Ω.Then*

$$\begin{array}{rl} & k_n.k_{n+1} = k_n \vee k_{n+1}.k_n = k_{n+1} \\ \Rightarrow & k_n = 0, k_{n+1} \in [0,1) \vee k_{n+1} = 0, k_n \in [0,1) \end{array} \tag{29}$$

and for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $n \geq n_0$ we have

$$\mid k_n - k_{n+1} \mid < \epsilon \tag{30}$$

is provided. So, for any $\epsilon > 0$ and for all $n \in \mathbb{N}$, that is $n \geq n_0$, $\mid k_n - 0 \mid < \epsilon$ that is $\{k_n\}$ is convergent to $0 \in \Omega$. So $(\Omega, \perp, \rho)$ is a complete orthogonal metric space. Remark that, (Ω, ρ) is not a complete sub-metric space of $(\mathbb{R}, \rho)$ because of Ω is not a closed subset of $(\mathbb{R}, \rho)$.
Let $\eta : [0,\infty) \to [0,\infty)$ be defined as $\eta(s) = s$ and let $\kappa : [0,\infty) \to [0,\infty)$ be defined as $\kappa(s) = \frac{s^2}{9}$. Also let $h : \Omega \to \Omega$ be defined as

$$h(k) = \begin{cases} k - \frac{k^2}{3} & , 0 \leq k \leq \frac{1}{2}, \\ \frac{k}{2} & , \frac{1}{2} < k < 1. \end{cases} \tag{31}$$

In this case, one can see that $\eta, \kappa \in \Psi$, η is a sub-additive function. Also h is orthogonal preserving mapping. Indeed,

$$\begin{array}{rl} k \perp l & \Rightarrow kl = k \vee kl = l \\ & \Rightarrow k = 0, l \in [0,1) \vee l = 0, k \in [0,1) \\ & \Rightarrow hk = 0 \vee hl = 0 \\ & \Rightarrow hk \perp hl \vee hl \perp hk. \end{array} \tag{32}$$

On the other hand, h is orthogonal continuous at $0 \in \Omega$. Indeed, assume that $\{k_n\}$ is an orthogonal sequence and $k_n \to 0$. In this case,

$$\begin{array}{rl} & k_n.k_{n+1} = k_n \vee k_{n+1}.k_n = k_{n+1} \\ \Rightarrow & k_n = 0, k_{n+1} \in [0,1) \vee k_{n+1} = 0, k_n \in [0,1) \end{array} \tag{33}$$

and for any $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$, for all $n \in \mathbb{N}$ that is $n > n_0$, $\mid k_n - 0 \mid < \epsilon$ is obtained. So, for all $n \in \mathbb{N}$ that is $n > n_0$, $k_n \in [0, \frac{1}{2}]$. Thus, from the definition of h, for the same $n_0 \in \mathbb{N}$ that is $n > n_0$, $\mid h(k_n) - h(0) \mid < \epsilon$ that is $h(k_n) \to h(0) = 0$. Now, it can be shown that h is a self mapping satisfying the inequality (7) for all $k, l \in \Omega$ where $k \perp l$ and $k \neq l$.
Assume that $k, l \in \Omega$ two element of Ω, $k \perp l$ and $k \neq l$. In this case

$$kl = k \vee kl = l \Rightarrow k = 0, l \in [0,1) \vee l = 0, k \in [0,1). \tag{34}$$

Without loss of generality, assume that $k = 0, l \in [0, 1)$.
Case I: If $k = 0, l \in (0, \frac{1}{2}]$, *then* $hk = 0, hl = l - \frac{l^2}{3}$. *And*

$$\begin{aligned} \eta(\rho(hk, hl)) &= \mid 0 - (l - \tfrac{l^2}{3}) \mid = l - \tfrac{l^2}{3} \leq l - \tfrac{l^2}{9} \\ &= \mid 0 - l \mid - \mid 0 - \tfrac{l^2}{9} \mid = \eta(\rho(k, l)) - \kappa(\rho(k, l)). \end{aligned} \tag{35}$$

Case II: If $k = 0, l \in (\frac{1}{2}, 1)$, *then* $hk = 0, hl = \frac{l}{2}$. *And*

$$\eta(\rho(hk, hl)) = \mid 0 - \tfrac{l}{2} \mid = \tfrac{l}{2} \leq l - \tfrac{l^2}{9} = \mid 0 - l \mid - \mid 0 - \tfrac{l^2}{9} \mid = \eta(\rho(k, l)) - \kappa(\rho(k, l)). \tag{36}$$

Consequently, h *is a self mapping satisfying the inequality (7) for all* $k, l \in \Omega$ *whenever* $k \perp l$ *and* $k \neq l$. *Thus, all hypothesis of Theorem 2 satisfy and so, it is obvious that* h *has a unique fixed point* $0 \in \Omega$.

3. Conclusion

In the first part of this study, as a result of a comprehensive literature review, the developments related to the existence of fixed points for mappings that provide the appropriate contraction conditions from the beginning of the fixed point theory studies are mentioned, and then the general subject of this study is emphasized.

In this paper, some fixed point theorems in orthogonal complete metric spaces are presented by employing altering distance functions. The results of this paper, not only generalize the analogous fixed point theorems but are relatively simpler and more natural than the related ones. The results of this paper are actually three-fold: a relatively more general contraction condition is used, the continuity of the involved mapping is weakened to orthogonal continuity, the comparability conditions used by previous authors between elements are replaced by orthogonal relatedness.

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