

SOME NEW CHARACTERIZATIONS OF PARTIAL ISOMETRIES IN RINGS WITH INVOLUTION

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ABSTRACT. In this paper, some characterizations of partial isometries, normal elements and strongly EP elements are given by the construction of EP elements. At the same time, the partial isometry elements are characterized by the existence of solutions of equations in rings in a given set, and also by the general form of solutions of given equations.

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1. Introduction

Let R be an associative ring with 1. An element $a \in R$ is said to be group invertible if there exists $a^\# \in R$ such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad aa^\# = a^\#a.$$

The element $a^\#$ is called the group inverse of a , which is uniquely determined by the above equations [1].

An involution $*$: $a \mapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in R$ satisfying $a^*a = aa^*$ is called normal [4].

An element a^+ in R is called the Moore-Penrose inverse (MP-inverse) of a , if

$$aa^+a = a, \quad a^+aa^+ = a^+, \quad (aa^+)^* = aa^+, \quad (a^+a)^* = a^+a.$$

If such a^+ exists, then it is unique [3]. Denote by $R^\#$ and R^+ the set of all group invertible elements of R and the set of all MP-invertible elements of R .

An element a is said to be EP if $a \in R^\# \cap R^+$ and satisfies $a^\# = a^+$. We denote by R^{EP} the set of all EP elements of R .

An element a is called a partial isometry if $a \in R^+$ and $aa^*a = a$. And a is said to be strongly EP if $a \in R^{EP}$ is a partial isometry. We denote the set of all partial isometry elements and strongly EP elements of R by R^{PI} and R^{SEP} , respectively.

There are many studies on EP element, PI element and strongly EP element, see for example [2,4,5,6,7,8,9,10].

In [11], EP elements are characterized by constructing the existence of solutions of appropriate equations in a given set, which is a new method to study generalized inverse in rings.

In this paper, some characterizations of partial isometries, normal elements and strongly EP elements are given by the construction of EP elements. At the same time, the partial isometry elements are characterized by the existence of solutions of equations in rings in a given set, and also by the general form of solutions of given equations.

2. Main Results

Lemma 2.1. *Let $a \in R^\# \cap R^+$ and $x \in R$.*

- (1) *If $a^+a^+x = 0$, then $a^+x = 0$;*
- (2) *If $xa^+a^+ = 0$, then $xa^+ = 0$.*

Proof. (1) Since $a^+a^+x = 0$, $a^*a^+x = (a^*aa^+)a^+x = a^*a(a^+a^+x) = 0$. Noting that $a^+ = (a^+a)^*a^+ = (a^+aa^\#a)^*a^+ = (a^\#)^*a^*a^+aa^+ = (a^\#)^*a^*a^+$, we have $a^+x = (a^\#)^*a^*a^+x = 0$.

Similarly, we can show (2). □

It is well known that for $a \in R^+$, $a \in R^{PI}$ if and only if $a^+ = a^*$. Hence we have the following Lemma.

Lemma 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation*

$$x(a^+ - a^*) = 0$$

has at least one solution in $\chi_a = \{a, a^\#, a^+, a^, (a^\#)^*, (a^+)^*\}$.*

Proof. “ \Rightarrow ” If $a \in R^{PI}$, then $a^+ = a^*$. This infers every element of χ_a is a solution of the equation (1).

“ \Leftarrow ” (1) If $x = a$ is a solution, then $a(a^+ - a^*) = 0$, that is $aa^+ = aa^*$. Hence $a \in R^{PI}$ by [5, Theorem 2.1].

(2) If $x = a^\#$ is a solution, then $a^\#(a^+ - a^*) = 0$. Pre-multiplying the equality by a^2 , we have $a(a^+ - a^*) = 0$. Hence $a \in R^{PI}$ by (1).

(3) If $x = a^+$ is a solution, then $a^+(a^+ - a^*) = 0$, this gives $a^+a^+(aa^+ - aa^*) = 0$. By Lemma 2.1, we have $a^+(aa^+ - aa^*) = 0$, this is $a^+ = a^*$. Hence $a \in R^{PI}$.

(4) If $x = a^*$ is a solution, then $a^*(a^+ - a^*) = 0$. Pre-multiplying the equality by $a^+(a^+)^*$, yielding $a^+(a^+ - a^*) = 0$. Hence $a \in R^{PI}$ by (3).

(5) If $x = (a^\#)^*$ is a solution, then $(a^\#)^*(a^+ - a^*) = 0$. Pre-multiplying the equality by $(a^*)^2$, we have $a^*(a^+ - a^*) = 0$. Hence $a \in R^{PI}$ by (4).

(6) If $x = (a^+)^*$ is a solution, then $(a^+)^*(a^+ - a^*) = 0$. Pre-multiplying the equality by aa^* , one gets $a(a^+ - a^*) = 0$. Hence $a \in R^{PI}$ by (1). \square

Similarly, we have the following Lemma.

Lemma 2.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation*

$$(a^+ - a^*)x = 0$$

has at least one solution in χ_a .

Theorem 2.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the equation*

$$x(a^+ - a^*)y = 0$$

has at least one solution in $\chi_a^2 = \{(x, y) | x, y \in \chi_a\}$.

Proof. “ \Rightarrow ” It is an immediate result of Lemma 2.2.

“ \Leftarrow ” (1) If $y = a$, we have the following equation

$$x(a^+ - a^*)a = 0.$$

Post-multiplying the equation (4) by a^+ , one gets $x(a^+ - a^*) = 0$. Hence $a \in R^{PI}$ by Lemma 2.2.

(2) If $y = a^\#$, then we have the following equation

$$x(a^+ - a^*)a^\# = 0.$$

Post-multiplying the equation (5) by a^2 , one has $x(a^+ - a^*)a = 0$. Hence $a \in R^{PI}$ by (1).

(3) If $y = a^+$, then we obtain the following equation

$$x(a^+ - a^*)a^+ = 0.$$

So

$$x(a^+a - a^*a)a^+a^+ = 0.$$

By Lemma 2.1 $x(a^+ - a^*) = 0$. Hence $a \in R^{PI}$ by Lemma 2.2.

(4) If $y = a^*$, then we have the following equation

$$x(a^+ - a^*)a^* = 0.$$

Post-multiplying the equation (7) by $(a^+)^*a^+$, one gets $x(a^+ - a^*)a^+ = 0$. Hence $a \in R^{PI}$ by (3).

(5) If $y = (a^\#)^*$, then we have the following equation

$$x(a^+ - a^*)(a^\#)^* = 0.$$

Post-multiplying the equation (8) by $(a^*)^2$, we have $x(a^+ - a^*)a^* = 0$. Hence $a \in R^{PI}$ by (4).

(6) If $y = (a^+)^*$, then we have the following equation

$$x(a^+ - a^*)(a^+)^* = 0.$$

Post-multiplying the equation (9) by a^*a , we have $x(a^+ - a^*)a = 0$. Hence $a \in R^{PI}$ by (1). \square

Remark 2.5. Equation (1) can be generalized to

$$xa^+ - ya^* = 0. \quad (1)$$

It is easy to prove that the general solution of the equation (10) is given by

$$\begin{cases} x = -pa + u - ua^+a \\ y = -p(a^+)^* + v - va^+a \end{cases}, \text{ where } p, u, v \in R. \quad (2)$$

Lemma 2.6. Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if $a^+ = a^+(a^+)^*a^+$.

Proof. “ \Rightarrow ” Assume that $a \in R^{PI}$, then $(a^+)^* = a$. Hence $a^+ = a^+(a^+)^*a^+$.

“ \Leftarrow ” Suppose that $a^+ = a^+(a^+)^*a^+$, then $a = aa^+a = aa^+(a^+)^*a^+a = (a^+)^*$.

Thus $a \in R^{PI}$. \square

Proposition 2.7. Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution of the equation (10) is given by

$$\begin{cases} x = -p(a^+)^* + u - ua^+a \\ y = -p(a^+)^* + v - va^+a \end{cases}, \text{ where } p, u, v \in R. \quad (3)$$

Proof. “ \Rightarrow ” This is an immediate corollary of the formula (11).

“ \Leftarrow ” If the general solution of the equation (10) is given by the formula (12), then $(-p(a^+)^* + u - ua^+a)a^+ - (-p(a^+)^* + v - va^+a)a^* = 0$, that is $p(a^+)^*a^+ = paa^+$ for any $p \in R$. Especially, choose $p = a^+$, one yields $a^+(a^+)^*a^+ = a^+aa^+ = a^+$. Hence $a \in R^{PI}$ by Lemma 2.6. \square

Similarly, we have the following Proposition.

Proposition 2.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution of the equation (10) is given by*

$$\begin{cases} x = -pa + u - ua^+a \\ y = -pa + v - va^+a \end{cases}, \text{ where } p, u, v \in R. \quad (4)$$

Lemma 2.9. *Let $a \in R^\# \cap R^+$. Then $(a^+)^* a^+ a^* a a^* \in R^{EP}$ with*

$$((a^+)^* a^+ a^* a a^*)^+ = (a^+)^* a^+ (a^\#)^* a a^*.$$

Proof. Since

$$\begin{aligned} ((a^+)^* a^+ a^* a a^*)((a^+)^* a^+ (a^\#)^* a a^*) &= (a^+)^* a^+ a^* (a a^* (a^+)^*) a^+ (a^\#)^* a a^* \\ &= (a^+)^* a^+ (a^* a a^+) (a^\#)^* a a^* \\ &= (a^+)^* a^+ a^* (a^\#)^* a a^* \\ &= (a^+)^* a^+ a a^* \\ &= (a^+)^* a^* \\ &= (a a^+)^* \\ &= a a^+, \end{aligned}$$

$$\begin{aligned} ((a^+)^* a^+ (a^\#)^* a a^*)((a^+)^* a^+ a^* a a^*) &= (a^+)^* a^+ (a^\#)^* (a a^* (a^+)^*) a^+ a^* a a^* \\ &= (a^+)^* a^+ ((a^\#)^* a a^+) a^* a a^* \\ &= (a^+)^* (a^+ (a^\#)^* a^*) a a^* \\ &= (a^+)^* a^+ a a^* \\ &= a a^+. \end{aligned}$$

Hence $(a^+)^* a^+ a^* a a^* \in R^{EP}$ with $((a^+)^* a^+ a^* a a^*)^+ = (a^+)^* a^+ (a^\#)^* a a^*$. \square

Proposition 2.10. *Let $a \in R^\# \cap R^+$. Then*

- (1) $a \in R^{PI}$ if and only if $((a^+)^* a^+ a^* a a^*)^+ = a a^+ (a^\#)^* a a^*$;
- (2) $a \in R^{PI}$ if and only if $((a^+)^* a^+ a^* a a^*)^+ = (a^+)^* a^+ (a^\#)^*$;
- (3) a is normal if and only if $((a^+)^* a^+ a^* a a^*)^+ = (a^+)^*$;
- (4) $a \in R^{SEP}$ if and only if $((a^+)^* a^+ a^* a a^*)^+ = (a^+)^* a^+ (a^+)^*$.

Proof. (1) “ \Rightarrow ” Since $a \in R^{PI}$, $(a^+)^* = a$. Hence, by Lemma 2.9, we have

$$((a^+)^* a^+ a^* a a^*)^+ = (a^+)^* a^+ (a^\#)^* a a^* = a a^+ (a^\#)^* a a^*.$$

“ \Leftarrow ” Assume that $((a^+)^* a^+ a^* a a^*)^+ = a a^+ (a^\#)^* a a^*$, then $(a^+)^* a^+ (a^\#)^* a a^* = a a^+ (a^\#)^* a a^*$ by Lemma 2.9. Post-multiplying the equality by $(a^+)^* a^+$, one yields

$(a^+)^* a^+ (a^\#)^* = aa^+ (a^\#)^*$. Pre-multiplying the equality by a^* , one has $a^+ (a^\#)^* = a^* (a^\#)^*$, this gives $a^+ = a^+ (a^\#)^* a^* = a^* (a^\#)^* a^* = a^*$. Thus $a \in R^{PI}$.

(2) “ \Rightarrow ” Since $a \in R^{PI}$, $a^* = a^+$. This infers $(a^\#)^* aa^* = (a^\#)^* aa^+ = (a^\#)^*$. Hence $((a^+)^* a^+ a^* aa^*)^+ = (a^+)^* a^+ (a^\#)^*$ by Lemma 2.9.

“ \Leftarrow ” Assume that $((a^+)^* a^+ a^* aa^*)^+ = (a^+)^* a^+ (a^\#)^*$, then $(a^+)^* a^+ (a^\#)^* aa^* = (a^+)^* a^+ (a^\#)^*$ by Lemma 2.9. Pre-multiplying by a^* , we have $a^+ (a^\#)^* aa^* = a^+ (a^\#)^*$. Again pre-multiplying by $(a^\#)^* a^* a$, one obtains $(a^\#)^* aa^* = (a^\#)^*$, this gives $a^\# = aa^* a^\#$. Hence $a \in R^{PI}$.

(3) “ \Rightarrow ” Since a is normal, $aa^* = a^* a$ and $a \in R^{EP}$. Hence, by Lemma 2.9, we obtain

$$\begin{aligned} ((a^+)^* a^+ a^* aa^*)^+ &= (a^+)^* a^+ (a^\#)^* aa^* \\ &= (a^+)^* a^+ (a^+)^* a^* a \\ &= (a^+)^* a^+ a \\ &= (a^+)^*. \end{aligned}$$

“ \Leftarrow ” Assume that $((a^+)^* a^+ a^* aa^*)^+ = (a^+)^* = (a^*)^+$, then $(a^+)^* a^+ a^* aa^* = a^*$. Post-multiplying the equality by $(a^+)^* a^+$, one gets $(a^+)^* a^+ a^* = a^+$, this gives $aa^+ a^+ = aa^+ (a^+)^* a^+ a^* = (a^+)^* a^+ a^* = a^+$. Hence $a \in R^{EP}$. Now $a^* a^\# = a^* a^+ = a^* (a^+)^* a^+ a^* = a^+ a^* = a^\# a^*$. Thus a is normal.

(4) “ \Rightarrow ” It is an immediate result of Lemma 2.9, because $a^+ = a^\#$ and $a \in R^{PI}$. “ \Leftarrow ” Assume that $((a^+)^* a^+ a^* aa^*)^+ = (a^+)^* a^+ (a^+)^*$, then by Lemma 2.9, one has $(a^+)^* a^+ (a^\#)^* aa^* = (a^+)^* a^+ (a^+)^*$. Post-multiplying the equality by $1 - aa^+$, one yields $(a^+)^* a^+ (a^+)^* (1 - aa^+) = 0$. Pre-multiplying the equality by $a^* aa^*$, one has $a^+ a (1 - aa^+) = 0$, this gives $a(1 - aa^+) = 0$. Hence $a \in R^{EP}$. It follows that $((a^+)^* a^+ (a^\#)^* aa^*)^+ = (a^+)^* a^+ (a^+)^* = (a^+)^* a^+ (a^\#)^*$. Hence $a \in R^{PI}$ by (2). Therefore $a \in R^{SEPI}$. \square

Lemma 2.11. *Let $a \in R^\# \cap R^+$. Then $aa^+ a^* aa^* \in R^{EP}$ with*

$$(aa^+ a^* aa^*)^+ = (a^+)^* a^+ (a^\#)^*.$$

Proof.

$$\begin{aligned} (aa^+ a^* aa^*)((a^+)^* a^+ (a^\#)^*) &= aa^+ a^* aa^+ (a^\#)^* \\ &= aa^+ a^* (a^\#)^* \\ &= aa^+, \end{aligned}$$

$$\begin{aligned}
((a^+)^* a^+ (a^\#)^*) (aa^+ a^* aa^*) &= (a^+)^* a^+ (a^\#)^* a^* aa^* \\
&= (a^+)^* a^+ aa^* \\
&= aa^+.
\end{aligned}$$

Hence $aa^+ a^* aa^* \in R^{EP}$ with $(aa^+ a^* aa^*)^+ = (a^+)^* a^+ (a^\#)^*$. \square

Proposition 2.12. *Let $a \in R^\# \cap R^+$. Then*

- (1) $a \in R^{PI}$ if and only if $((a^+)^* a^+ (a^\#)^*)^+ = aa^+ a^*$;
- (2) $a \in R^{EP}$ if and only if $((a^+)^* a^+ (a^\#)^*)^+ = a^* aa^*$;
- (3) $a \in R^{SEP}$ if and only if $((a^+)^* a^+ (a^\#)^*)^+ = a^*$.

Proof. (1) “ \Rightarrow ” Since $a \in R^{PI}$, $a^* = a^+$. Hence, by Lemma 2.11, we get

$$((a^+)^* a^+ (a^\#)^*)^+ = aa^+ a^* aa^* = aa^+ a^* aa^+ = aa^+ a^*.$$

“ \Leftarrow ” Assume that $((a^+)^* a^+ (a^\#)^*)^+ = aa^+ a^*$, then $aa^+ a^* aa^* = aa^+ a^*$ by Lemma 2.11. Pre-multiplying the equality by $(a^\#)^* a^*$, one yields $a^* aa^* = a^*$, this gives $a = aa^* a$. Hence $a \in R^{PI}$.

- (2) “ \Rightarrow ” Since $a \in R^{EP}$, $aa^+ = a^+ a$. Hence

$$((a^+)^* a^+ (a^\#)^*)^+ = aa^+ a^* aa^* = a^+ aa^* aa^* = a^* aa^*.$$

“ \Leftarrow ” Assume that $((a^+)^* a^+ (a^\#)^*)^+ = a^* aa^*$. Then $aa^+ a^* aa^* = a^* aa^*$ by Lemma 2.11. Post-multiplying the equality by $(a^+)^* a^+ (a^+)^*$, one gets $aa^+ a^+ a = a^+ a$. Hence $a \in R^{EP}$.

- (3) “ \Rightarrow ” Since $a \in R^{SEP}$, $a \in R^{EP}$ and $a^* = a^+$. By (2) $((a^+)^* a^+ (a^\#)^*)^+ = a^* aa^* = a^* aa^+ = a^*$.

“ \Leftarrow ” Assume that $((a^+)^* a^+ (a^\#)^*)^+ = a^*$, then $aa^+ a^* aa^* = a^*$ by Lemma 2.11. Post-multiplying the equality by $(a^+)^* a^+$, one obtains $aa^+ a^* = a^+$. Hence $a \in R^{SEP}$ by [1, Theorem 2.3]. \square

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References

- [1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd. ed., Springer, New York, 2003.
- [2] S. Karanasios, *EP elements in rings and semigroup with involution and C^* -algebras*, *Serdica Math. J.*, 41 (2015), 83-116.
- [3] J. J. Koliha and P. Patricio, *Elements of rings with equal spectral idempotents*, *J. Aust. Math. Soc.*, 72 (2002), 137-152.

- [4] D. Mosić and D. S. Djordjević, *Partial isometries and EP elements in rings with involution*, Electron. J. Linear Algebra, 18 (2009), 761-772.
- [5] D. Mosić and D. S. Djordjević, *Further results on partial isometries and EP elements in rings with involution*, Math. Comput. Modelling, 54 (2011), 460-465.
- [6] D. Mosić, D. S. Djordjević and J. J. Koliha, *EP elements in rings*, Linear Algebra Appl., 431 (2009), 527-535.
- [7] Y. C. Qu, J. C. Wei and H. Yao, *Characterizations of normal elements in rings with involution*, Acta. Math. Hungar., 156(2) (2018), 459-464.
- [8] Z. C. Xu, R. J. Tang and J. C. Wei, *Strongly EP elements in a ring with involution*, Filomat, 34(6) (2020), 2101-2107.
- [9] S. Z. Xu, J. L. Chen and J. Benítez, *EP elements in rings with involution*, Bull. Malays. Math. Sci. Soc., 42 (2019), 3409-3426.
- [10] R. J. Zhao, H. Yao and J. C. Wei, *Characterizations of partial isometries and two special kinds of EP elements*, Czechoslovak Math. J., 70(145) (2020), 539-551.
- [11] R. J. Zhao, H. Yao and J. C. Wei, *EP elements and the solutions of equation in rings with involution*, Filomat, 32(13) (2018), 4537-4542.

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