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The Bounds for the First General Zagreb Index of a Graph

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Abstract

The first general Zagreb index of a graph G is defined as the sum of the αth powers of the vertex degrees of G, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$. In this note, for $\alpha > 1$, we present upper bounds involving chromatic and clique numbers for the first general Zagreb index of a graph; for an integer $\alpha \geq 2$, we present a lower bound involving the independence number for the first general Zagreb index of a graph.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges, where $V = \{v_1, v_2, ..., v_n\}$. We assume that the vertices in G are arranged such that $\Delta(G) = d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n) = \delta(G)$, where $d_G(v_i)$, for each i with $1 \leq i \leq n$, is the degree of vertex v_i in G. The chromatic number, denoted $\chi(G)$, of a graph G is the smallest number of colors which can be assigned to $V(G)$ so that the adjacent vertices in G are colored differently. A clique of a graph G is a complete subgraph of G. A clique of largest possible size is called a maximum clique. The clique number, denoted $\omega(G)$, of a graph G is the number of vertices in a maximum clique of G. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of largest possible size. The independence number, denoted $\beta(G)$, of a graph G is the cardinality of a maximum independent set in G. If H is any graph of order n with degree sequence $d_H(v_1) \geq d_H(v_2) \geq \cdots \geq d_H(v_n)$, and if H^* is any graph of order n with degree sequence $d^*_H(v_1) \geq d^*_H(v_2) \geq \cdots \geq d^*_H(v_n)$, such that $d_H(v_i) \leq d^*_H(v_i)$ (for each i with $1 \leq i \leq n$), then H^* is said to dominate H. We use $C(n,r)$ to denote the number of r-element subsets of a set of size n, where n and r are nonnegative integers such that $0 \leq r \leq n$.

The first Zagreb index was introduced by Gutman and Trinajstić in [8]. For a graph G, its first Zagreb index is defined as $\sum_{i=1}^{n} d_G^2(v_i)$. Li and Zheng in [9] further extended the first Zagreb index of a graph and introduced the concept of the first general Zagreb index of a graph. The first general Zagreb index, denoted $M_\alpha(G)$, of a graph G is defined as $\sum_{i=1}^{n} d_G^{\alpha}(v_i)$, where α is a real number such that $\alpha \neq 0$ and $\alpha \neq 1$.

In this note, we will present upper bounds involving chromatic and cliques numbers for the first general Zagreb index of a graph when $\alpha > 1$ and a lower bound involving the independent number for the first general Zagreb index of a graph when α is an integer at least 2. The main results of this note are as follows.

Theorem 1.1. *Let G be a graph of order n. Assume α is a real number such that $\alpha > 1$. Then*

$$(1)\ M_\alpha \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).$$

Equality holds if and only if G is K_n.

$$(2)\ M_\alpha \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega}\right).$$

Equality holds if and only if G is K_n.

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Theorem 1.2. *Let G be a graph of order n. Assume α is an integer which is at least 2. Then*

$$M_\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

Equality holds if and only if G is a disjoint union of β complete graphs of order $\Delta+1$.

2. Lemmas

In order to prove Theorem 1 and Theorem 2, we need the following results as our lemmas. The first one is a theorem proved by Erdős in [6]. Its proofs in English can be found in [1].

Lemma 2.1. *If H is any graph of order n, then there exists a graph H^* of order n, where $\chi(H^*) \leq \omega(H)$, such that H^* dominates H.*

The second one can be found in [4] and [10].

Lemma 2.2. *If G is a graph, then*

$$\beta \geq \sum_{v \in V} \frac{1}{d(v)+1}.$$

Equality holds if and only if each component of G is complete.

3. Proofs

Next, we will prove Theorem 1.1. The ideas from the proofs of Theorem 3 on Page 53 in [5] will be used in the proofs of Theorem 1.1 below.

Proof of (1) in Theorem 1.1 Let us partition the vertex set V of G into the pairwise disjoint nonempty subsets of V_1, V_2, ..., V_χ such that V_i is independent for each i with $1 \leq i \leq \chi$. Set $|V_i| := n_i$ for each i with $1 \leq i \leq \chi$. Then we have that $n = \sum_{i=1}^{\chi} n_i$ and $d(x) \leq n - n_i$ for each vertex x in V_i and each i with $1 \leq i \leq \chi$. Without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_\chi$. From Cauchy-Schwarz inequality, we have that

$$\sum_{i=1}^{\chi} n_i^2 \geq \frac{\left(\sum_{i=1}^{\chi} n_i\right)^2}{\chi} = \frac{n^2}{\chi}.$$

Now

$$\begin{aligned} M_\alpha &= \sum_{v \in V} d^\alpha(v), \\ &= \sum_{i=1}^{\chi} \sum_{v \in V_i} d^\alpha(v), \\ &\leq \sum_{i=1}^{\chi} n_i (n-n_i)^\alpha, \\ &= \sum_{i=1}^{\chi} n_i (n-n_i)(n-n_i)^{\alpha-1} \leq \sum_{i=1}^{\chi} n_i (n-n_i)(n-n_1)^{\alpha-1}, \\ &= (n-n_1)^{\alpha-1} \sum_{i=1}^{\chi} n_i(n-n_i) \leq (n-1)^{\alpha-1}(n^2 - \sum_{i=1}^{\chi} n_i^2) \leq (n-1)^{\alpha-1}\left(n^2 - \frac{n^2}{\chi}\right) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right). \end{aligned}$$

If

$$M_\alpha = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right),$$

we, from the above proofs, have that $n_1 = n_2 = \cdots = n_\chi = 1$ and $d(v) = n-1$ for each vertex v in V. Thus G is K_n. If G is K_n, it is easy to verify that

$$M_\alpha = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi}\right).$$

This completes the proof of (1) in Theorem 1.1.

Proof of (2) in Theorem 1.1 From Lemma 2.1, we can find a graph G^* dominating G and $\chi(G^*) \leq \omega(G)$. From (1) of this theorem, we have that

$$M_\alpha(G) \leq M_\alpha(G^*) \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi(G^*)}\right) \leq n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega(G)}\right).$$

If

$$M_\alpha(G) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega}\right),$$

then

$$M_\alpha(G^*) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\chi(G^*)}\right).$$

From (1) of this theorem, we have that G^* is K_n and $\chi(G^*) = n$. Thus $\omega(G) \geq \chi(G^*) = n$. Hence G is K_n. If G is K_n, it is again easy to verify that

$$M_\alpha(G) = n^2(n-1)^{\alpha-1}\left(1 - \frac{1}{\omega}\right).$$

This completes the proof of (2) in Theorem 1.1.

Next, we will prove Theorem 1.2 which is motivated by Theorem 3.1 on Page 309 in [7].

Proof of Theorem 1.2 From Lemma 2.2 and the inequalities on the power means, arithmetic means, and harmonic means of n positive integers, we have that

$$\left(\frac{(1+d_1)^\alpha + (1+d_2)^\alpha + \cdots + (1+d_n)^\alpha}{n}\right)^{\frac{1}{\alpha}} \geq \frac{(1+d_1)+(1+d_2)+\cdots+(1+d_n)}{n} \geq \frac{n}{\frac{1}{1+d_1}+\frac{1}{1+d_2}+\cdots+\frac{1}{1+d_n}} \geq \frac{n}{\beta}.$$

Then

$$(1+d_1)^\alpha + (1+d_2)^\alpha + \cdots + (1+d_n)^\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha}.$$

It is easy to check that for each i with $1 \leq i \leq n$ we have

$$(1+d_i)^\alpha = \sum_{k=0}^{\alpha} C(\alpha,k)d_i^k \leq \sum_{k=0}^{\alpha} C(\alpha,k)\Delta^k - \Delta^\alpha + d_i^\alpha = (1+\Delta)^\alpha - \Delta^\alpha + d_i^\alpha.$$

Equality holds if and only if $d_i = \Delta$. Thus

$$(1+\Delta)^\alpha - \Delta^\alpha + d_1^\alpha + (1+\Delta)^\alpha - \Delta^\alpha + d_2^\alpha + \cdots + (1+\Delta)^\alpha - \Delta^\alpha + d_n^\alpha \geq (1+d_1)^\alpha + (1+d_2)^\alpha + \cdots + (1+d_n)^\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha}.$$

Therefore

$$M_\alpha \geq \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

If

$$M_\alpha = \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha),$$

then $d_1 = d_2 = \cdots = d_n = \Delta$. From Lemma 2, we have that G is a union of β complete graphs of order $\Delta+1$. If G is a union of β complete graphs of order $\Delta+1$, then $(\Delta+1)\beta = n$. It is easy to verify that

$$M_\alpha = \frac{n^{\alpha+1}}{\beta^\alpha} + n(\Delta^\alpha - (1+\Delta)^\alpha).$$

This completes the proof of Theorem 1.2.

Remark 3.1. *Let G be a graph with n vertices and e edges. Notice that*

$$n + 4e + M_2 = \sum_{i=1}^{n}(1+d_i)^2 \geq \frac{n^3}{\beta^2}.$$

We have that

$$M_2 \geq \frac{n^3}{\beta^2} - n - 4e.$$

It can be verified that $M_2 = \frac{n^3}{\beta^2} - n - 4e$ if and only if G is a disjoint union of β complete graphs of order $\Delta+1$.

Remark 3.2. *Let G be a graph with n vertices and e edges. Notice that*

$$n + 6e + 3M_2 + M_3 = \sum_{i=1}^{n}(1+d_i)^3 \geq \frac{n^4}{\beta^3}.$$

We have that

$$M_3 \geq \frac{n^4}{\beta^3} - n - 6e - 3U,$$

where U is an upper bound for M_2. A variety of concrete expressions for U can be found in [3].

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Author's contributions

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