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Multi-Parametric Families of Real and Non Singular Solutions of the Kadomtsev-Petviasvili I Equation

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Abstract

Multi-parametric solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of Fredholm determinants are constructed in function of exponentials. A representation of these solutions as a quotient of wronskians of order $2N$ in terms of trigonometric functions is deduced. All these solutions depend on $2N-1$ real parameters.
A third representation in terms of a quotient of two real polynomials depending on $2N-2$ real parameters is given; the numerator is a polynomial of degree $2N(N+1)-2$ in x, y and t and the denominator is a polynomial of degree $2N(N+1)$ in x, y and t. The maximum absolute value is equal to $2(2N+1)^2-2$.
We explicitly construct the expressions for the first third orders and we study the patterns of their absolute value in the plane (x,y) and their evolution according to time and parameters. It is relevant to emphasize that all these families of solutions are real and non singular.

1. Introduction

We consider the Kadomtsev-Petviashvili I equation (KPI)

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1.1}$$

where subscripts x, y and t denote partial derivatives.
This equation was introduced by Kadomtsev and Petviashvili [1] in 1970. It is considered as a model in hydrodynamic for surface and internal water waves [2] or in nonlinear optics [3]. Dryuma showed in 1974 how the KP equation could be written in Lax form [4].
Manakov, Zakharov, Bordag and Matveev first constructed rational solutions in 1977 [5] and two month later Krichever published other solutions [6].
In the frame of algebraic geometry, Krichever constructed for the first time in 1976 [7] the solutions to KPI equation in terms of Riemann theta functions and a little later, it was done by Dubrovin [8].
Others rational solutions of the KPI equation were obtained. For example, one can quote of the studies of Krichever in 1978 [9], Satsuma and Ablowitz in 1979 [10], Matveev in 1979 [11], Freeman and Nimmo in 1983 [12, 13], Matveev in 1987 [14], Pelinovsky and Stepanyants in 1993 [15], Pelinovsky in 1994 [16], Ablowitz, Villarroel, Chakravarty, Trubatch [17–19] in 1997-2000, Biondini and Kodama [20–22] in 2003-2007.

We give in the following three types of representations of the solutions to the KPI equation : first, in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters in function of exponentials, then in terms of wronskians of order $2N$ with $2N-1$ real parameters in function of some trigonometric functions.
In a third representation, real rational solutions of order N depending on $2N-2$ real parameters are constructed and they can be written as a ratio of two polynomials; the numerator is a polynomial in x, y and t of degree $2N(N+1)-2$ and the denominator a polynomial in x, y and t of degree $2N(N+1)$.
So we get rational real and non singular solutions to the KPI equation at each order N depending on $2N-2$ real parameters. We present explicit rational solutions and the representations of their absolute value in the plane of the coordinates (x,y) according to the $2N-2$ real parameters a_i and b_i $(1 \le i \le N-1)$ and time t for the first three orders.

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2. Families of solutions of order N depending on $2N-1$ real parameters in terms of Fredholm determinants to the KPI equation

We define the numbers λ_ν, κ_ν, δ_ν, γ_ν, $x_{r,\nu}$, e_ν depending on a real number ε by

$$\begin{array}{l} \lambda_j = 1-2\varepsilon^2 j^2, \lambda_{N+j} = -\lambda_j, \quad \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j\lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \\ x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, r = 1,3, \quad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2}, \\ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1}, \quad x_{r,N+j} = -x_{r,j},, \quad \tau_{N+j} = \tau_j \\ e_j = 2i\left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} - i\sum_{k=1}^{1/2M-1} b_K (je)^{2k+1}\right), \\ e_{N+j} = 2i\left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} + i\sum_{k=1}^{1/2M-1} b_k (je)^{2k+1}\right), \\ a_j, b_j \text{ real numbers }, \quad \varepsilon_j = 1, \quad \varepsilon_{N+j} = 0, \quad \varphi \text{ a real number} \\ 1 \le j \le N. \end{array} \tag{2.1}$$

Then we have the following statement:

Theorem 2.1. *Let $v(x,y,t)$ be the expression defined by*

$$v(x,y,t) = \frac{\det(I + D_3(x,y,t))}{\det(I + D_1(x,y,t))}, \tag{2.2}$$

with I the unit matrix and $D_r = (d_{jk}^{(r)})_{1\le j,k\le 2N}$ the matrix

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu}\prod_{\eta\neq\mu}\left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right)\exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{2.3}$$

Then the function defined by

$$u(x,y,t) = -2(|v(x+3t,y,t)|^2 - 1) \tag{2.4}$$

is a solution to the KPI equation (1.1) depending on $2N-1$ real parameters a_k, b_k, $1 \le k \le N-1$ and ε.

Proof. We have proven in [23] that the function w defined by (2.5)

$$w(x,y) = \frac{\det(I + D_3(x,y,0))}{\det(I + D_1(x,y,0))}\exp(2iy - i\varphi) \tag{2.5}$$

is a solution to the nonlinear Schrödinger equation (2.6)

$$iw_y + w_{xx} + 2|w|^2 w = 0. \tag{2.6}$$

It can then be similarly proven that the function v defined by

$$\tilde{v}(x,y,t) = v(x,y,t)\times\exp(2iy - i\varphi) = \frac{\det(I + D_3(x,y,t))}{\det(I + D_1(x,y,t))}\times\exp(2iy - i\varphi)$$

is a solution to the NLS equation (2.6) by considering t as a parameter. We can then deduce that the function u defined by (3)

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = -2(|\tilde{v}(x+3t,y,t)|^2 - 1)$$

is a solution to the KPI equation, which proves the result. $\square$

3. Families of solutions of order N depending on $2N-1$ real parameters in terms of wronskians to the KPI equation

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1},\ldots,\phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\mu\in[1,\ldots,2N]}]. \tag{3.1}$$

We consider the matrix $D_r = (d_{\nu\mu}^{(r)})_{\nu,\mu\in[1,\ldots,2N]}$ defined in (2.3).
We consider the real parameters a_k, b_k $1 \le k \le N-1$ and ε, and κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν defined in the previous section.
Then we have the following statement

Theorem 3.1. *Let $\Phi_{r,\nu}$ be the functions defined by*

$$\phi_{r,\nu}(x,y,t,w)=\sin\left(\frac{\kappa_\nu x}{2}+i\delta_\nu y-i\frac{x_{r,\nu}}{2}-i\frac{\tau_\nu}{2}t+\gamma_\nu w-i\frac{e_\nu}{2}\right),\quad 1\le\nu\le N,$$
$$\phi_{r,\nu}(x,y,t,w)=\cos\left(\frac{\kappa_\nu x}{2}+i\delta_\nu y-i\frac{x_{r,\nu}}{2}-i\frac{\tau_\nu}{2}t+\gamma_\nu w-i\frac{e_\nu}{2}\right),\quad N+1\le\nu\le 2N,\quad r=1,3,$$

Let v be the expression defined by

$$v(x,y,t)=\frac{W_3(\phi_{3,1},\dots,\phi_{3,2N})(x,y,t,0)}{W_1(\phi_{1,1},\dots,\phi_{1,2N})(x,y,t,0)} \tag{3.2}$$

Then the function defined by

$$u(x,y,t)=-2(|v(x+3t,y,t)|^2-1) \tag{3.3}$$

is a solution to the KPI equation (1.1) depending on $2N-1$ real parameters a_k, b_k, $1\le k\le N-1$ and ε.

Proof. We have proven in [24] that the function v defined by (3.4)

$$w(x,y)=\frac{W_3(\phi_{3,1},\dots,\phi_{3,2N})(x,y,0,0)}{W_1(\phi_{1,1},\dots,\phi_{1,2N})(x,y,0,0)}\exp(2iy-i\varphi) \tag{3.4}$$

is a solution to the nonlinear Schrödinger equation (3.5)

$$iw_y+w_{xx}+2|w|^2w=0. \tag{3.5}$$

We can similarly prove that the function v defined by

$$\tilde{v}(x,y,t)=v(x,y,t)\times\exp(2iy-i\varphi)$$

is a solution of the NLS equation by considering t as a parameter. We can then deduce that the function u defined by

$$u(x,t)=-2(|v(x+3t,y,t)|^2-1)=-2(|\tilde{v}(x+3t,y,t)|^2-1)$$

is a solution to the KPI equation which proves the result. □

4. Real and non singular rational solutions to the KPI equation of order N depending on $2N-2$ real parameters

We construct in this section rational solutions to the KPI equation as a quotient of two determinants.
We define functions of the following arguments:

$$X_\nu=\frac{\kappa_\nu x}{2}+i\delta_\nu y-i\frac{x_{3,\nu}}{2}-i\frac{\tau_\nu}{2}t-i\frac{e_\nu}{2}, \tag{4.1}$$
$$Y_\nu=\frac{\kappa_\nu x}{2}+i\delta_\nu y-i\frac{x_{1,\nu}}{2}-i\frac{\tau_\nu}{2}t-i\frac{e_\nu}{2}, \tag{4.2}$$

for $1\le\nu\le 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in the first section.
We consider the following functions:

$$\begin{array}{l}
\varphi_{4j+1,k}=\gamma_k^{4j-1}\sin X_k,\quad \varphi_{4j+2,k}=\gamma_k^{4j}\cos X_k,\\
\varphi_{4j+3,k}=-\gamma_k^{4j+1}\sin X_k,\quad \varphi_{4j+4,k}=-\gamma_k^{4j+2}\cos X_k,\\
\varphi_{4j+1,N+k}=\gamma_k^{2N-4j-2}\cos X_{N+k},\quad \varphi_{4j+2,N+k}=-\gamma_k^{2N-4j-3}\sin X_{N+k},\\
\varphi_{4j+3,N+k}=-\gamma_k^{2N-4j-4}\cos X_{N+k},\quad \varphi_{4j+4,N+k}=\gamma_k^{2N-4j-5}\sin X_{N+k},\\
\psi_{4j+1,k}=\gamma_k^{4j-1}\sin Y_k,\quad \psi_{4j+2,k}=\gamma_k^{4j}\cos Y_k,\\
\psi_{4j+3,k}=-\gamma_k^{4j+1}\sin Y_k,\quad \psi_{4j+4,k}=-\gamma_k^{4j+2}\cos Y_k,\\
\psi_{4j+1,N+k}=\gamma_k^{2N-4j-2}\cos Y_{N+k},\quad \psi_{4j+2,N+k}=-\gamma_k^{2N-4j-3}\sin Y_{N+k},\\
\psi_{4j+3,N+k}=-\gamma_k^{2N-4j-4}\cos Y_{N+k},\quad \psi_{4j+4,N+k}=\gamma_k^{2N-4j-5}\sin Y_{N+k},\\
1\le k\le N
\end{array} \tag{4.3}$$

Then we get the following result

Theorem 4.1. *Let v be the expression defined by*

$$v(x,y,t)=\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{4.4}$$

where

$$
\begin{aligned}
&n_{j1}=\varphi_{j,1}(x,y,t,0), \quad n_{jk}=\frac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&n_{jN+1}=\varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k}=\frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{j1}=\psi_{j,1}(x,y,t,0), \quad d_{jk}=\frac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{jN+1}=\psi_{j,N+1}(x,y,t,0), \quad d_{jN+k}=\frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&2\le k\le N, \quad 1\le j\le 2N,
\end{aligned}
\tag{4.5}
$$

the functions φ and ψ being defined in (4.3).
Then the function defined by

$$u(x,t)=-2(|v(x+3t,y,t)|^2-1) \tag{4.6}$$

is a solution to the KPI equation (1.1) depending on $2N-2$ parameters a_k, b_k, $1\le k\le N-1$.

Proof. It is still a consequence of our previous works. Precisely, we have proven in [25] that the function w defined by

$$w(x,y)=\frac{\det(n_{jk})_{j,k\in[1,2N]_{t=0}}}{\det(d_{jk})_{j,k\in[1,2N]_{t=0}}}\times\exp(2iy-i\varphi) \tag{4.7}$$

is a solution to the nonlinear Schrödinger equation (4.8)

$$iw_y+w_{xx}+2|w|^2w=0. \tag{4.8}$$

We can prove in the same way that the function v defined by

$$\tilde{v}(x,y,t)=v(x,y,t)\times\exp(2iy-i\varphi)$$

is a solution of the NLS equation by considering t as a parameter. Then, we can deduce that the function u defined by

$$u(x,t)=-2(|v(x+3t,y,t)|^2-1)=-2(|\tilde{v}(x+3t,y,t)|^2-1)$$

is a solution to the KPI equation which proves the result. □

5. The structure of the solutions to the KPI equation

The structure of the rational solutions to the KPI equation is given by the following result

Theorem 5.1. *Let u the function defined by*

$$u(x,t)=-2(|v(x+3t,y,t)|^2-1)=\frac{n(x,y,t)}{d(x,y,t)},$$

with

$$v(x,y,t)=\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}$$

$$
\begin{aligned}
&n_{j1}=\varphi_{j,1}(x,y,t,0), \quad n_{jk}=\frac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&n_{jN+1}=\varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k}=\frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{j1}=\psi_{j,1}(x,y,t,0), \quad d_{jk}=\frac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&d_{jN+1}=\psi_{j,N+1}(x,y,t,0), \quad d_{jN+k}=\frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
&2\le k\le N, \quad 1\le j\le 2N,
\end{aligned}
$$

Then the function v is a rational solution to the KPI equation (1.1) quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters a_j and b_j, $1\le j\le N-1$.
n is a polynomial of degree $2N(N+1)-2$ in x, y and t.
d is a polynomial of degree $2N(N+1)$ in x, y and t.

Proof. It is already proven in the previous section that this function is a solution to the KPI equation.
The proof of the structure of the solution is similar to this given in [26]. The difference in this present case is due to the reduction of the fraction which cancel the terms in $x^{2N(N+1)}$, $y^{2N(N+1)}$, $t^{2N(N+1)}$ in the numerator, these terms having the same maximal power in numerator and denominator, and the fact that the elevation by the power 2 makes that the succeeding terms in x, y and t are to the power $2N(N+1)-2$. □

Theorem 5.2. *Let u the function defined by*

$$u(x,t) = -2(|v(x+3t,y,t)|^2 - 1) = \frac{n(x,y,t)}{d(x,y,t)},$$

with

$$v(x,y,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}$$

$$\begin{array}{l}
n_{j1} = \varphi_{j,1}(x,y,t,0), \quad n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
n_{jN+1} = \varphi_{j,N+1}(x,y,t,0), \quad n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
d_{j1} = \psi_{j,1}(x,y,t,0), \quad d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
d_{jN+1} = \psi_{j,N+1}(x,y,t,0), \quad d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial\varepsilon^{2k-2}}(x,y,t,0),\\
2 \le k \le N, \quad 1 \le j \le 2N,
\end{array}$$

Then the function v_0 defined by

$$v_0(x,y,t) = v(x,y,t)_{(a_j=b_j=0,\,1\le j\le N-1)} \tag{5.1}$$

is the solution of order N solution to the KPI equation (1.1) whose highest amplitude in modulus is equal to $2(2N+1)^2-2$.

Proof. The proof of this result is similar to this given in [26]. We do not give more details. The reader can do by himself the rewriting of this proof. □

6. Explicit expressions and patterns of the rational solutions to the KPI equation in function of parameters and time

We have explicitly constructed rational solutions to the KPI equation of order N depending on $2N-2$ parameters for $1 \le N \le 3$.
In the following, we only give patterns of the modulus of the solutions in the plane (x,y) of coordinates in function of the parameters a_i, and b_i, for $1 \le i \le N-1$ for $2 \le N \le 3$, and time t.
We present the solutions using the following notations $X = 2x$, $Y = 4y$, $T = 2t$,

$$u_N(X,Y,T) = 1 - \frac{G_N(X,Y,T)}{Q_N(X,Y,T)}$$

with

$$\begin{array}{l}
G_N(X,Y,T) = \sum_{k=0}^{2N(N+1)} g_k(Y,T)X^k,\\
Q_N(X,Y,T) = \sum_{k=0}^{2N(N+1)} q_k(Y,T)X^k.
\end{array}$$

By construction, all these solutions constructed in this study are real. Moreover, we know from the study of the NLS equation that the solutions constructed by ourself were non singular. From the construction, the denominators of the solutions to the KPI equation being the square of those of the solutions of the NLS equation, we get the non singularity of all these families of solutions to the KPI equation.

6.1. Case $N = 1$

The polynomials Q_1 and G_1 are given by
$\mathbf{q}_4 = 1$, $\mathbf{q}_3 = -12T$, $\mathbf{q}_2 = 54T^2+2Y^2+2$, $\mathbf{q}_1 = -108T^3+(-12Y^2-12)T$, $\mathbf{q}_0 = 81T^4+Y^4+\left(18Y^2+18\right)T^2+2Y^2+1$
$\mathbf{g}_4 = 1$, $\mathbf{g}_3 = -12T$, $\mathbf{g}_2 = 54T^2+2Y^2-14$, $\mathbf{g}_1 = -108T^3+(-12Y^2+84)T$, $\mathbf{g}_0 = 81T^4+Y^4+\left(18Y^2-126\right)T^2+18Y^2+17$
This type of solution to the KPI equation is different from our previous works.
In our previous works [26–31], we constructed solution of order 1 to KPI equation and got
$\tilde{v}_1(X,Y,T)$
$= -2\,\frac{9-6X^2+72XT+X^4+1296T^4+216X^2T^2-216T^2+10Y^2+Y^4-24XTY^2-24X^3T+2X^2Y^2-864XT^3+72T^2Y^2}{(X^2-12XT+36T^2+Y^2+1)^2}.$
The solution of order 1 obtained in this paper can be rewritten as

$$v_1(X,Y,T) = 16\,\frac{-1+X^2-6XT+9T^2-Y^2}{X^2-6XT+9T^2+Y^2+1)^2} = 16\,\frac{-1+(X-3T)^2-Y^2}{(1+(X-3T)^2+Y^2)^2}.$$

It can be easily seen in this example that these two solutions are different and non singular. Moreover, we can verify that the maximum of the absolute value of v_1 is equal to $2(2N+1)^2-2 = 16$ obtained when $X = Y = T = 0$.

In the case $N = 1$, one obtains a peak which the height decreases very quickly as t increases.

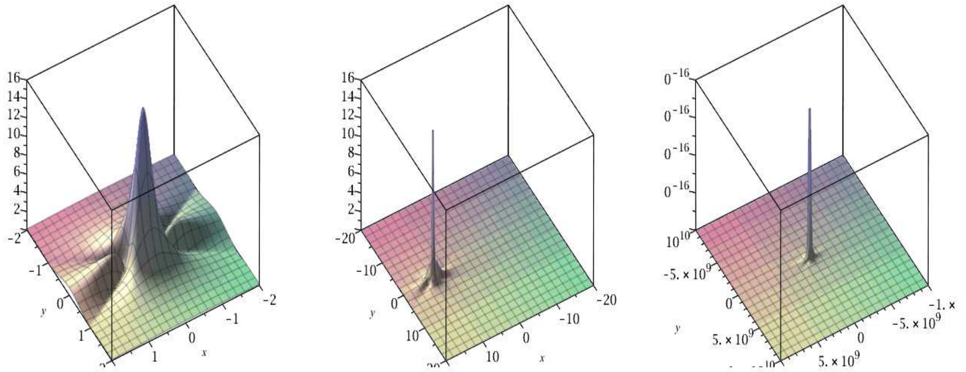

Figure 1. Solution of order 1 to the KPI equation, on the left for $t = 0$; in the center for $t = 4$; on the right for $t = 10^8$.

6.2. Case $N = 2$

In the case $N = 2$, the polynomials G_2 and G_2 more complex are given by

$\mathbf{q}_{12} =,1$ $\mathbf{f}_{11} = -36T$, $\mathbf{q}_{10} = 594T^2+6Y^2+6$, $\mathbf{q}_9 = -5940T^3+(-180Y^2+12)T-12a_1$, $\mathbf{q}_8 = 40095T^4+6Y^4+(2430Y^2-2754)T^2+324Ta_1-36Y^2+36Yb_1+(3Y^2+3)^2+54$, $\mathbf{q}_7 = -192456T^5+(-19440Y^2+42768)T^3-3888T^2a_1+36Y^2a_1+(-144Y^4+288Y^2-864Yb_1-1872+2(-36Y^2+60)(3Y^2+3))T+36a_1-12(3Y^2+3)a_1$, $\mathbf{q}_6 = 673596T^6+2Y^6+(102060Y^2-333396)T^4+27216T^3a_1+54Y^4-12Y^3b_1+(1512Y^4+3024Y^2+9072Yb_1+30312+2(162Y^2-702)(3Y^2+3)+(-36Y^2+60)^2)T^2+198Y^2-108Yb_1+54a_1^2+18b_1^2+(-756Y^2a_1-1332a_1-12(-36Y^2+60)a_1+108(3Y^2+3)a_1)T+2(3Y^4-18Y^2+18Yb_1+27)(3Y^2+3)+18$, $\mathbf{q}_5 = -1732104T^7+(-367416Y^2+1592136)T^5-122472T^4a_1+(-9072Y^4-54432Y^2-54432Yb_1-273456+2(162Y^2-702)(-36Y^2+60)+2(-324Y^2+2268)(3Y^2+3))T^3+(6804Y^2a_1+17172a_1-324(3Y^2+3)a_1-12(162Y^2-702)a_1+108(-36Y^2+60)a_1)T^2+(-36Y^6-972Y^4+216Y^3b_1-3564Y^2+1944Yb_1-972a_1^2-324b_1^2+2(-18Y^4-180Y^2-108Yb_1-450)(3Y^2+3)-324+2(3Y^4-18Y^2+18Yb_1+27)(-36Y^2+60))T-12(3Y^4-18Y^2+18Yb_1+27)a_1+2(18Y^2a_1+18a_1)(3Y^2+3)$, $\mathbf{q}_4 = 3247695T^8+(918540Y^2-4960116)T^6+367416T^5a_1+(34020Y^4+340200Y^2+204120Yb_1+1472580+(162Y^2-702)^2+2(243Y^2-2349)(3Y^2+3)+2(-324Y^2+2268)(-36Y^2+60))T^4+(-34020Y^2a_1-12(-324Y^2+2268)a_1-111780a_1-324(-36Y^2+60)a_1+108(162Y^2-702)a_1+324(3Y^2+3)a_1)T^3+(270Y^6+7290Y^4-1620Y^3b_1+26730Y^2-14580Yb_1+7290a_1^2+2430b_1^2+2430+2(27Y^4+702Y^2+162Yb_1+3411)(3Y^2+3)+2(-18Y^4-180Y^2-108Yb_1-450)(-36Y^2+60)+2(3Y^4-18Y^2+18Yb_1+27)(162Y^2-702))T^2+(-12(-18Y^4-180Y^2-108Yb_1-450)a_1+108(3Y^4-18Y^2+18Yb_1+27)a_1+2(-54Y^2a_1-342a_1)(3Y^2+3)+2(18Y^2a_1+18a_1)(-36Y^2+60))T-12(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(3Y^2+3)+(3Y^4-18Y^2+18Yb_1+27)^2$, $\mathbf{q}_3 = -4330260T^9+(-1574640Y^2+10182672)T^7-734832T^6a_1+(-81648Y^4-1143072Y^2-489888Yb_1-4856112+2(243Y^2-2349)(-36Y^2+60)+2(-324Y^2+2268)(162Y^2-702))T^5+(102060Y^2a_1+413100a_1-324(162Y^2-702)a_1+108(-324Y^2+2268)a_1-12(243Y^2-2349)a_1+324(-36Y^2+60)a_1)T^4+(-1080Y^6-29160Y^4+6480Y^3b_1-106920Y^2+58320Yb_1-29160a_1^2-9720b_1^2-9720+2(27Y^4+702Y^2+162Yb_1+3411)(-36Y^2+60)+2(-18Y^4-180Y^2-108Yb_1-450)(162Y^2-702)+2(-324Y^2+2268)(3Y^4-18Y^2+18Yb_1+27))T^3+(-12(27Y^4+702Y^2+162Yb_1+3411)a_1+108(-18Y^4-180Y^2-108Yb_1-450)a_1-324(3Y^4-18Y^2+18Yb_1+27)a_1+2(-54Y^2a_1-342a_1)(-36Y^2+60)+2(18Y^2a_1+18a_1)(162Y^2-702))T^2+(-12(-54Y^2a_1-342a_1)a_1+108(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(-36Y^2+60)+2(-18Y^4-180Y^2-108Yb_1-450)(3Y^4-18Y^2+18Yb_1+27))T-12(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)a_1+2(18Y^2a_1+18a_1)(3Y^4-18Y^2+18Yb_1+27)$, $\mathbf{q}_2 = 3897234T^{10}+(1771470Y^2-13345074)T^8+944784T^7a_1+(122472Y^4+2204496Y^2+734832Yb_1+(-324Y^2+2268)^2+9640296+2(243Y^2-2349)(162Y^2-702))T^6+(-183708Y^2a_1-883548a_1+324(162Y^2-702)a_1+108(243Y^2-2349)a_1-324(-324Y^2+2268)a_1)T^5+(2430Y^6+65610Y^4-14580Y^3b_1+240570Y^2-131220Yb_1+65610a_1^2+21870b_1^2+21870+2(27Y^4+702Y^2+162Yb_1+3411)(162Y^2-702)+2(243Y^2-2349)(3Y^4-18Y^2+18Yb_1+27)+2(-18Y^4-180Y^2-108Yb_1-450)(-324Y^2+2268))T^4+(324(3Y^4-18Y^2+18Yb_1+27)a_1-324(-18Y^4-180Y^2-108Yb_1-450)a_1+108(27Y^4+702Y^2+162Yb_1+3411)a_1+2(-54Y^2a_1-342a_1)(162Y^2-702)+2(18Y^2a_1+18a_1)(-324Y^2+2268))T^3+(-324(18Y^2a_1+18a_1)a_1+108(-54Y^2a_1-342a_1)a_1+(-18Y^4-180Y^2-108Yb_1-450)^2+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(162Y^2-702)+2(27Y^4+702Y^2+162Yb_1+3411)(3Y^4-18Y^2+18Yb_1+27))T^2+(108(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)a_1+2(-54Y^2a_1-342a_1)(3Y^4-18Y^2+18Yb_1+27)+2(18Y^2a_1+18a_1)(-18Y^4-180Y^2-108Yb_1-450))T+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(3Y^4-18Y^2+18Yb_1+27)+(18Y^2a_1+18a_1)^2$, $\mathbf{q}_1 = -2125764T^{11}+(-1180980Y^2+10156428)T^9-708588T^8a_1+(-104976Y^4-2309472Y^2-629856Yb_1+2(243Y^2-2349)(-324Y^2+2268)-10602576)T^7+(183708Y^2a_1+324(-324Y^2+2268)a_1-324(243Y^2-2349)a_1+1023516a_1)T^6+(-2916Y^6-78732Y^4+17496Y^3b_1-288684Y^2+157464Yb_1-78732a_1^2-26244b_1^2+2(243Y^2-2349)(-18Y^4-180Y^2-108Yb_1-450)+2(27Y^4+702Y^2+162Yb_1+3411)(-324Y^2+2268)-26244)T^5+(-324(27Y^4+702Y^2+162Yb_1+3411)a_1+324(-18Y^4-180Y^2-108Yb_1-450)a_1+2(-54Y^2a_1-342a_1)(-324Y^2+2268)+2(243Y^2-2349)(18Y^2a_1+18a_1))T^4+(-324(-54Y^2a_1-342a_1)a_1+324(18Y^2a_1+18a_1)a_1+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(-324Y^2+2268)+2(27Y^4+702Y^2+162Yb_1+3411)(-18Y^4-180Y^2-108Yb_1-450))T^3+(-324(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)a_1+2(-54Y^2a_1-342a_1)(-18Y^4-180Y^2-108Yb_1-450)+2(27Y^4+702Y^2+162Yb_1+3411)(18Y^2a_1+18a_1))T^2+(2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(-18Y^4-180Y^2-108Yb_1-450)+2(-54Y^2a_1-342a_1)(18Y^2a_1+18a_1))T+2(Y^6+27Y^4-6Y^3b_1+99Y^2-54Yb_1+9a_1^2+9b_1^2+9)(18Y^2a_1+18a_1)$, $\mathbf{q}_0 = 531441T^{12}+Y^{12}+(354294Y^2-3424842)T^{10}+236196T^9a_1+54Y^{10}-12Y^9b_1+(98415Y^4-118098Y^2+236196Yb_1+10491039)T^8-1259712T^7a_1+927Y^8-432Y^7b_1+18Y^6a_1^2+54Y^6b_1^2+(14580Y^6+253692Y^4+69984Y^3b_1-1495908Y^2-839808Yb_1+39366a_1^2+13122b_1^2-16011756)T^6+5364Y^6-4104Y^5b_1+486Y^4a_1^2+1134Y^4b_1^2-108Y^3a_1^2b_1-108Y^3b_1^3+(-17496Y^4a_1+314928Y^2a_1+52488Ya_1b_1+2711880a_1)T^5+(1215Y^8+46332Y^6+5832Y^5b_1+598266Y^4+229392Y^3b_1-13122Y^2a_1^2+30618Y^2b_1^2+4328316Y^2+1358856Yb_1-153090a_1^2-42282b_1^2+11592639)T^4+10287Y^4-10800Y^3b_1+1782Y^2a_1^2+4698Y^2b_1^2-972Ya_1^2b_1-972Yb_1^3+81a_1^4+162a_1^2b_1^2+81b_1^4+(-2592Y^6a_1-85536Y^4a_1-19440Y^3a_1b_1-816480Y^2a_1-128304Ya_1b_1+$

$2916a_1{}^3+2916a_1b_1{}^2-2330208a_1)T^3+(54Y^{10}+2862Y^8+50076Y^6-2592Y^5b_1+3402Y^4a_1{}^2-1458Y^4b_1{}^2+323676Y^4-84672Y^3b_1+49572Y^2a_1{}^2-4860Y^2b_1{}^2+2916Ya_1{}^2b_1+2916Yb_1{}^3+688014Y^2-365472Yb_1+178362a_1{}^2+61398b_1{}^2+61398)T^2+1782Y^2-972Yb_1+162a_1{}^2+162b_1{}^2+(-108Y^8a_1-3600Y^6a_1+648Y^5a_1b_1-29160Y^4a_1+9936Y^3a_1b_1-972Y^2a_1{}^3-972Y^2a_1b_1{}^2-68688Y^2a_1+36936Ya_1b_1-6156a_1{}^3-6156a_1b_1{}^2-6156a_1)T+81$

$\mathbf{g}_{12}=1,\quad \mathbf{g}_{12}=1,\quad \mathbf{g}_{11}=-36T,\quad \mathbf{g}_{10}=594T^2+6Y^2-42,\quad \mathbf{g}_9=-5940T^3+(-180Y^2+1452)T-12a_1,\quad \mathbf{g}_8=40095T^4+15Y^4+(2430Y^2-22194)T^2+324Ta_1-162Y^2+36Yb_1-81,\quad \mathbf{g}_7=-192456T^5+(-19440Y^2+198288)T^3-3888T^2a_1+(-360Y^4+3888Y^2-864Yb_1+1944)T,\quad \mathbf{g}_6=673596T^6+20Y^6+(102060Y^2-1149876)T^4+27216T^3a_1-132Y^4+96Y^3b_1+(3780Y^4-40824Y^2+9072Yb_1-6588)T^2-1728Ta_1-1476Y^2+1152Yb_1+54a_1{}^2+18b_1{}^2+1620,\quad \mathbf{g}_5=-1732104T^7+(-367416Y^2+4531464)T^5-122472T^4a_1+72Y^4a_1+(-22680Y^4+244944Y^2-54432Yb_1-126360)T^3+31104T^2a_1+3888Y^2a_1-216Ya_1b_1+(-360Y^6+1224Y^4-1728Y^3b_1-35640Y^2-17280Yb_1-972a_1{}^2-324b_1{}^2-81000)T+3240a_1,\ quad\mathbf{g}_4=3247695T^8+15Y^8+(918540Y^2-12308436)T^6+367416T^5a_1+156Y^6+72Y^5b_1+(85050Y^4-918540Y^2+204120Yb_1+1406970)T^4-233280T^3a_1+8442Y^4-3312Y^3b_1-162Y^2a_1{}^2+378Y^2b_1{}^2+(2700Y^6-540Y^4+12960Y^3b_1+692388Y^2+103680Yb_1+7290a_1{}^2+2430b_1{}^2+1286604)T^2+23004Y^2-13176Yb_1+1134a_1{}^2+2214b_1{}^2+(-1080Y^4a_1-53136Y^2a_1+3240Ya_1b_1-84888a_1)T+1647,\quad \mathbf{g}_3=-4330260T^9+(-1574640Y^2+22779792)T^7-734832T^6a_1+96Y^6a_1+(-204120Y^4+2204496Y^2-489888Yb_1-6362712)T^5+933120T^4a_1+1440Y^4a_1+720Y^3a_1b_1+(-10800Y^6-32400Y^4-51840Y^3b_1-4304016Y^2-311040Yb_1-29160a_1{}^2-9720b_1{}^2-8581680)T^3-864Y^2a_1+4752Ya_1b_1-108a_1{}^3-108a_1b_1{}^2+(6480Y^4a_1+287712Y^2a_1-19440Ya_1b_1+644112a_1)T^2+(-180Y^8-3408Y^6-864Y^5b_1-124344Y^4+28224Y^3b_1+1944Y^2a_1{}^2-4536Y^2b_1{}^2-262224Y^2+82080Yb_1-8424a_1{}^2-24840b_1{}^2-144180)T+7776a_1,\quad \mathbf{g}_2=3897234T^{10}+6Y^{10}+(1771470Y^2-27516834)T^8+944784T^7a_1+270Y^8+(306180Y^4-3306744Y^2+734832Yb_1+15142788)T^6-2099520T^5a_1+9468Y^6-2592Y^5b_1+378Y^4a_1{}^2-162Y^4b_1{}^2+(24300Y^6+150660Y^4+116640Y^3b_1+12763332Y^2+466560Yb_1+65610a_1{}^2+21870b_1{}^2+27660204)T^4+26460Y^4-1728Y^3b_1+2916Y^2a_1{}^2-4860Y^2b_1{}^2+324Ya_1{}^2b_1+324Yb_1{}^3+(-19440Y^4a_1-769824Y^2a_1+58320Ya_1b_1-2087856a_1)T^3+(810Y^8+22248Y^6+3888Y^5b_1+759996Y^4-75168Y^3b_1-8748Y^2a_1{}^2+20412Y^2b_1{}^2+1864296Y^2+55728Yb_1+14580a_1{}^2+104004b_1{}^2+2079594)T^2+51678Y^2-33696Yb_1+3402a_1{}^2+3078b_1{}^2+(-864Y^6a_1-25056Y^4a_1-6480Y^3a_1b_1-85536Y^2a_1-53136Ya_1b_1+972a_1{}^3+972a_1b_1{}^2-178848a_1)T-8586,\quad \mathbf{g}_1=-2125764T^{11}+(-1180980Y^2+19604268)T^9-708588T^8a_1+36Y^8a_1+(-262440Y^4+2834352Y^2-629856Yb_1-18738216)T^7+2519424T^6a_1-144Y^6a_1-216Y^5a_1b_1+(-29160Y^6-274104Y^4-139968Y^3b_1-18563256Y^2-279936Yb_1-78732a_1{}^2-26244b_1{}^2-42766056)T^5+1080Y^4a_1-5616Y^3a_1b_1+324Y^2a_1{}^3+324Y^2a_1b_1{}^2+(29160Y^4a_1+1014768Y^2a_1-87480Ya_1b_1+2991816a_1)T^4+(-1620Y^8-58320Y^6-7776Y^5b_1-2114424Y^4+46656Y^3b_1+17496Y^2a_1{}^2-40824Y^2b_1{}^2-7917264Y^2-1127520Yb_1+17496a_1{}^2-192456b_1{}^2+527148)T^3-27216Y^2a_1+8424Ya_1b_1-2268a_1{}^3-2268a_1b_1{}^2+(2592Y^6a_1+111456Y^4a_1+19440Y^3a_1b_1+785376Y^2a_1+190512Ya_1b_1-2916a_1{}^3-2916a_1b_1{}^2-878688a_1)T^2+(-36Y^{10}-2196Y^8-54504Y^6+19008Y^5b_1-2268Y^4a_1{}^2+972Y^4b_1{}^2-176040Y^4+100224Y^3b_1-33048Y^2a_1{}^2+23976Y^2b_1{}^2-1944Ya_1{}^2b_1-1944Yb_1{}^3+125388Y^2+67392Yb_1+88452a_1{}^2+17820b_1{}^2+212220)T-10044a_1,\quad \mathbf{g}_0=,531441T^{12}+Y^{12}+(354294Y^2-6259194)T^{10}+236196T^9a_1+102Y^{10}-12Y^9b_1+(98415Y^4-1062882Y^2+236196Yb_1+9546255)T^8-1259712T^7a_1+1935Y^8-432Y^7b_1+18Y^6a_1{}^2+54Y^6b_1{}^2+(14580Y^6+183708Y^4+69984Y^3b_1+10681308Y^2+39366a_1{}^2+13122b_1{}^2+25348788)T^6+2772Y^6+2808Y^5b_1-1674Y^4a_1{}^2-162Y^4b_1{}^2-108Y^3a_1{}^2b_1-108Y^3b_1{}^3+(-17496Y^4a_1-524880Y^2a_1+52488Ya_1b_1-1487160a_1)T^5+(1215Y^8+54108Y^6+5832Y^5b_1+2176794Y^4+42768Y^3b_1-13122Y^2a_1{}^2+30618Y^2b_1{}^2+12189852Y^2+1732104Yb_1-48114a_1{}^2+132678b_1{}^2-11229921)T^4-6129Y^4+30672Y^3b_1-5994Y^2a_1{}^2-13446Y^2b_1{}^2+1620Ya_1{}^2b_1+1620Yb_1{}^3+81a_1{}^4+162a_1{}^2b_1{}^2+81b_1{}^4+(-2592Y^6a_1-147744Y^4a_1-19440Y^3a_1b_1-1562976Y^2a_1-221616Ya_1b_1+2916a_1{}^3+2916a_1b_1{}^2+2708640a_1)T^3+(54Y^{10}+4158Y^8+82908Y^6-33696Y^5b_1+3402Y^4a_1{}^2-1458Y^4b_1{}^2-138564Y^4-312768Y^3b_1+72900Y^2a_1{}^2-28188Y^2b_1{}^2+2916Ya_1{}^2b_1+2916Yb_1{}^3-2375730Y^2+515808Yb_1-171558a_1{}^2-39690b_1{}^2-186138)T^2+41958Y^2-21708Yb_1+1458a_1{}^2+4050b_1{}^2+(-108Y^8a_1-144Y^6a_1+648Y^5a_1b_1+50328Y^4a_1+20304Y^3a_1b_1-972Y^2a_1{}^3-972Y^2a_1b_1{}^2+273456Y^2a_1-77112Ya_1b_1+1620a_1{}^3+1620a_1b_1{}^2-16524a_1)T+3969$

For $N=2$, the formation of three peaks is obtained when the parameters a_1 or b_1 are not equal to 0.

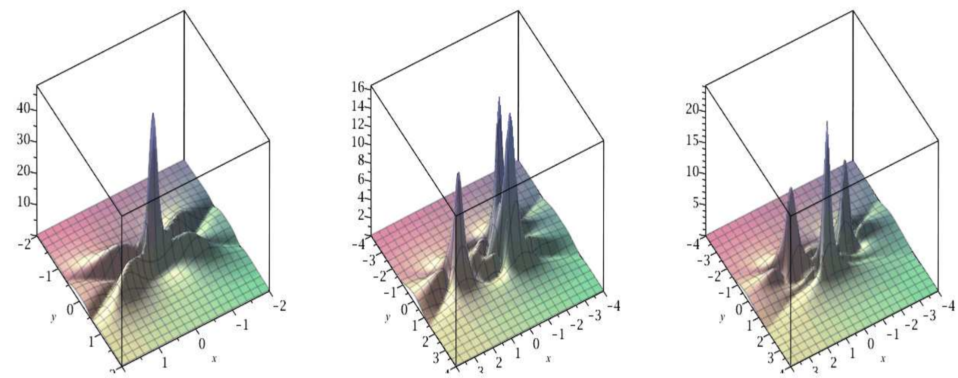

Figure 2. Solution of order 2 to the KPI equation for $t=0$, on the left $a_1=0$, $b_1=0$,; in the center $a_1=10$, $b_1=0$,; on the right $a_1=10$, $b_1=10$.

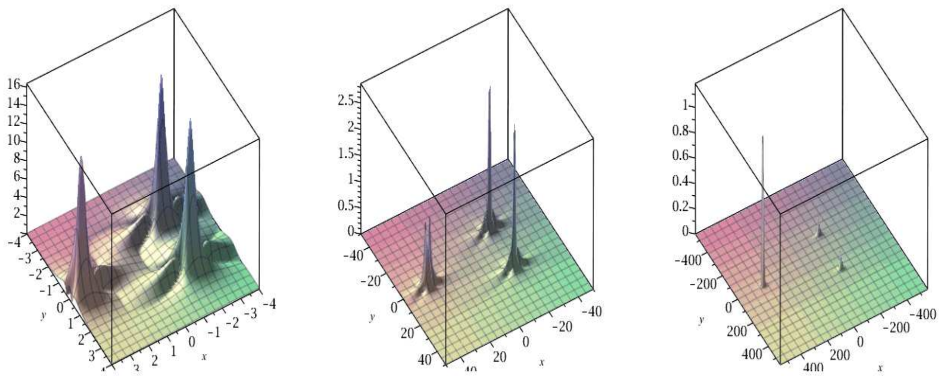

Figure 3. Solution of order 2 to the KPI equation for $t = 0$, on the left $a_1 = 10^2$, $b_1 = 0$; in the center $a_1 = 10^4$, $b_1 = 0$; on the right for $t = 10$, $a_1 = 10^8$, $b_1 = 0$.

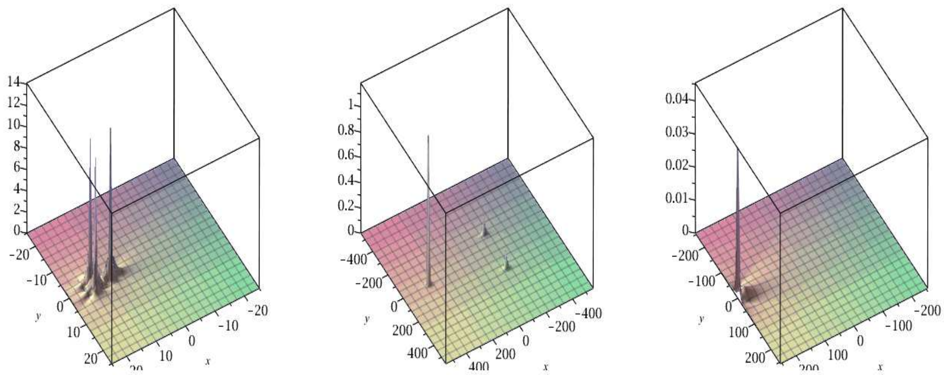

Figure 4. Solution of order 2 to the KPI equation, on the left for $t = 5$, $a_1 = 0$, $b_1 = 0$; in the center for $t = 10$, $a_1 = 0$, $b_1 = 0$; on the right for $t = 100$, $a_1 = 0$, $b_1 = 0$.

6.3. Case $N = 3$

In this case, polynomials G_3 and Q_3 depending on 4 parameters being too complex, we cannot give their explicit expressions. Even without parameters, due to the length of the solution, the explicit expression cannot be given here.

In the case $N = 3$, for $a_1 \neq 0$ or $b_1 \neq 0$ and the other parameters equal to zero, we obtain a triangle with 6 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, and other parameters equal to zero, we obtain a concentric rings of 5 peaks with a peak in the center.

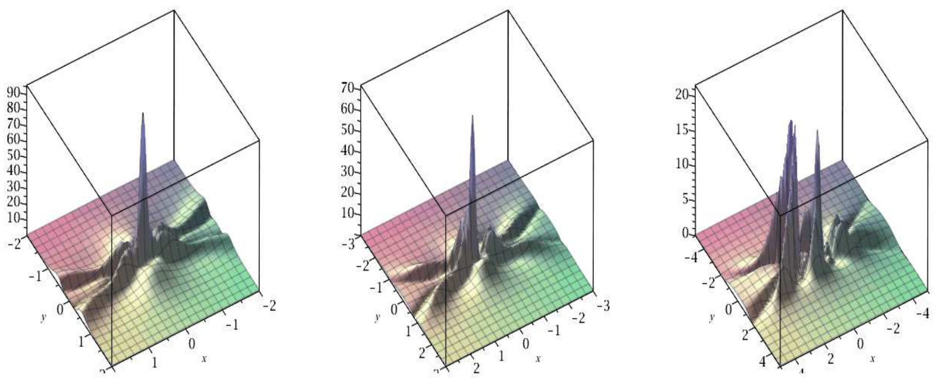

Figure 5. Solution of order 3 to the KPI equation, on the left for $t = 0$; in the center for $t = 0,01$; on the right for $t = 0,1$; all the parameters are equal to 0.

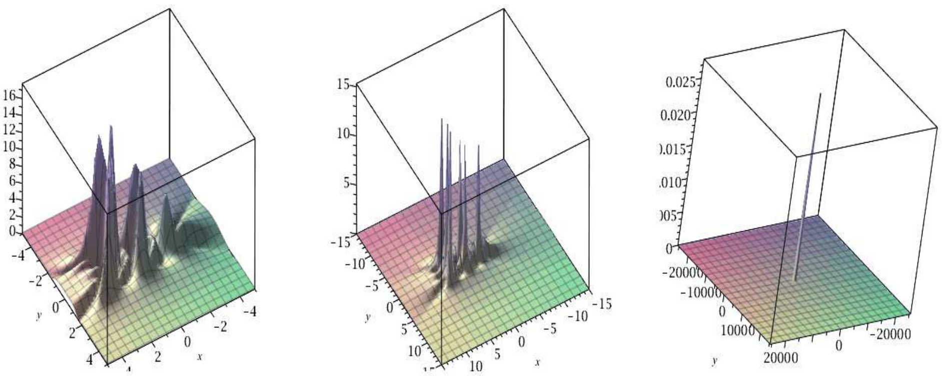

Figure 6. Solution of order 3 to the KPI equation, on the left for $t = 0,2$; in the center for $t = 10^0$; on the right for $t = 10^1$; all the parameters are equal to 0.

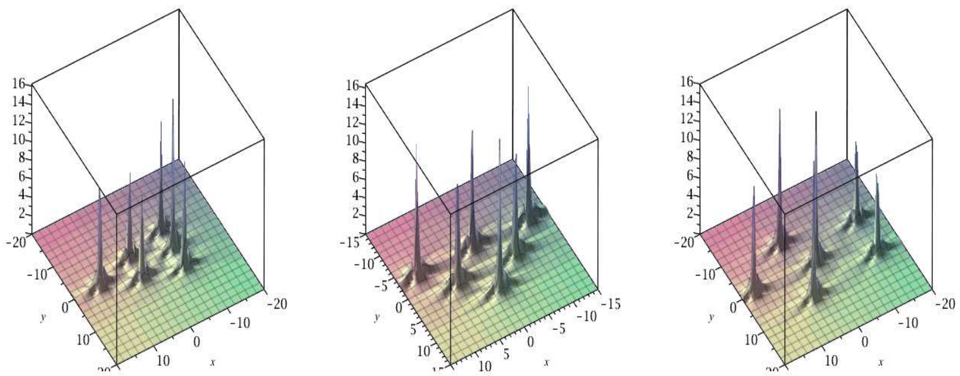

Figure 7. Solution of order 3 to the KPI equation, on the left for $a_1 = 10^3$; in the center for $b_1 = 10^3$; on the right for $a_2 = 10^6$; here $t = 0$.

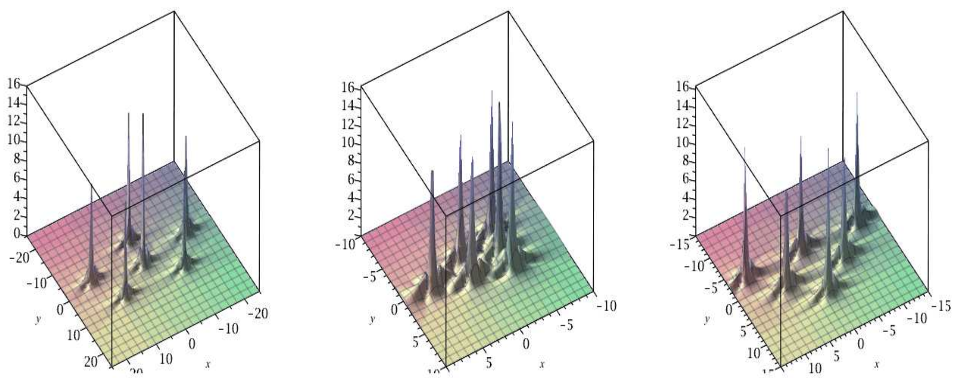

Figure 8. Solution of order 3 to the KPI equation, on the left for $t = 0$, $b_2 = 10^6$; in the center for $t = 0,01$, $a_1 = 10^3$ all the other parameters are equal to 0; on the right for $t = 0,1$, $b_1 = 10^3$ all the parameters are equal to 0.

7. Conclusion

We have given three representations of the solutions to the KPI equation: in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters in function of exponentials; in terms of wronskians of order $2N$ depending on $2N-1$ real parameters by means of trigonometric functions; in terms of real rational solutions as a quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ of degrees $2N(N+1)-2$ in x, y, t and $2N(N+1)$ in x, y, t respectively and depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.
The maximum of the modulus of those solutions is equal to $2(2N+1)^2 - 2$. That gives a new approach to find explicit solutions for higher orders and try to describe the structure of those rational solutions.
In the (x,y) plane of coordinates, different structures appear.
All the solutions described in this study are different from those constructed in previous works [26–31] .
It will be relevant to go on this study for higher orders to try to understand the structure of those rational solutions.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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