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Wilker−type Inequalities for k−Fibonacci Hyperbolic Functions

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Abstract. In this paper, we introduce the Wilker−Anglesio's inequality and parameterized Wilker inequality for the k−Fibonacci hyperbolic functions using classical analytical techniques.

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1. Introduction

The inequalities

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

and

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x$$

were formulated by Wilker [15], where $0 < x < \frac{\pi}{2}$ and c is constant. Several proofs of Wilker's inequality were introduced by Sumner et al. [14], Guo et al. [6], Zhang and Zhu [19] and Zhu [20]. Moreover, Anglesio proposed the sharp inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \frac{16}{\pi^4} x^3 \tan x, \tag{1.2}$$

where $0 < x < \frac{\pi}{2}$ and the constant $\frac{16}{\pi^4}$ is best possible and it can not be changed with a larger number. Also, Huygens [8] proved an important inequality, that is

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3.$$

In recent years, some authors have studied the generalization and some applications of the Wilker (1.1) inequality and Wilker-Anglesio (1.2) inequality [5, 12, 16, 21, 22]. Also, Wu and Srivastava [16] presented a generalization of Wilker's inequality as

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$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^{p}+\frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^{q}>1,$$

where $0 < x < \frac{\pi}{2}$, $\lambda > 0$, $\mu > 0$, $p \leq \frac{2q\mu}{\lambda}$, $q > 0$ or $q \leq \min\{\frac{-\lambda}{\mu}, -1\}$. Recently, some authors have studied the applications of Wilker and Anglesio type inequalities for hyperbolic functions [1, 11, 17]. Wu and Debnath [17] introduced the Wilker-Anglesio and parameterized Wilker inequality for hyperbolic functions as follows:

$$\left(\frac{\sinh x}{x}\right)^{2}+\frac{\tanh x}{x}>2+\frac{8}{45}x^{3}\tanh x,$$

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^{p}+\frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^{q}>1,$$

where $0 < x < \frac{\pi}{2}$, $\lambda > 0$, $\mu > 0$, $p \leq \frac{2q\mu}{\lambda}$, $q > 0$ or $q \leq \min\{\frac{-\lambda}{\mu}, -1\}$. Moreover, Bahşi, in [1], studied the Wilker−Anglesio and parameterized Wilker inequality for Fibonacci hyperbolic functions. In this paper, our purpose is to establish the Wilker-Anglesio and parameterized Wilker's inequality for $k-$Fibonacci hyperbolic functions and extend the study in [1] for different k values.

2. Preliminaries

Fibonacci sequence, F_n, is one of the most popular sequences in mathematics. The classical Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$, for $n \in \mathbb{N}$, with initial conditions $F_0 = 0$, $F_1 = 1$. Until now, several authors have studied the applications and generalizations of the Fibonacci sequence [3, 4, 9, 10, 13, 18]. For $n \geq 1$ and any positive real number k, a remarkable generalization of the Fibonacci sequence, the $k-$Fibonacci sequence, $\{F_{k,n}\}_{n\in\mathbb{N}}$, was defined by,

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}$$

with the initial conditions $F_{k,0} = 0, F_{k,1} = 1$ in [2]. The characteristic equation of $F_{k,n}$ is

$$r^2 - kr - 1 = 0. \tag{2.1}$$

The zeros of the Eq. (2.1) are $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$ and $\gamma_k = \frac{k-\sqrt{k^2+4}}{2}$. Recently, some authors have studied the generalizations and relations with the special inequalities of hyperbolic functions. Stakhov and Rozin [13] defined a new class of hyperbolic functions, the symmetrical Fibonacci hyperbolic functions, as:

$$sFs(x) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}$$

and

$$cFs(x) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}},$$

where α is positive root of the characteristic equation of the Fibonacci sequence. Falcon and Plaza [4] defined $k-$Fibonacci hyperbolic functions as:

$$sF_kh(x) = \frac{\sigma_k^x - \sigma_k^{-x}}{\sqrt{k^2+4}}$$

and

$$cF_kh(x) = \frac{\sigma_k^x + \sigma_k^{-x}}{\sqrt{k^2+4}}.$$

In addition, the $k-$Fibonacci hyperbolic functions $sF_kh(x)$ and $cF_kh(x)$ are increasing on $(0, +\infty)$. Also some properties, which we use in this study, for the $k-$Fibonacci hyperbolic functions are as follows [4]:

- $sF_kh(x) = -sF_kh(-x)$,
- $cF_kh(x) = cF_kh(-x)$,

- $tF_kh(x) = -tF_kh(-x)$,
- $[cF_kh(x)]^2 - [sF_kh(x)]^2 = \frac{4}{k^2+4}$,
- $tF_kh(x) = \frac{sF_kh(x)}{cF_kh(x)}$.

Also, the derivative of the k−Fibonacci hyperbolic functions, with respect to x, are

$$[cF_kh(x)]^{(m)} = \begin{cases} \ln(\sigma_k)^m sF_kh(x), & \text{odd } m \\ \ln(\sigma_k)^m cF_kh(x), & \text{even } m, \end{cases}$$

and

$$[sF_kh(x)]^{(m)} = \begin{cases} \ln(\sigma_k)^m cF_kh(x), & \text{odd } m \\ \ln(\sigma_k)^m sF_kh(x), & \text{even } m. \end{cases}$$

3. Some Lemmas

In order to prove the main results in Sections 4 and 5, we first introduce the following lemmas.

Lemma 3.1 ([7]). *If $x_i > 0, \lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$, then*

$$\sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n {x_i}^{\lambda_i}.$$

Lemma 3.2. *For all nonzero real numbers x and any positive real number k, the following inequality holds:*

$$\frac{2}{\sqrt{k^2+4}} \leq cF_kh(x) \leq \frac{k^2+4}{4\ln(\sigma_k^3)} \left(\frac{sF_kh(x)}{x}\right)^3. \tag{3.1}$$

Proof. From $cF_kh(0) = \frac{2}{\sqrt{k^2+4}}$, $cF_kh(x) = cF_kh(-x)$, and $cF_kh(x)$ is increasing on $(0, +\infty)$, the left hand side of the equation (3.1) is true. Now we prove the right hand side of the inequality of (3.1).

Case (I) : For $x > 0$, define a function $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = \frac{sF_kh^3(x)}{x^3 cF_kh(x)}.$$

By differentiating with respect to x, we have

$$\begin{aligned} f'(x) &= \frac{sF_kh(x)^2}{x^4 cF_kh(x)^2}\Big(2\ln(\sigma_k)x sF_kh(x)^2 + \frac{12\ln(\sigma_k)x}{k^2+4} - 3sF_kh(x)cF_kh(x)\Big) \\ &= \frac{sF_kh(x)^2}{x^4 cF_kh(x)^2} f_1(x), \\ f_1'(x) &= 4\ln(\sigma_k)sF_kh(x)cF_kh(x)\Big(x\ln(\sigma_k) - \frac{sF_kh(x)}{cF_kh(x)}\Big) \\ &= 4\ln(\sigma_k)sF_kh(x)cF_kh(x)f_2(x), \\ f_2'(x) &= \ln(\sigma_k)\left(\frac{sF_kh(x)}{cF_kh(x)}\right)^2 > 0. \end{aligned}$$

This means that $f_2(x)$ is increasing on $(0, +\infty)$. Hence, we conclude from $f_2(0) = f_1(0)$ that $f_2(x) > 0$ and $f_1(x)$ is increasing and positive on $(0, +\infty)$. Therefore, $f(x)$ is increasing on $(0, +\infty)$. By using

$$\lim_{x\to 0^+} f(x) = \frac{4\ln(\sigma_k)^3}{k^2+4},$$

we conclude that

$$cF_kh(x) < \frac{k^2+4}{4\ln(\sigma_k)^3}\left(\frac{sF_kh(x)}{x}\right)^3.$$

Case (II) : Let $x < 0$ or $-x > 0$. Since $sF_kh(x) = -sF_kh(-x)$, $cF_kh(x) = cF_kh(-x)$, the proof is the same as in the Case (I). Therefore, the proof is completed. □

4. Wilker-Anglesio's Inequality for k−Fibonacci Hyperbolic Functions

Theorem 4.1. *For nonzero real number x and any positive real number k, the following inequality holds:*

$$\left(\frac{sF_kh(x)}{x}\right)^2 + \left(\frac{tF_kh(x)}{x}\right) > \frac{8\ln(\sigma_k)^2}{k^2+4} + \frac{32\ln(\sigma_k)^5}{45(k^2+4)}x^3 tF_kh(x).$$

Proof. From $sF_kh(-x) = -sF_kh(x)$ and $tF_kh(-x) = -tF_kh(x)$, we get

$$\left(\frac{sF_kh(-x)}{-x}\right)^2 + \left(\frac{tF_kh(-x)}{-x}\right) = \left(\frac{sF_kh(x)}{x}\right)^2 + \left(\frac{tF_kh(x)}{x}\right).$$

Hence, it is enough to prove that Theorem 4.1 is true for $x > 0$. Now we define a function $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(x) = \frac{\frac{k^2+4}{4\ln(\sigma_k)^2}\left(\frac{sF_kh(x)}{x}\right)^2 + \frac{1}{\ln(\sigma_k)}\frac{tF_kh(x)}{x} - 2}{x^3 tF_kh(x)}. \tag{4.1}$$

Then, differentiating the Eq. (4.1) with respect to x, we have

$$\begin{aligned}
g'(x) &= \frac{1}{4\sqrt{k^2+4}\ln(\sigma_k)x^6 sF_kh^2(x)}\Big(2xcF_kh(4x) + 24x^2\ln(\sigma_k)sF_kh(2x) - \frac{5}{\ln(\sigma_k)}sF_kh(4x)\\
&\quad + \frac{10}{\ln(\sigma_k)}sF_kh(2x) - 20xcF_kh(2x) + \frac{36x}{\sqrt{k^2+4}} + \frac{32x^3\ln(\sigma_k)^2}{\sqrt{k^2+4}}\Big)\\
&= \frac{g_1(x)}{4\sqrt{k^2+4}\ln(\sigma_k)x^6 sF_kh^2(x)},\\
g_1'(x) &= 8\sqrt{k^2+4}cF_kh^2(x)\Big(6x^2\ln(\sigma_k)^2 - \frac{18}{8}(k^2+4)sF_kh^2(x)\\
&\quad + x\ln(\sigma_k)sF_kh(x)\left((k^2+4)cF_kh(x) - \frac{1}{cF_kh(x)}\right)\Big)\\
&= 8\sqrt{k^2+4}cF_4h^2(x)g_2(x),\\
g_2'(x) &= \frac{1}{cF_kh^2(x)}\Big(\frac{-7}{2}(k^2+4)\ln(\sigma_k)sF_kh(x)cF_kh^3(x) - \ln(\sigma_k)sF_kh(x)cF_kh(x)\\
&\quad + 2(k^2+4)x\ln(\sigma_k)^2cF_kh^4(x) + 8x\ln(\sigma_k)^2cF_kh^2(x) - \frac{4}{k^2+4}x\ln(\sigma_k)^2\Big)\\
&= \frac{g_3(x)}{cF_kh^2(x)},\\
g_3'(x) &= 4\ln(\sigma_k)^2 sF_kh(2x)\Big(\frac{8}{\sqrt{k^2+4}}x\ln(\sigma_k) - 3sF_kh(2x) + 2x\ln(\sigma_k)cF_kh(2x)\Big)\\
&= 4\ln(\sigma_k)^2 sF_kh(2x)g_4(x),\\
g_4'(x) &= 4\ln(\sigma_k)sF_kh(2x)\left(x\ln(\sigma_k) - \frac{sF_kh(x)}{cF_kh(x)}\right)\\
&= 4\ln(\sigma_k)sF_kh(2x)g_5(x),\\
g_5'(x) &= \ln(\sigma_k)\left(\frac{sF_kh(x)}{cF_kh(x)}\right)^2 > 0.
\end{aligned}$$

Hence, we can conclude that $g_5(x)$ is increasing on the interval $(0, +\infty)$. From $g_5(0) = g_4(0) = g_3(0) = g_2(0) = g_1(0) = 0$, we see that the functions $g_5(x), g_4(x), g_3(x), g_2(x)$ and $g_1(x)$ are increasing and positive on $(0, +\infty)$. Therefore, $g(x)$

is increasing on $(0, +\infty)$. Moreover, we use

$$\lim_{x\to 0^+} g(x) = \frac{8}{45}\ln(\sigma_k)^3.$$

Hence, we conclude from

$$\left(\frac{\frac{k^2+4}{4\ln(\sigma_k)^2}\left(\left(\frac{sF_kh(x)}{x}\right)^2+\frac{tF_kh(x)}{x}\right)-2}{x^3tF_kh(x)}\right) > \left(\frac{\frac{k^2+4}{4\ln(\sigma_k)^2}\left(\frac{sF_kh(x)}{x}\right)^2+\frac{tF_kh(x)}{x\ln(\sigma_k)}-2}{x^3tF_kh(x)}\right)$$

that

$$\left(\frac{sF_kh(x)}{x}\right)^2+\left(\frac{tF_kh(x)}{x}\right) > \frac{8\ln(\sigma_k)^2}{k^2+4}+\frac{32\ln(\sigma_k)^5}{45(k^2+4)}x^3tF_kh(x).$$

This proves the theorem.

□

5. Parameterized Wilker's Inequality for k-Fibonacci Hyperbolic Functions

Next theorem establishes parameterized Wilker's inequality for k−Fibonacci hyperbolic functions.

Theorem 5.1. *For the k−Fibonacci hyperbolic functions, the following inequality holds:*

$$\frac{\lambda}{\lambda+\mu}\left(\frac{sF_kh(x)}{x}\right)^p+\frac{\mu}{\lambda+\mu}\left(\frac{tF_kh(x)}{x}\right)^q > \left(\frac{2\ln(\sigma_k)}{\sqrt{k^2+4}}\right)^{\frac{p\lambda+q\mu}{\lambda+\mu}},$$

where $x \neq 0$, $\lambda > 0$, $\mu > 0$, $p \geq \frac{2q\mu}{\lambda}$ and $q > 0$.

Proof. From Lemma 3.1 and Theorem 4.1, we get

$$\begin{aligned}
&\frac{\lambda}{\lambda+\mu}\left(\frac{sF_kh(x)}{x}\right)^p+\frac{\mu}{\lambda+\mu}\left(\frac{tF_kh(x)}{x}\right)^q \geq \left(\frac{sF_kh(x)}{x}\right)^{\frac{p\lambda}{\lambda+\mu}}\left(\frac{tF_kh(x)}{x}\right)^{\frac{q\mu}{\lambda+\mu}}\\
&=\left(\frac{sF_kh(x)}{x}\right)^{\frac{p\lambda}{\lambda+\mu}}\left(\frac{sF_kh(x)}{x}\right)^{\frac{q\mu}{\lambda+\mu}}\left(\frac{1}{cF_kh(x)}\right)^{\frac{q\mu}{\lambda+\mu}} > \left(\frac{sF_kh(x)}{x}\right)^{\frac{p\lambda+q\mu}{\lambda+\mu}}\left(\frac{sF_kh(x)}{x}\right)^{\frac{-3q\mu}{\lambda+\mu}}\left(\frac{k^2+4}{4\ln(\sigma_k)^3}\right)^{\frac{-q\mu}{\lambda+\mu}}\\
&=\left(\frac{sF_kh(x)}{x}\right)^{\frac{p\lambda-2q\mu}{\lambda+\mu}}\left(\frac{k^2+4}{4\ln(\sigma_k)^3}\right)^{\frac{-q\mu}{\lambda+\mu}} > \left(\frac{2\ln(\sigma_k)}{\sqrt{k^2+4}}\right)^{\frac{p\lambda-2q\mu}{\lambda+\mu}}\left(\frac{4\ln(\sigma_k)^3}{k^2+4}\right)^{\frac{q\mu}{\lambda+\mu}} > \left(\frac{2\ln(\sigma_k)}{\sqrt{k^2+4}}\right)^{\frac{p\lambda+q\mu}{\lambda+\mu}}.
\end{aligned}$$

□

Now we give some applications of the Wilker−type inequalities for k−Fibonacci hyperbolic functions.

Corollary 5.2. *Let $x \neq 0$, $\lambda \geq \mu > 0$ and $(p, q) = (2, 1)$. Then,*

$$\frac{\lambda}{\lambda+\mu}\left(\frac{sF_kh(x)}{x}\right)^2+\frac{\mu}{\lambda+\mu}\left(\frac{tF_kh(x)}{x}\right) > \left(\frac{4\ln(\sigma_k)^2}{k^2+4}\right).$$

Corollary 5.3. *Let $x \neq 0$, $p \geq q > 0$ and $(\lambda, \mu) = (2, 1)$. Then,*

$$2\left(\frac{sF_kh(x)}{x}\right)^p+\left(\frac{tF_kh(x)}{x}\right)^q > 3\left(\frac{2\ln(\sigma_k)}{\sqrt{k^2+4}}\right)^p.$$

6. Results and Discussion

We formulate the Wilker−type inequality and Wilker−Anglesio type inequality for k−Fibonacci hyperbolic functions. In particular, if we get $k = 1$, our results reduce to the study in [1]. As a result, this study contributes to the literature by providing essential information for the extension of Wilker and Wilker−Anglesio type inequalities for the hyperbolic functions.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author have read and agreed to the published version of the manuscript.

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