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ON THE SPECTRUM OF THE UPPER TRIANGULAR DOUBLE BAND MATRIX $U(a_0, a_1, a_2; b_0, b_1, b_2)$ OVER THE SEQUENCE SPACE c

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Abstract. The upper triangular double band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ is defined on a Banach sequence space by

$$U(a_0, a_1, a_2; b_0, b_1, b_2)\,(x_n) = (a_n x_n + b_n x_{n+1})_{n=0}^{\infty}$$

where $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod 3)$. The class of the operator

$$U(a_0, a_1, a_2; b_0, b_1, b_2)$$

includes, in particular, the operator $U(r, s)$ when $a_k = r$ and $b_k = s$ for all $k \in \mathbb{N}$, with $r, s \in \mathbb{R}$ and $s \neq 0$. Also, it includes the upper difference operator; $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}$. In this paper, we completely determine the spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum, and the compression spectrum of the operator $U(a_0, a_1, a_2; b_0, b_1, b_2)$ over the sequence space c.

1. Introduction

Spectral theory is an important branch of mathematics. It also has many applications in physics. It is used, for example, to determine atomic energy levels in quantum mechanics. The resolvent set, which is the complement of the spectrum set of band matrices, can be used in such problems.

In this paper, we will calculate spectral decomposition of $U(a_0, a_1, a_2; b_0, b_1, b_2)$ matrix. $U(a_0, a_1, a_2; b_0, b_1, b_2)$ matrix is studied in c_0 sequence space by Durna and Kılıç [9] therefore some result is omit becuse it is similar with [9].

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$A : X \to Y$ be a bounded linear operator where X and Y are two Banach spaces. We will show the image set of A with set $R(A) = \{y \in Y : y = Ax,\ x \in X\}$. $B(X)$ is defined as in $B : X \to X$ all bounded, linear operators.

$A : D(A) \to X$ is a linear operator including $D(A) \subset X$, where $D(A)$ show the domain of A and X is a complex normed space. Let $A_\lambda := \lambda I - A$ for $A \in B(X)$ and $\lambda \in \mathbb{C}$ where I show the identity operator. A_λ^{-1} is defined as the resolvent operator of A.

The resolvent set of A consist from the set of complex numbers λ of A such that A_λ^{-1} exists, is continuous and, is defined on a set which is dense in X, signified by $\rho(A, X)$. The complement of $\rho(A, X)$ i.e. $\sigma(A, X) = \mathbb{C} \backslash \rho(A, X)$ is the spectrum of A.

Spectrum $\sigma(A, X)$ is the union of three sets which are disjoint, as follows: If A_λ^{-1} does not exist $\lambda \in \mathbb{C}$ belongs to the point spectrum. If A_λ^{-1} is defined on a dense subspace of X and is unbounded then $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(A, X)$ of A. If A_λ^{-1} exists, but its domain of definition is not dense in X then A_λ^{-1} may be bounded or unbounded. In this case $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(A, X)$.

$$\sigma(A, X) = \sigma_p(A, X) \cup \sigma_c(A, X) \cup \sigma_r(A, X) \tag{1}$$

is obtained by from above definitons and these sets are two by two discrete between them.

The all, bounded, convergent, null and bounded variation sequences are denoted by w, ℓ_∞, c, c_0 and bv, respectively. Moreover the spaces of all $p-$absolutely summable sequences and $p-$bounded variation sequences are denoted by ℓ_p, bv_p, respectively.

We notice that the dual space of c is norm isomorphic to the Banach space

$$\ell_1 = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k| < \infty \right\}.$$

Many Authors studied the spectrum and fine spectrum of linear operators on some sequence spaces. Some of the operators studied on the spectrum are as follows: The q-Cesáro matrices with $0 < q < 1$ on c_0 was studied by Yıldırım [19] in 2020, the difference operator over the sequence space bv_p by Akhmedov and Başar [1] in 2007 and forward difference operator on the Hahn space by Yeşilkayagil and Kirişci [16] in 2016.

2. Fine Spectrum

The upper triangular double band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ is defined on a Banach sequence space by

$$U(a_0, a_1, a_2; b_0, b_1, b_2)(x_n) = (a_n x_n + b_n x_{n+1})_{n=0}^{\infty}$$

where $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod 3)$. The class of the operator $U(a_0, a_1, a_2; b_0, b_1, b_2)$ includes, in particular, the operator $U(r, s)$ when $a_k = r$ and $b_k = s$ for all $k \in \mathbb{N}$, with $r, s \in \mathbb{R}$ and $s \neq 0$. Also, it includes the upper difference operator; $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}$. These operators have been studied in [14] and [11], respectively. $U(a_0, a_1, a_2; b_0, b_1, b_2)$ is an infinite matrix of form

$$U(a_0, a_1, a_2; b_0, b_1, b_2) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & b_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (b_0, b_1, b_2 \neq 0). \tag{2}$$

In this work, we will calculate spectral decomposition of above matrix.

Lemma 1 ([3], p.6)**.** *The matrix $B = (b_{nk})$ gives rise to a bounded linear operator $T \in (c; c)$ from c to itself if and only if*
(i) the rows of B are in ℓ_1 and their ℓ_1 norm are bounded,
(ii) the colums of B are in c,
(iii) the squence of row sums of B is in c.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1. *$U(a_0, a_1, a_2; b_0, b_1, b_2) : c \to c$ is a bounded linear operator and the norm is $\|U(a_0, a_1, a_2; b_0, b_1, b_2)\| = \max\{|a_0| + |b_0|, |a_1| + |b_1|, |a_2| + |b_2|\}$.*

Notation 1. *Throughout this study we will demonstrate as*

$$M = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\},$$

∂M is the boundary of the set M and $\mathring{M}$ is interior of the set M.

Theorem 1. *$\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \mathring{M}$.*

Proof. Proof is similar to proof of [9, Theorem 1]. □

Lemma 2 ([3], p.267)**.** *Let $T : c \to c$ be a bounded linear operator. If $T^* : \ell_1 \to \ell_1$, $T^* g = g \circ T$, $g \in c^* \cong \ell_1$, then T and T^* have matrix representations $B = (b_{nk})$ and B^* respectively. In here*

$$B^* = \begin{pmatrix} \bar{\chi} & v_0 - \bar{\chi} & v_1 - \bar{\chi} & v_2 - \bar{\chi} & \cdots \\ u_0 & b_{00} - u_0 & b_{10} - u_0 & b_{20} - u_0 & \cdots \\ u_1 & b_{01} - u_1 & b_{11} - u_1 & b_{21} - u_1 & \cdots \\ u_2 & b_{02} - u_2 & b_{12} - u_2 & b_{22} - u_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$u_n = \lim_{m\to\infty} b_{m,n} \quad , \quad v_n = \sum_{m=0}^{\infty} b_{n,m}$$

and

$$\bar{\chi} = \lim_{n\to\infty} v_n.$$

In this section, we will take $a_n + b_n = a_{n+1} + b_{n+1} = s$, herein $a_x = a_y$, $b_x = b_y$, $x \equiv y \,(\mathrm{mod}\, 3)$.

From Lemma 2 the adjoint of $U(a_0, a_1, a_2; b_0, b_1, b_2) : c \to c$ is the matrix

$$U(a_0, a_1, a_2; b_0, b_1, b_2)^* = \begin{pmatrix} s & 0 \\ 0 & U^t \end{pmatrix}$$

and $U(a_0, a_1, a_2; b_0, b_1, b_2) \in B\,(\ell_1)$.

Lemma 3 (Goldberg [13, p.59])**.** *T has a dense range $\Leftrightarrow$ T^* is 1-1.*

Lemma 4 (Goldberg [13, p.60])**.** *T has a bounded inverse $\Leftrightarrow$ T^* is onto.*

Theorem 2. *$\sigma_p(U\,(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \{s\}$.*

Proof. Let η be an eigenvalue of the operator $U\,(a_0, a_1, a_2; b_0, b_1, b_2)^*$. Then there exists $u \neq \theta = (0, 0, 0, ...)$ in ℓ_1 such that $U\,(a_0, a_1, a_2; b_0, b_1, b_2)^*\, u = \eta u$.
Then, we obtain

$$s u_0 = \eta u_0 \tag{3}$$

$$a_0 u_1 = \eta u_1 \tag{4}$$

$$b_0 u_1 + a_1 u_2 = \eta u_2 \tag{5}$$

$$b_1 u_2 + a_2 u_3 = \eta u_3 \tag{6}$$

$$b_2 u_3 + a_0 u_4 = \eta u_4 \tag{7}$$

$$\vdots$$

Then we have if $\eta = s$, then from (3) $u_0 \in \mathbb{C}$, from (4) and etc. $u_1 = u_2 = u_3 = \cdots = u_n = \cdots = 0$. If $\eta \neq s$, then from (3) $u_0 = 0$, from (4) $\eta = a_0$. Therefore from (7) $u_3 = 0$, from (6) $u_2 = 0$, from (5) $u_1 = 0$ and etc. So $u_0 = u_1 = u_2 = \cdots = u_n = \cdots = 0$. Hereby, $\sigma_p(U\,(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \{s\}$. □

Theorem 3. *$\sigma_r(U\,(a_0, a_1, a_2; b_0, b_1, b_2)\,, c) = \{s\}$.*

Proof. Owing to $\sigma_r(A, c) = \sigma_p(A^*, c^* \cong \ell_1) \backslash \sigma_p(A, c)$, required result is given by Theorems 1 and 2 □

Lemma 5.

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{3n+t} a_k b_{nk} \right) = \sum_{k=1}^{\infty} a_{3k+t} \left(\sum_{n=k}^{\infty} b_{n,3k+t} \right) \ , \ t = 0, 1, 2$$

herein (a_k) and (b_{nk}) are real numbers.

Proof.

$$\begin{aligned}
& \sum_{n=1}^{\infty}\left(\sum_{k=0}^{3n+t} a_k b_{nk}\right) \\
= & \sum_{k=0}^{3+t} a_k b_{1k} + \sum_{k=0}^{6+t} a_k b_{2k} + \sum_{k=0}^{9+t} a_k b_{3k} + \cdots + \sum_{k=0}^{3n+t} a_k b_{nk} + \cdots \\
= & a_0 b_{10} + a_1 b_{11} + a_2 b_{12} + a_3 b_{13} + a_4 b_{14} + a_5 b_{15} \\
& + a_0 b_{20} + a_1 b_{21} + a_2 b_{22} + a_3 b_{23} + a_4 b_{24} + a_5 b_{25} + a_6 b_{26} + a_7 b_{27} + a_8 b_{28} \\
& + a_0 b_{30} + a_1 b_{31} + a_2 b_{32} + a_3 b_{33} + a_4 b_{34} + a_5 b_{35} + a_6 b_{36} + a_7 b_{37} + a_8 b_{38} \\
& + a_9 b_{39} + a_{10} b_{3,10} + a_{11} b_{3,11} \\
& + \ldots \\
& + a_0 b_{n0} + a_1 b_{n1} + \cdots + a_{3n+2} b_{n,3n+2} \\
& + \cdots \\
= & a_0 \sum_{n=1}^{\infty} b_{n0} + a_1 \sum_{n=1}^{\infty} b_{n1} + a_2 \sum_{n=1}^{\infty} b_{n2} + a_{3+t} \sum_{n=1}^{\infty} b_{n,3+t} + a_{6+t} \sum_{n=2}^{\infty} b_{n,6+t} \\
& + \cdots + a_{3k+t} \sum_{n=k}^{\infty} b_{n,3k+t} \\
= & \sum_{k=0}^{\infty} a_{3k+t}\left(\sum_{n=k}^{\infty} b_{n,3k+t}\right)
\end{aligned}$$

□

Theorem 4.

$$\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \partial M \setminus \{s\} \quad \textit{and} \quad \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M.$$

Proof. Let $v = (v_n) \in \ell_1$ be such that $(U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I)^* u = v$ for some $u = (u_n)$. Then we get following system of linear equations:

$$\begin{gathered}
(s-\lambda)u_0 = v_0 \\
(a_0-\lambda)u_1 = v_1 \\
b_0 u_1 + (a_1-\lambda)u_2 = v_2 \\
\vdots
\end{gathered}$$

$$\left.\begin{aligned}
(s-\lambda)u_0 &= v_0 \\
(a_0-\lambda)u_1 &= v_1 \\
b_2 u_{3n} + (a_0-\lambda)u_{3n+1} &= v_{3n+1} \\
b_0 u_{3n+1} + (a_1-\lambda)u_{3n+2} &= v_{3n+2} \\
b_1 u_{3n+2} + (a_2-\lambda)u_{3n+3} &= v_{3n+3} \\
&\vdots
\end{aligned}\right. \quad , n \geq 0$$

Solving above equations, we have

$$u_0 = \frac{1}{s-\lambda}v_0$$
$$u_{3n+t} = \frac{1}{a_{t+2}-\lambda}\left[\sum_{k=1}^{3n+t}(-1)^{3n+t-k}v_k\prod_{v=0}^{3n+t-k-1}\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right],\quad t=0,1,2;\ n=1,2,\ldots$$

Herein $a_x=a_y$, $b_x=b_y$ for $x\equiv y(mod3)$ and we accept that $\prod_{v=0}^{-1}\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}=1.$ Therefore we get

$$\begin{aligned}\sum_{n=0}^{\infty}|u_n| &= |u_0|+|u_1|+|u_2|+|u_3|+\cdots\\
&= |u_0|+|u_1|+|u_2|+\sum_{n=1}^{\infty}|u_{3n+t}|\\
&= |u_0|+|u_1|+|u_2|\\
&\quad+\sum_{n=1}^{\infty}\left|\frac{1}{a_{t+2}-\lambda}\left[\sum_{k=1}^{3n+t}(-1)^{3n+t-k}v_k\prod_{\nu=0}^{3n+t-k-1}\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right]\right|\\
&\le \left|\frac{1}{s-\lambda}v_0\right|+\left|\frac{1}{a_0-\lambda}v_1\right|+\left|\frac{1}{a_1-\lambda}v_2-\frac{b_0}{(a_0-\lambda)(a_1-\lambda)}v_1\right|\\
&\quad+\frac{1}{|a_{t+2}-\lambda|}\sum_{n=1}^{\infty}\left[\sum_{k=1}^{3n+t}|v_k|\prod_{\nu=0}^{3n+t-k-1}\left|\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right|\right]\end{aligned}$$

Thus the inequality is gotten;

$$\left|\sum_{n=0}^{\infty}u_n\right|\le G+\max_{m=0}^{2}\frac{1}{|a_m-\lambda|}\sum_{n=1}^{\infty}\left[\sum_{k=1}^{3n+t}|v_k|\prod_{\nu=0}^{3n+t-k-1}\left|\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right|\right]\qquad(8)$$

where

$$G=\left|\frac{1}{s-\lambda}v_0\right|+\left|\frac{1}{a_0-\lambda}v_1\right|+\left|\frac{1}{a_1-\lambda}v_2-\frac{b_0}{(a_0-\lambda)(a_1-\lambda)}v_1\right|$$

Now, we consider the sum $\sum_{n=1}^{\infty}\left[\sum_{k=1}^{3n+t}|v_k|\prod_{\nu=0}^{3n+t-k-1}\left|\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right|\right]$. In Lemma 5 if we take $a_k=|v_k|$ and $b_{nk}=\prod_{\nu=0}^{3n+t-k-1}\left|\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right|$ then we have

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{3n+t}|v_k|\prod_{\nu=0}^{3n+t-k-1}\left|\frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda}\right|\right]$$

$$= \sum_{k=1}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} \prod_{\nu=0}^{3n+t-(3k+t)-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| \right]$$
$$= \sum_{k=1}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} \prod_{\nu=0}^{3n-3k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| \right].$$

Also since $\prod_{\nu=0}^{3n-3k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| = |d|^{n-k}, t = 0, 1, 2$ setting $d = \frac{b_2 b_1 b_0}{(a_2 - \lambda)(a_1 - \lambda)(a_0 - \lambda)}$ while $|d| < 1$, the last equation turns into the sum

$$\sum_{k=0}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} \prod_{\nu=0}^{3n-3k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| \right] = \sum_{k=0}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} |d|^{n-k} \right]$$
$$= \sum_{k=0}^{\infty} |v_{3k+t}| \left(\frac{1}{1 - |d|} \right)$$
$$\frac{1}{1 - |d|} \|v\|_{\ell_1}.$$

Then since $|d| < 1$ we get

$$\left| \sum_{n=0}^{\infty} u_n \right| \leq G + \max_{m=0}^{2} \frac{1}{|a_m - \lambda|} \frac{1}{1 - |d|} \|v\|_{\ell_1}.$$

So, we have $v = (v_n) \in \ell_1$, $u = (u_n) \in \ell_1$ if $|d| = \left| \frac{b_2 b_1 b_0}{(a_2 - \lambda)(a_1 - \lambda)(a_0 - \lambda)} \right| < 1$. Consequently, if for $\lambda \in \mathbb{C}$, $|a_2 - \lambda| |a_1 - \lambda| |a_0 - \lambda| > |b_2| |b_1| |b_0|$, then $(u_n) \in \ell_1$. Thus, the operator $(U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I)^*$ is onto if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| > |b_0| |b_1| |b_2|$. Then by Lemma 4, $U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I$ has a bounded inverse if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| > |b_0| |b_1| |b_2|$. Therefore,

$$\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}.$$

Owing to $\sigma(A, c)$ is the disjoint union of $\sigma_p(A, c)$, $\sigma_r(A, c)$ and $\sigma_c(A, c)$, thence

$$\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}.$$

By Theorem 1, we get

$$\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\} = \sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c)$$
$$\subset \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c).$$

Since, $\sigma(A, c)$ is closed

$$\overline{\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\}} \subset \overline{\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)}$$
$$= \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c),$$

and hence $\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\} \subset \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)$. Therefore, $\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M$ and so $\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M \setminus \left(\mathring{M} \cup \{s\}\right) = \partial M \setminus \{s\}$. □

3. Subdivision of the Spectrum

Subdivision of the spectrum; consists of three subsets of the spectrum that need not be discrete as follows:

The sequence $(x_n) \in X$ that satisfy the conditions of $\|x_n\| = 1$ and $\|Ax_n\| \to 0$ as $n \to \infty$ is called a Weyl sequence for A.

The set

$$\sigma_{ap}(A, X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - A\} \tag{9}$$

show the approximate point spectrum of A. The set

$$\sigma_{\delta}(A, X) := \{\lambda \in \sigma(A, X) : \lambda I - A \text{ is not surjective}\} \tag{10}$$

show defect spectrum of A. Finally, the set

$$\sigma_{co}(A, X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - A)} \neq X\} \tag{11}$$

show compression spectrum in the literature.

The below Proposition is extremely important for obtaining the subdivision of the spectrum of $U(a_0, a_1, a_2; b_0, b_1, b_2)$ in c.

Proposition 1 ([2], Proposition 1.3)**.** *The spectrum and subspectrum of an operator $A \in B(X)$ and its adjoint $A^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(A^*, X^*) = \sigma(A, X)$, *(b)* $\sigma_c(A^*, X^*) \subseteq \sigma_{ap}(A, X)$,
(c) $\sigma_{ap}(A^*, X^*) = \sigma_{\delta}(A, X)$, *(d)* $\sigma_{\delta}(A^*, X^*) = \sigma_{ap}(A, X)$,
(e) $\sigma_p(A^*, X^*) = \sigma_{co}(A, X)$, *(f)* $\sigma_{co}(A^*, X^*) \supseteq \sigma_p(A, X)$,
(g) $\sigma(A, X) = \sigma_{ap}(A, X) \cup \sigma_p(A^*, X^*) = \sigma_p(A, X) \cup \sigma_{ap}(A^*, X^*)$.

Goldberg's Classification of Spectrum

If $A \in B(X)$, then there are three cases for $R(A)$:

(I) $R(A) = X$, (II) $\overline{R(A)} = X$, but $R(A) \neq X$, (III) $\overline{R(A)} \neq X$
and three cases for A^{-1}:

(1) A^{-1} exists and bounded, (2) A^{-1} exists but boundless, (3) A^{-1} does not exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 (see [13]).

$\sigma(A, X)$ can be divided into subdivisions $I_2\sigma(A, X) = \emptyset$, $I_3\sigma(A, X)$, $II_2\sigma(A, X)$, $II_3\sigma(A, X)$, $III_1\sigma(A, X)$, $III_2\sigma(A, X)$, $III_3\sigma(A, X)$. For example, if $T = \lambda I - A$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(A, X)$.

By the definitions given above and introduction, we can write following Table 1.

TABLE 1. Subdivisions of the spectrum of a linear operator

		1	2	3
		A_λ^{-1} exists and is bounded	A_λ^{-1} exists and is unbounded	A_λ^{-1} does not exists
I	$R(\lambda I - A) = X$	$\lambda \in \rho(A, X)$	–	$\lambda \in \sigma_p(A, X)$ $\lambda \in \sigma_{ap}(A, X)$
II	$\overline{R(\lambda I - A)} = X$	$\lambda \in \rho(A, X)$	$\lambda \in \sigma_c(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$	$\lambda \in \sigma_p(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$
III	$\overline{R(\lambda I - A)} \neq X$	$\lambda \in \sigma_r(A, X)$ $\lambda \in \sigma_\delta(A, X)$ $\lambda \in \sigma_{co}(A, X)$	$\lambda \in \sigma_r(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$ $\lambda \in \sigma_{co}(A, X)$	$\lambda \in \sigma_p(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$ $\lambda \in \sigma_{co}(A, X)$

The articles mentioned in the Section 2, are related to the discretization of the spectrum defined by Goldberg. However, subdivision of the spectrum was examined on certain sequence space in [4], [6], [7]. Moreover, the spectrum and fine spectrum was calculated in [5], [8], [10] , [12], [15], [17], [18].

Theorem 5. *If* $|\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| < |b_0|\,|b_1|\,|b_2|$, *then*

$$\lambda \in I_3\sigma(U\,(a_0, a_1, a_2; b_0, b_1, b_2)\,, c).$$

Proof. Proof is similar to proof of [9, Theorem 5]. □

Corollary 2. $III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$, $III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \{s\}$.

Proof. If $|\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| > |b_0|\,|b_1|\,|b_2|$ then the operator $U\,(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I$ has a bounded inverse from proof of Theorem 3 and $\lambda = s$ does not satisfy the inequality $|\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| > |b_0|\,|b_1|\,|b_2|$. Owing to

$$\begin{aligned} \sigma_r(U(a_0, a_1, a_2; b_0, b_1, b_2), c) &= III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \\ &\cup III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \end{aligned}$$

from Table 1, we obtain $III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$,
$III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \{s\}$. □

Corollary 3. $II_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = III_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$.

Proof. Since

$$\begin{aligned}\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c) &= I_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)\\ &\cup II_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)\\ &\cup III_3\sigma\left(U(a_0, a_1, a_2; b_0, b_1, b_2), c\right)\end{aligned}$$

in Table 1, $\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = I_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)$ from Theorem 1 and Theorem 5. Thus
$II_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = III_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$. □

Theorem 6. *(a)* $\sigma_\delta(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \partial M$,
(b) $\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M$,
(c) $\sigma_{co}(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \{s\}$.

Proof. (a) From Table 1, we obtain

$$\sigma_\delta(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \sigma\left(U(a_0, a_1, a_2; b_0, b_1, b_2), c\right)\backslash I_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c).$$

So using Theorem 4 and 5 with $a_n + b_n = a_{n+1} + b_{n+1} = S$, the required result is gotten.

(b) From Table 1, we obtain

$$\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \sigma((U(a_0, a_1, a_2; b_0, b_1, b_2), c)\backslash III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c).$$

And so $\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M$ from Corollary 2.

(c) By Proposition 1 (e), we obtain

$$\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^*) = \sigma_{co}(U(a_0, a_1, a_2; b_0, b_1, b_2), c).$$

Using Theorem 2 with $a_n + b_n = a_{n+1} + b_{n+1}$, the required result is gotten. □

Corollary 4. *(a)* $\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \partial M$,
(b) $\sigma_\delta(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = M$.

Proof. By Proposition 1 (c) and (d), we obtain

$$\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \sigma_\delta(U(a_0, a_1, a_2; b_0, b_1, b_2), c)$$

and

$$\sigma_\delta(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c).$$

from Theorem 6 (a) and (b) with $a_n + b_n = a_{n+1} + b_{n+1} = S$, the required results are gotten. □

4. Results

We can generalize our operator

$$U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1})=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & b_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $b_0,b_1,\ldots,b_{n-1}\neq 0$.

In parallel with our study, the following results are valid for the n-entry upper triangular double band matrix above.

Theorem 7. *The following results are valid, where* $T=\left\{\lambda\in\mathbb{C}:\prod_{k=0}^{n-1}\left|\frac{\lambda-a_k}{b_k}\right|\leq 1\right\}$, $\mathring{T}$ *be the interior of the set* T *and* ∂T *be the boundary of the set* T *and for* $a_n+b_n=a_{n+1}+b_{n+1}=t$

(1) $\sigma_p(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=\mathring{T}$,
(2) $\sigma_p(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1})^*,c^*\cong\ell_1)=\{t\}$,
(3) $\sigma_r(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=\{t\}$,
(4) $\sigma_c(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=\partial T\setminus\{t\}$,
(5) $\sigma(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=T$.

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