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Special Mean and Total Curvature of a Dual Surface in Isotropic Spaces

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ABSTRACT

The study of the geometry surfaces in spaces with a degenerate metric is one of the urgent problems of modern geometry since its results find numerous applications in problems of mechanics and quantum mechanics.
In this paper, we study the properties of the total and mean curvatures of a surface and its dual image in an isotropic space. We prove the equality of the mean curvature and the second quadratic forms. The relation of the mean curvature of a surface to its dual surface is found. The superimposed space method is used to investigate the geometric characteristics of a surface relative to the normal and special normal.

Keywords: *Isotropic space, mean curvature, total curvature, dual surface, special mean curvature, special total curvature.*
AMS Subject Classification (2020): *Primary: 53A35; Secondary:51D99; 53A25; 53A40.*

1. Introduction

Differential geometry of isotropic spaces was first investigated by K. Strubecker [21, 22]. Interest in the geometry of isotropic space was renewed at the beginning of the XXI century due to its connection with classical mechanics and quantum theory. The works by Turkish mathematicians E.M. Aydin [6, 7, 8, 9], M.S. Lone and M.K. Karacan [18] are devoted to the reconstruction of a surface with constant total curvature. The work by H.Sachs [14] is devoted to groups of the motion of an isotropic space, and M. Dede [11] investigated the recovery problem in Galilean space.

The degeneracy of the scalar product does not make it possible to determine the analogue of the vector product in an isotropic space. Therefore, the superimposed space method is used to determine the surface normal. In addition to this normal, the surface has one more special normal. The surface has different geometric characteristics with respect to these normals. In this work, we study the properties of these geometric characteristics.

2. Preliminaries results

2.1. Basic concepts of an isotropic space R_3^2

Consider an affine space A_3 with the coordinate system *Oxyz*. Let $\vec{X}(x_1, y_1, z_1)$ and $\vec{Y}(x_2, y_2, z_2)$ be vectors of A_3.

Definition 2.1. If the scalar product of the vectors $\vec{X}$ and $\vec{Y}$ is defined by the formula

$$\begin{cases} (X,Y)_1 = x_1x_2 + y_1y_2 & \text{if} \quad x_1x_2 + y_1y_2 \neq 0, \\ (X,Y)_2 = z_1z_2 & \text{if} \quad x_1x_2 + y_1y_2 = 0, \end{cases} \tag{2.1}$$

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then A_3 is said to be an isotropic space R_3^2. [2, 4]

It is known, the norm of a vector $\overrightarrow{X}$ is defined by the formula $|\overrightarrow{X}| = \sqrt{(\overrightarrow{X}, \overrightarrow{X})}$, and the distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is defined as the norm of the vector: $|\overrightarrow{AB}| = \sqrt{(\overrightarrow{AB}, \overrightarrow{AB})}$. From this we obtain the definition of the distance d between A and B

$$d = \begin{cases} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} & \text{if} \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \neq 0, \\ |z_2 - z_1| & \text{if} \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0. \end{cases} \tag{2.2}$$

The metric defined in this way is called a degenerate metric, and spaces with projective metrics that have degeneracy were studied in the monograph by B.A.Rosenfeld [13].

Geometry in a plane of an isotropic space will be Euclidean if it is not parallel to the *oz* axis. When a plane is parallel to *oz*, the geometry on it will be Galilean.

Since an isotropic space has an affine structure, there is an affine transformation that preserves the distance determined by formula (2.2). This motion of an isotropic space is given by the formula [14, 15]

$$\begin{cases} x' = x\cos\alpha - y\sin\alpha + a \\ y' = x\sin\alpha + y\cos\alpha + b \\ z' = Ax + By + z + c \end{cases} \tag{2.3}$$

The matrix of this transformation has the form:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ A & B & 1 \end{pmatrix}$$

The sphere of an isotropic space, that is, the set of points in the space equidistant from one point, has the equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{2.4}$$

here (x_0, y_0, z) is the center of the sphere (2.4), r is its radius. When the center coincides with the origin, the equation of the sphere is $x^2 + y^2 = r^2$.

2.2. *Surface theory in isotropic space, duality*

Let a regular surface from the class C^2 be given in an isotropic space by the vector equality

$$r(u,v) = (x(u,v); y(u,v); z(u,v)), \qquad (u,v) \in D. \tag{2.5}$$

We introduce the concept of an isotropic surface normal vector [22] $\overrightarrow{n}_m(0,0,1)$ and a normal to a surface as in Euclidean space [19]: $\overrightarrow{n} = [r_u, r_v]$. In this case, we will use the superimposed space method, that is, the *Oxyz* coordinate system is considered the Euclidean Cartesian coordinate system. Moreover, the surface is considered as a surface of Euclidean space.

By analogy with Euclidean space, we define the first and second quadratic forms of the surface (2.5). The first quadratic form

$$I = ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{2.6}$$

where

$$\begin{cases} E = r_u^2 = x_u^2 + y_u^2 \\ G = r_v^2 = x_v^2 + y_v^2 \\ F = r_u r_v = x_u x_v + y_u y_v. \end{cases} \tag{2.7}$$

The second quadratic form

$$II = (d^2 r, n) = Ldu^2 + 2Mdudv + Ndv^2 \tag{2.8}$$

and

$$\begin{cases} L = (r_{uu}, n) = \dfrac{(r_{uu}, r_u, r_v)}{\sqrt{EG - F^2}} \\ M = (r_{uv}, n) = \dfrac{(r_{uv}, r_u, r_v)}{\sqrt{EG - F^2}} \\ N = (r_{vv}, n) = \dfrac{(r_{vv}, r_u, r_v)}{\sqrt{EG - F^2}} \end{cases}.$$

Consider also the second quadratic form of the surface (2.5) with respect to the isotropic normal to the surface

$$II = (d^2r, n_m) = L_m du^2 + 2M_m dudv + N_m dv^2 \tag{2.9}$$

where

$$\begin{cases} L_m = (r_{uu}, n_m) = z_{uu} \\ M_m = (r_{uv}, n_m) = z_{uv} \\ N_m = (r_{vv}, n_m) = z_{vv} \end{cases}.$$

Define the normal section of the surface in the given direction by a plane passing in the given direction and the isotropic normal n_m.

The normal curvature of a curve is defined as the curvature of the curve resulting from the normal section. Since the isotropic normal n_m is directed along the *Oz* axis, the geometry of the normal plane will be Galilean [12].

The normal curvature of a curve is defined by the formula [2]

$$k_n = \frac{II_m}{I}. \tag{2.10}$$

In an isotropic space, there are two kinds of spheres. The definition of the first of them we gave (2.4) in the previous paragraph.

The second sphere is defined as a surface with the constant normal curvature. This sphere of the unit radius has the equation [21]

$$x^2 + y^2 = 2z, \tag{2.11}$$

we call it the isotropic sphere.

Let a plane π be given in R_3^2, which is not parallel to the *oz* axis of the space. Consider the section of the isotropic sphere by the plane π and denote it by Γ. Since an isotropic sphere is a paraboloid of revolution, the section Γ by a plane is a closed curve. It was proved in [4] that Γ is an ellipse.

Draw tangent planes to isotropic sphere (2.11) through points$M \in \Gamma$. Denote the set of tangent planes to points *F* by $\{\pi\}$.

The following statement holds.

Theorem 2.1. *All planes of the set* $\{\pi\}$ *intersect at one point. [1]*

If a plane π_0 is given by the equation

$$z = Ax + By + C, \tag{2.12}$$

then the intersection point of the planes of the set $\{\pi\}$ will be $(A, B, -C)$.

Definition 2.2. The point $(A, B, -C)$ will be called dual to plane (2.12) with respect to isotropic sphere (2.11). [16]

Consider the plane $z = H$ and its section Γ by an isotropic sphere. Let the surface *F* be given by the equation

$$F : \{z = f(x, y) \,|\, (x, y) \in D'\}, \tag{2.13}$$

and the edge of the surface be the curve Γ. The surface (2.13) itself is convex and contained in the inner part of the space bounded by the plane and the isotropic sphere.

Let us draw the tangent plane π_M to the surface *F* at the point $M(x_0, y_0, z_0)$. Denote by M^* the dual image of the tangent space π_M with respect to the isotropic sphere. When the point $M \in F$ changes on the surface F, its dual image describes a surface F^*.

Definition 2.3. . The surface F^* is said to be the dual surface to the surface F in an isotropic space. [16]

When F is given by the equation $z = f(x, y)$, F^* has the equations

$$\begin{cases} x^*(u, v) = f'_u(u, v) \\ y^*(u, v) = f'_v(u, v) \\ z^*(u, v) = u \cdot f'_u(u, v) + v \cdot f'_u(u, v) - f(u, v) \end{cases} \tag{2.14}$$

3. Main results

3.1. Geometric characteristics of a surface in an isotropic space

We mean by the geometric characteristics of a surface the total and mean curvatures. Moreover, we define them in an isotropic and superimposed space. It will be connected with choosing the appropriate normal to the surface.

In addition, we will assume that a surface is regular as many times as required by the characteristics to be determined.

Lemma 3.1. *When the total curvature of a surface $K = 0$, its dual image is a point or a curve.*

Proof. Indeed, in the case of $K = 0$, the surface can be a plane, a cylinder, a cone, or another developing surface. In all these cases, there are straight generatrices on the surface. Tangent planes do not change along rectilinear generatrices; therefore, the dual image is a set of one-parameter points, that is, a curved line. □

For the total curvature of a surface and its dual surface (2.14), the following statement holds.

Theorem 3.1. *The product of the total curvatures of the surface $F$and the dual surface F^* of the isotropic space is equal to unity:*

$$K \cdot K^* = 1. \tag{3.1}$$

Proof. The total curvature of the surface F, given by the equation $z = f(x, y)$, is calculated in the isotropic space by the formula

$$K = f''_{uu} f''_{vv} - f''_{uv}{}^2. \tag{3.2}$$

We calculate the total curvature of the dual surface (2.14) by its formula.

Find the required derivatives of the function:

$$r_u = \begin{pmatrix} f''_{uu} & f''_{uv} & uf''_{uu} + vf''_{uv} \end{pmatrix} \qquad r_v = \begin{pmatrix} f''_{uv} & f''_{vv} & uf''_{uv} + vf''_{vv} \end{pmatrix}$$

$$r_{uu} = \begin{pmatrix} f'''_{uuu} & f'''_{uuv} & f''_{uu} + uf'''_{uuu} + vf'''_{uuv} \end{pmatrix}$$
$$r_{vv} = \begin{pmatrix} f'''_{uvv} & f'''_{vvv} & f''_{vv} + uf'''_{uvv} + vf'''_{vvv} \end{pmatrix}$$
$$r_{uv} = \begin{pmatrix} f'''_{uuv} & f'''_{uvv} & f''_{uv} + uf'''_{uuv} + vf'''_{uvv} \end{pmatrix}$$

The coefficients of the first quadratic form (2.6)

$$E^* = f''_{uu}{}^2 + f''_{uv}{}^2$$
$$F^* = f''_{uu} f''_{uv} + f''_{uv} f''_{vv}$$
$$G^* = f''_{uv}{}^2 + f''_{vv}{}^2$$

The form itself has the form

$$I^* = (f''_{uu}{}^2 + f''_{uv}{}^2)d^2u + (f''_{uu} f''_{uv} + f''_{uv} f''_{vv})dudv + (f''_{uv}{}^2 + f''_{vv}{}^2)d^2v. \tag{3.3}$$

Find the discriminant of the first quadratic form (2.6)

$$\begin{aligned} W = E^*G^* - (F^*)^2 &= \left(f''_{uu}{}^2 + f''_{uv}{}^2\right)\left(f''_{uv}{}^2 + f''_{vv}{}^2\right) - \left(f''_{uu} f''_{uv} + f''_{uv} f''_{vv}\right)^2 = \\ &= f''_{uu}{}^2 f''_{uv}{}^2 + f''_{uu}{}^2 f''_{vv}{}^2 + f''_{uv}{}^4 + f''_{uv}{}^2 f''_{vv}{}^2 - f''_{uu}{}^2 f''_{uv}{}^2 - 2f''_{uu} f''_{uv}{}^2 f''_{vv} - f''_{uv}{}^2 f''_{vv}{}^2 = \\ &= f''_{uu}{}^2 f''_{vv}{}^2 + f''_{uv}{}^4 - 2f''_{uu} f''_{uv}{}^2 f''_{vv} = \left[f''_{uu} f''_{vv} - f''_{uv}{}^2\right]^2 \end{aligned} \tag{3.4}$$

We use the following notations

$f''_{uu} f''_{vv} - f''_{uv}{}^2 = A$

$$\begin{vmatrix} f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix} = -u\left(f''_{uu} f''_{vv} - f''_{uv}{}^2\right) = -uA$$

$$\begin{vmatrix} f''_{uu} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & uf''_{uv}+vf''_{vv} \end{vmatrix} = v\left(f''_{uu}f''_{vv}-f''_{uv}{}^2\right)=vA$$
$$\begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix} = f''_{uu}f''_{vv}-f''_{uv}{}^2=A$$

$$L^*=\frac{\begin{vmatrix} f'''_{uuu} & f'''_{uuv} & f''_{uu}+uf'''_{uuu}+vf'''_{uuv} \\ f''_{uu} & f''_{uv} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & f''_{vv} & uf''_{uv}+vf''_{vv} \end{vmatrix}}{\sqrt{W}}=\frac{1}{\sqrt{W}}\left\{f'''_{uuu}\begin{vmatrix} f''_{uv} & uf''_{uu}+vf''_{uv} \\ f''_{vv} & uf''_{uv}+vf''_{vv} \end{vmatrix}-\right.$$
$$\left.f'''_{uuv}\begin{vmatrix} f''_{uu} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & uf''_{uv}+vf''_{vv} \end{vmatrix}+(f''_{uu}+uf'''_{uuu}+vf'''_{uuv})\begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix}\right\}=$$
$$\frac{1}{\sqrt{W}}\left\{-uf'''_{uuu}A-vf'''_{uuv}A+(f''_{uu}+uf'''_{uuu}+vf'''_{uuv})A\right\}=\frac{f''_{uu}A}{A}=f''_{uu}$$

$$M^*=\frac{\begin{vmatrix} f'''_{uuv} & f'''_{uvvv} & f''_{uv}+uf'''_{uuv}+vf'''_{uvv} \\ f''_{uu} & f''_{uv} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & f''_{vv} & uf''_{uv}+vf''_{vv} \end{vmatrix}}{\sqrt{W}}=\frac{1}{\sqrt{W}}\left\{f'''_{uuv}\begin{vmatrix} f''_{uv} & uf''_{uu}+vf''_{uv} \\ f''_{vv} & uf''_{uv}+vf''_{vv} \end{vmatrix}-\right.$$
$$\left.-f'''_{uvv}\begin{vmatrix} f''_{uu} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & uf''_{uv}+vf''_{vv} \end{vmatrix}+(f''_{uv}+uf'''_{uuv}+vf'''_{uvv})\begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix}\right\}=$$
$$=\frac{1}{\sqrt{W}}\left\{-uf'''_{uuv}A-vf'''_{uvv}A+(f''_{uv}+uf'''_{uuv}+vf'''_{uvv})A\right\}=\frac{f''_{uv}A}{A}=f''_{uv}$$

$$N^*=\frac{\begin{vmatrix} f'''_{uvv} & f'''_{vvv} & f''_{vv}+uf'''_{uvv}+vf'''_{vvv} \\ f''_{uu} & f''_{uv} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & f''_{vv} & uf''_{uv}+vf''_{vv} \end{vmatrix}}{\sqrt{W}}=\frac{1}{\sqrt{W}}\left\{f'''_{uvv}\begin{vmatrix} f''_{uv} & uf''_{uu}+vf''_{uv} \\ f''_{vv} & uf''_{uv}+vf''_{vv} \end{vmatrix}-\right.$$
$$\left.-f'''_{vvv}\begin{vmatrix} f''_{uu} & uf''_{uu}+vf''_{uv} \\ f''_{uv} & uf''_{uv}+vf''_{vv} \end{vmatrix}+(f''_{vv}+uf'''_{uvv}+vf'''_{vvv})\begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix}\right\}=$$
$$=\frac{1}{\sqrt{W}}\left\{-uf'''_{uvv}A-vf'''_{vvv}A+(f''_{vv}+uf'''_{uvv}+vf'''_{vvv})A\right\}=\frac{f''_{vv}A}{A}=f''_{vv}$$

Find the discriminant of the second quadratic form (2.8)

$$LN-M^2=f''_{uu}f''_{vv}-f''_{uv}{}^2=A$$

$$K^*=\frac{L^*N^*-(M^*)^2}{E^*G^*-(F^*)^2}=\frac{f''_{uu}f''_{vv}-f''_{uv}{}^2}{(f''_{uu}f''_{vv}-f''_{uv}{}^2)^2}=\frac{1}{f''_{uu}f''_{vv}-f''_{uv}{}^2}=\frac{1}{K}$$

Hence it follows that $K^*\cdot K=1$. Theorem is proved. □

The proved theorem implies the following statement.

Corollary 3.1. *The second quadratic forms of the surface and its dual image are equal, that is*

$$II=II^*. \tag{3.5}$$

The proof follows from the equality of the coefficients

$$L=f''_{uu}=L^*$$
$$M=f''_{uv}=M^*$$
$$N=f''_{vv}=N^*$$

which are, respectively, the coefficients of the second quadratic forms (2.8) of the surfaces (2.13).

Taking into account that the mean curvature of the dual surface (2.14) is calculated by the formula [8]

$$H^* = \frac{EN - 2FM + LG}{EG - F^2}, \tag{3.6}$$

substituting the corresponding values of the coefficients, we obtain the following equality

$$H^* = \frac{H}{K}, \tag{3.7}$$

where H and K are, respectively, the total and mean curvatures of the surface.

Now we determine the formula for the mean curvature of the surface with respect to the isotropic normal of the surface $\overrightarrow{n}_m(0,0,1)$. This mean curvature will be called the special mean curvature of the surface.

Theorem 3.2. *The mean curvatures defined with respect to the normal and the special normal are equal:* $H_m^* = H^*$.

Proof. In the proof of this theorem, we use some lemmas. To determine the mean curvature of the dual surface with respect to the isotropic normal, consider the following two auxiliary functions given by the equations

$$\overrightarrow{R_1}(u,v) = f_u \cdot \overrightarrow{i} + f_v \cdot \overrightarrow{j} + f_u \cdot \overrightarrow{k}, \tag{3.8}$$

$$\overrightarrow{R_2}(u,v) = f_u \cdot \overrightarrow{i} + f_v \cdot \overrightarrow{j} + f_v \cdot \overrightarrow{k}. \tag{3.9}$$

Lemma 3.2. *The special mean curvatures of the surfaces, given by the functions* $\overrightarrow{R_1}(u,v)$ *and* $\overrightarrow{R_2}(u,v)$*, are calculated, respectively, by the formulas*

$$H_m(R_1) = \frac{f_{uvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uuv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuu}\left(f_{uv}^2 + f_{vv}^2\right)}{\left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2}, \tag{3.10}$$

$$H_m(R_2) = \frac{f_{vvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uvv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuv}\left(f_{uv}^2 + f_{vv}^2\right)}{\left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2}. \tag{3.11}$$

Proof. The surface given by the equation $\overrightarrow{R}_1(u,v)$ and the surface F^* have the same first quadratic forms, since the coefficients at the basis vectors i, j are the same for them. They are as follows

$$E\left(R_1\right) = {f''_{uu}}^2 + {f''_{uv}}^2 = E^*$$

$$F\left(R_1\right) = f''_{uu}f^?_{uv} + f''_{uv}f''_{vv} = F^*$$

$$G(R_1) = {f''_{uv}}^2 + {f''_{vv}}^2 = G^*$$

Calculate now the coefficients of the second quadratic forms of these surfaces with respect to the normal $\overrightarrow{n_m} = (0,\ 0,\ 1)$:

$$L_m^*\left(R_1\right) = (r_{uu}, n) = f''_{uu} + uf'''_{uuu} + vf'''_{uuv}$$

$$N_m^*\left(R_1\right) = (r_{vv}, n) = f''_{vv} + uf'''_{uvv} + vf'''_{vvv}$$

$$M_m^*\left(R_1\right) = (r_{uv}, n) = f''_{uv} + uf'''_{uuv} + vf'''_{uvv}$$

Substituting the values of the coefficients of the first and second quadratic forms into the mean curvature formula (3.6), we obtain

$$H_m(R_1) = \frac{f_{uvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uuv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuu}\left(f_{uv}^2 + f_{vv}^2\right)}{\left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2}.$$

By analogous reasoning, we calculate the mean curvature of the surface $\overrightarrow{R}_2(u,v)$ (3.9):

$$H_m(R_2) = \frac{f_{vvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uvv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuv}\left(f_{uv}^2 + f_{vv}^2\right)}{\left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2}.$$

□

Lemma 3.3. *The mean curvatures of the surfaces, given by the functions* $\overrightarrow{R_1}(u,v)$ *and* $\overrightarrow{R_2}(u,v)$*, are equal to zero.*

Proof. For a given vector function $\overrightarrow{R_1}(u,v)$, the first and third components of the vector are equal to each other. Hence, we can conclude that this is the surface $x=z$, that is, it can be specified by the system of equations $\begin{cases} x-z=0 \\ y=f_v(u,v) \end{cases}$.

Hence, the surface is degenerated and is a domain on a plane, what implies that its mean curvature is equal to zero, since the principal curvatures vanish. Lemma is proved

When considering the surface defined by the function $\overrightarrow{R_2}(u,v)$, its second component is equal to the third one. By analogous reasoning we obtain the proof of Lemma. □

Lemma 3.4. *The mean curvature and special mean curvature of the dual surface* (2.14) *and the surfaces* $R_1(u,v)$, $R_2(u,v)$ *are connected by the equality:*

$$H_m^* = H^* + u\cdot H_m(R_1) + v\cdot H_m(R_2). \tag{3.12}$$

Proof. Calculate the coefficients of the second quadratic forms with respect to the special normal $\overrightarrow{n}=(0,\,0,\,1)$:

$$\begin{array}{c} L_m^* = (r_{uu}, n) = f''_{uu} + uf'''_{uuu} + vf'''_{uuv} \\ N_m^* = (r_{vv}, n) = f''_{vv} + uf'''_{uvv} + vf'''_{vvv} \\ M_m^* = (r_{uv}, n) = f''_{uv} + uf'''_{uuv} + vf'''_{uvv} \end{array}$$

Calculate the special mean curvature

$$H_m{}^* = \frac{E^*N_m^* - 2F^*M_m{}^* + L_m^*G^*}{E^*G^* - (F^*)^2} =$$

$$= \frac{\left(f''_{uu}{}^2 + f''_{uv}{}^2\right)\left(f''_{vv} + uf'''_{uvv} + vf'''_{vvv}\right) - 2\left(f''_{uu}f''_{uv} + f''_{uv}f''_{vv}\right)\left(f''_{uv} + uf'''_{uuv} + vf'''_{uvv}\right) +}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2}$$

$$+\frac{\left(f''_{uv}{}^2 + f''_{vv}{}^2\right)\left(f''_{uu} + uf'''_{uuu} + vf'''_{uuv}\right)}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2} =$$

$$= \frac{f''_{uu}{}^2 f''_{vv} + uf'''_{uvv}f''_{uu}{}^2 + vf'''_{vvv}f''_{uu}{}^2 + f''_{uv}{}^2 f''_{vv} + f''_{uv}{}^2 uf'''_{uvv} + f''_{uv}{}^2 vf'''_{vvv}}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2} +$$

$$-\frac{2f''_{uu}f''_{uv}{}^2 + 2uf'''_{uuv}f''_{uu}f''_{uv} + 2vf'''_{uvv}f''_{uu}f''_{uv} + 2f''_{uv}{}^2 f''_{vv} + 2uf'''_{uuv}f''_{uv}f''_{vv} + 2vf'''_{uvv}f''_{uv}f''_{vv}}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2} +$$

$$+\frac{f''_{uu}f''_{uv}{}^2 + uf'''_{uuu}f''_{uv}{}^2 + vf'''_{uuv}f''_{uv}{}^2 + f''_{uu}f''_{vv}{}^2 + uf'''_{uuu}f''_{vv}{}^2 + vf'''_{uuv}f''_{vv}{}^2}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2} =$$

$$= \frac{\left(f_{uu} + f_{vv}\right)\left(f''_{uu}f''_{vv} - f''_{uv}{}^2\right)}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2}$$

$$+\frac{u\left[f_{uvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uuv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuu}\left(f_{uv}^2 + f_{vv}^2\right)\right]}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2}$$

$$+\frac{v\left[f_{vvv}\left(f_{uu}^2 + f_{uv}^2\right) - 2f_{uvv}\left(f_{uu}f_{uv} + f_{uv}f_{vv}\right) + f_{uuv}\left(f_{uv}^2 + f_{vv}^2\right)\right]}{\left[f''_{uu}f''_{vv} - f''_{uv}{}^2\right]^2}.$$

Taking into account the formula obtained in Lemma 2, we have

$$H_m^* = \frac{H}{K} + u\cdot H_m(R_1) + v\cdot H_m(R_2)$$

It was shown earlier (3.7) that $H^* = \frac{H}{K}$, from this we obtain the required equality

$$H_m^* = H^* + u\cdot H_m(R_1) + v\cdot H_m(R_2), \tag{3.13}$$

where $H_m(R_1)$ and $H(R_2)$ are special mean curvatures of the surfaces defined by the functions $R_1(u,v)$ and $R_2(u,v)$.

Lemma is proved. □

Using results of Lemmas 2, 3, and 4, it is easy to prove Theorem. In equality (3.13), the terms vanish, taking this into account, we obtain the equality from the statement of Theorem: $H_m^* = H^*$. Theorem is proved. □

Consider the surface F and its dual surface F^*, as well as the surfaces given by the equations

$$Z_1 = f_u(u, v) \tag{3.14}$$

and

$$Z_2 = f_v(u, v). \tag{3.15}$$

Denote by Ω the following determinant

$$\Omega = \begin{vmatrix} f_{uuu} & f_{uvv} \\ f_{uuv} & f_{vvv} \end{vmatrix} = f'''_{uuu} f'''_{vvv} - f'''_{uuv} f'''_{uvv}. \tag{3.16}$$

Theorem 3.3. *If $\Omega = 0$, then the special total curvature of the surface F^* is expressed in terms of the special total curvatures of the surfaces F, Z_1, and Z_2.*

Proof. Consider the discriminant of the second quadratic form

$$LN - M^2 = f''_{uu} f''_{vv} - {f''_{uv}}^2 + u^2 \left(f'''_{uuu} f'''_{uvv} - {f'''_{uuv}}^2 \right) + v^2 \left(f'''_{uuv} f'''_{vvv} - {f'''_{uvv}}^2 \right) +$$

$$+ u \left(f''_{uu} f'''_{uvv} + f''_{vv} f'''_{uuu} - 2 f''_{uv} f'''_{uuv} \right) + v \left(f''_{uu} f'''_{vvv} + f''_{vv} f'''_{uuv} - 2 f''_{uv} f'''_{uvv} \right) +$$

$$+ uv \left[f'''_{uuu} f'''_{vvv} - f'''_{uuv} f'''_{uvv} \right]$$

Taking into account the discriminant of the first quadratic form (2.6)

$${K_m}^* = \frac{LN - M^2}{EG - F^2} = \frac{f''_{uu} f''_{vv} - {f''_{uv}}^2 + u^2 \left(f'''_{uuu} f'''_{uvv} - {f'''_{uuv}}^2 \right) + v^2 \left(f'''_{uuv} f'''_{vvv} - {f'''_{uvv}}^2 \right)}{\left(f''_{uu} f''_{vv} - {f''_{uv}}^2 \right)^2} +$$

$$+ \frac{u \left(f''_{uu} f'''_{uvv} + f''_{vv} f'''_{uuu} - 2 f''_{uv} f'''_{uuv} \right) + v \left(f''_{uu} f'''_{vvv} + f''_{vv} f'''_{uuv} - 2 f''_{uv} f'''_{uvv} \right) + uv \left[f'''_{uuu} f'''_{vvv} - f'''_{uuv} f'''_{uvv} \right]}{\left(f''_{uu} f''_{vv} - {f''_{uv}}^2 \right)^2}$$

$${K_m}^* = \frac{K + u^2 K_{z_1} + v^2 K_{z_2} + u \frac{\partial K}{\partial u} + v \frac{\partial K}{\partial v} + uv \cdot \Omega}{K^2}$$

where K_{z_1} and K_{z_2} are, respectively, special total curvatures of the surfaces (3.14) and (3.15), we obtain at $\Omega = 0$ the formula

$${K_m}^* = \frac{K + u^2 K_{z_1} + v^2 K_{z_2} + u \frac{\partial K}{\partial u} + v \frac{\partial K}{\partial v}}{K^2},$$

which shows the validity of the statement of Theorem. Theorem is proved □

Let us study the class of surfaces $\{\Phi\}$ defined by the equation

$$\Phi : \ \{ z = a_{11} x^2 + 2 a_{12} xy + a_{22} y^2 + 2 a_{31} x + 2 a_{32} y + a_{33} \} \tag{3.17}$$

Theorem 3.4. *If $\Phi_0 \in \{\Phi\}$, then its total and special total curvatures are equal if they are nonzero.*

Proof. When the total curvature of a surface $K = 0$, it will be a cylindrical surface or a conical surface. Then its dual surface can be a point or a curve. Therefore, we exclude the equality of the total curvature to zero.

When the total curvature is nonzero, Theorem is proved by direct calculations. We have

$$\Phi^* : \begin{cases} x^* = 2 a_{11} x + 2 a_{12} y + 2 a_{31} \\ y^* = 2 a_{12} x + 2 a_{22} y + 2 a_{32} \\ z = a_{11} x^2 + 2 a_{12} xy + a_{22} y^2 - a_{33} \end{cases} \tag{3.18}$$

The special total curvature of the surface $\Phi_0 : K = 4(a_{11} a_{22} - a_{12}^2)$. Hence, by Theorem 2. the total curvature of the dual surface (3.18)

$$K^* = \frac{1}{4(a_{11} a_{22} - a_{12}^2)}.$$

Calculate the special total curvature of the surface Φ^* by Theorem 4. We get the equalities $K = 4(a_{11} a_{22} - a_{12}^2)$ $\frac{\partial K}{\partial u} = 0, \ \frac{\partial K}{\partial v} = 0$ for the surface Φ_0.

$K_{z_1} = 0$ and $K_{z_2} = 0$ for Z_1 and Z_2.

And the value of the operator is $\Omega = 0$. Calculate the special total curvature of the dual surface (3.18)

$$\Phi^* : K_m^* = \frac{4(a_{11}a_{22} - a_{12}^2) + u^2 \cdot 0 + v^2 \cdot 0 + u \cdot 0 + v \cdot 0 + u \cdot v \cdot 0}{[4(a_{11}a_{22} - a_{12}^2)]^2} = \frac{1}{4(a_{11}a_{22} - a_{12}^2)} = K^*.$$

The special total curvature of the dual to $\{\Phi\}$ surface in the isotropic space is equal to the total curvature of the dual surface. Theorem is proved. □

Conclusions

In the paper, we obtained formulas and study properties of the total and mean curvatures of a surface defined with respect to the isotropic normal and special normal.

We proved that the mean curvatures of the surface, defined with respect to the normal and the singular normal, are equal to each other.

The total curvature of the surface and its dual image have the opposite meaning, they will be equal only for parabolic surfaces.

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