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Research Article

Geometric properties of normalized Rabotnov function

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Abstract

In the present paper, our aim is to study geometric properties of normalized Rabotnov functions. For this purpose, we determined sufficient conditions for univalency, close-to-convexity, convexity and starlikeness of the normalized Rabotnov functions in the open unit disk.

Mathematics Subject Classification (2020). 33E12, 33E20 30C45, 30C55

Keywords. Rabotnov function, univalent, starlike, convex and close-to-convex

1. Introduction

In 1948, Yu. N. Rabotnov, who worked in solid mechanics included plasticity, creep theory, hereditary mechanics, failure machanics, nonelastic stability, composites and shell theory, introduced a special function applied in viscoelasticity [12]. This function, known today as the Rabotnov fractional exponential function or briefly Rabotnov function, is defined as follows

$$R_{\alpha,\beta}(z) = z^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k}{\Gamma\left((k+1)(1+\alpha)\right)} z^{k(1+\alpha)}.$$

The convergence of this series at any values of the argument is evident. Noting that for $\alpha = 0$ it reduces to the standard exponential $\exp(\beta z)$. Rabotnov function is the particular case of the familiar Mittag-Leffler function widely used in fractional calculus. The relation between the Rabotnov function and Mittag-Leffler function can be written as follows

$$R_{\alpha,\beta}(z) = z^{\alpha} E_{1+\alpha,1+\alpha}(\beta z^{1+\alpha}),$$

where E is Mittag-Leffler function and $\alpha, \beta, z \in \mathbb{C}$.

Our aim in this study is to determine geometric properties of Rabotnov function. For this we need the following well-known definitions of geometric function theory.

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Let $\mathcal{A}$ denotes the class of functions f which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Thus, each function $f \in \mathcal{A}$, has the following series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ (z \in \mathbb{U}). \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is called starlike (with respect to the origin), denoted by $f \in \mathcal{S}^*$, if f is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a starlike domain with respect to the origin. The analytic characterization of $\mathcal{S}^*$ is

$$\mathcal{S}^* = \left\{ f : \ f \in \mathcal{A}, \ Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{U} \right\}.$$

A function $f \in \mathcal{A}$ that maps $\mathbb{U}$ onto a convex domain is called convex function. We denote by $\mathcal{C}$ the class of all functions $f \in \mathcal{A}$ that are convex. The analytic characterization of $\mathcal{C}$ is

$$\mathcal{C} = \left\{ f : \ f \in \mathcal{A}, \ Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in \mathbb{U} \right\}.$$

A function $f \in \mathcal{A}$ is called close-to-convex, if the range $f(\mathbb{U})$ is close-to-convex, i.e. the complement of $f(\mathbb{U})$ can be written as the union of nonintersecting half-lines. We denote by $\mathcal{K}$ all close-to-convex functions. The class $\mathcal{K}$ can be analytically characterized as follows:

$$\mathcal{K} = \left\{ f : \ f \in \mathcal{A}, \ Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \ z \in \mathbb{U}, \ g \in \mathcal{C} \right\}.$$

Every convex function is close-to-convex. More generally, every starlike function is close-to-convex. Furthermore the Noshiro-Warschawski Theorem implies that, every close-to-convex function is univalent in $\mathbb{U}$. These remarks can be given by the following chain of proper inclusions: $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. For more details we refer to [4, 6].

The starlikeness, convexity, close-to-convexity and some other geometric properties of special functions such as Bessel, Struve, Wright, Mittag-Leffler etc. have been studied by many mathematicians recently (see for example [1–3, 10, 11, 13, 14]). However, there are no studies in the literature on the geometric properties of the Rabotnov function.

Throughout this paper, we shall restrict our attention to the case of real-valued $\alpha \geq 0$, $\beta > 0$ and $z \in \mathbb{U}$. It is clear that the Rabotnov function $R_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Rabotnov functions:

$$\begin{aligned} \mathbb{R}_{\alpha,\beta}(z) &= z^{1/(1+\alpha)}\Gamma(1+\alpha)R_{\alpha,\beta}(z^{1/(1+\alpha)}) \\ &= z + \sum_{k=2}^{\infty} \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^k. \end{aligned} \tag{1.2}$$

In order to present our results we need the following Lemmas.

Lemma 1.1. ([9]) *Let f define by (1.1) and suppose that*

$$1 \geq 2a_2 \geq \cdots \geq ka_k \geq \cdots \geq 0$$

or

$$1 \leq 2a_2 \leq \cdots \leq ka_k \leq \cdots \leq 2.$$

Then f is regular and univalent in $\mathbb{U}$.

Following the proof of Ozaki it can be proved that if a function f satisfies the conditions given in Lemma 1.1, then f is close-to-convex with respect to the convex function $-log(1-z)$.

Lemma 1.2. ([5]) *If* $a_k \geq 0$, $\{ka_k\}$ *and* $\{ka_k - (k+1)a_{k+1}\}$ *both are non-increasing, i.e.,* $\{ka_k\}$ *is monotone of order* 2*, then* f *defined by (1.1) is in* $\mathcal{S}^*$.

Lemma 1.3. ([9]) *Let* f *define by (1.1) and suppose that one of the four conditions*

$$1 \geq 3a_3 \geq 5a_5 \geq \cdots \geq (2k+1)a_{2k+1} \geq \cdots \geq 2a_2 \geq 4a_4 \geq \cdots \geq 2ka_{2k} \geq \cdots \geq 0$$

$$1 \leq 3a_3 \leq 5a_5 \leq \cdots \leq (2k+1)a_{2k+1} \leq \cdots \leq 2a_2 \leq 4a_4 \leq \cdots \leq 2ka_{2k} \leq \cdots \leq 2$$

$$1 \geq 3a_3 \geq 5a_5 \geq \cdots \geq (2k+1)a_{2k+1} \geq \cdots \geq 2ka_{2k} \geq \cdots \geq 4a_4 \geq 2a_2 \geq 0$$

$$1 \leq 3a_3 \leq 5a_5 \leq \cdots \leq (2k+1)a_{2k+1} \leq \cdots \leq 2ka_{2k} \leq \cdots \leq 4a_4 \leq 2a_2 \leq 2$$

is verified. Then f *is regular and univalent in* $\mathbb{U}$.

From the Lemma 1.3, we can easily write that, if f is an odd function (i.e., a_{2k} in (1.1) is zero for each $k \geq 1$) such that

$$1 \geq 3a_3 \geq \cdots \geq (2k+1)a_{2k+1} \geq \cdots \geq 0, \tag{1.3}$$

or

$$1 \leq 3a_3 \leq \cdots \leq (2k+1)a_{2k+1} \leq \cdots \leq 2 \tag{1.4}$$

then the function f is univalent in $\mathbb{U}$.

We can verify directly that if an odd function f satisfies (1.3) or (1.4), then f is close-to-convex with respect to the convex function $2^{-1}log(\frac{1+z}{1-z})$.

Lemma 1.4. ([7]) *If the function* $f \in \mathcal{A}$*, satisfy* $|(f(z)/z) - 1| < 1$ *for each* $z \in \mathbb{U}$*, then* f *is univalent and starlike in* $\mathbb{U}_{1/2} = \{z : |z| < 1/2\}$.

Lemma 1.5. ([8]) *If the function* $f \in \mathcal{A}$*, satisfy* $|f'(z) - 1| < 1$ *for each* $z \in \mathbb{U}$*, then* f *is convex in* $\mathbb{U}_{1/2}$.

2. Main results

Theorem 2.1. *Let* $\alpha \geq 0$ *and* $\beta > 0$*. If* $\alpha \geq 2\beta - 1$*, then normalized Rabotnov function* $\mathbb{R}_{\alpha,\beta}(z)$ *is close-to-convex with respect to* $-log(1-z)$ *and hence univalent in* $\mathbb{U}$.

Proof. The function $\mathbb{R}_{\alpha,\beta}(z)$ defined by (1.2) can be rewritten as

$$\mathbb{R}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

where

$$a_k = \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}, \quad \text{for} \quad k \geq 2 \quad \text{and} \quad a_1 = 1. \tag{2.1}$$

We note that under the stated conditions $a_k \geq 0$ for all $k \geq 1$ and $2a_2 = \frac{2\beta\Gamma(1+\alpha)}{\Gamma(2(1+\alpha))} \leq 1$. We use Lemma 1.1 to prove that $\mathbb{R}_{\alpha,\beta}(z)$ is close-to-convex with respect to $-log(1-z)$. Therefore, we need to show that $\{ka_k\}$ is a decreasing sequence. For $\alpha \geq 0$, we can write

$$\begin{aligned} ka_k-(k+1)a_{k+1} &= \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}-\frac{(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)(k+1))} \\ &\geq \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}-\frac{(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} \\ &= \frac{k^2(1+\alpha)\beta^{k-1}\Gamma(1+\alpha)}{(\Gamma(1+\alpha)k+1)}-\frac{(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} \\ &= \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)}X(k) \end{aligned}$$

where $X(k)=k^2(1+\alpha)-(k+1)\beta$. Using the fact that $k^2\geq 2k-1$, for all $k\geq 1$ we obtain

$$\begin{aligned} X(k) &= k^2(1+\alpha)-(k+1)\beta \\ &\geq (2\alpha-\beta+2)k-\alpha-\beta-1 \\ &\geq 1+\alpha-2\beta\geq 0, \end{aligned}$$

under the hypotheses of the theorem. Thus, $\{ka_k\}$ is a decreasing sequence. This completes the proof of the theorem. □

Example 2.2. The function $\mathbb{R}_{0,\frac{1}{2}}(z)=z+\sum_{k=2}^{\infty}\frac{\left(\frac{1}{2}\right)^{k-1}}{\Gamma(k)}z^k$ is close-to-convex with respect to $-log(1-z)$ and hence univalent in $\mathbb{U}$.

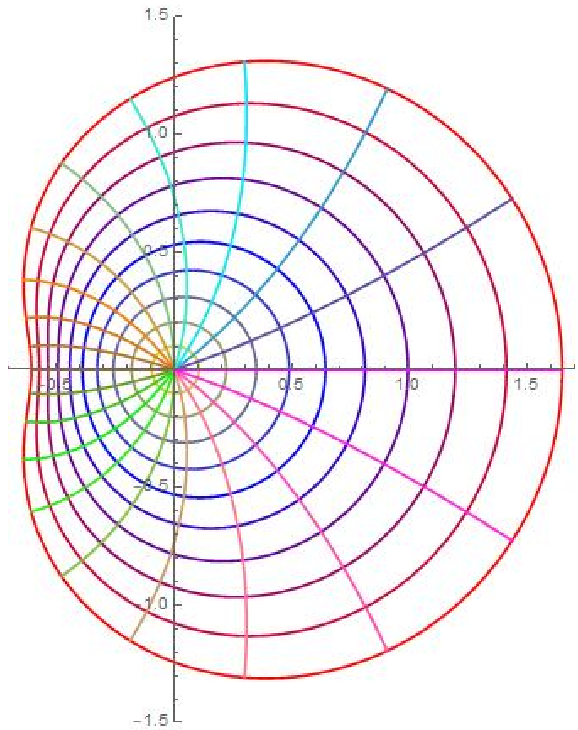

Figure 1. Mapping of $\mathbb{R}_{0,\frac{1}{2}}(z)$ over $\mathbb{U}$

Theorem 2.3. *Let $\alpha\geq 0$ and $\beta>0$. If $\alpha\geq 4\beta-1$, then normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is starlike in $\mathbb{U}$.*

Proof. We will use Lemma 1.2 in the proof of Theorem. By the proof of Theorem 2.1, the condition $\alpha \geq 4\beta - 1$ implies that the sequence $\{ka_k\}$ is non-increasing. We need to show that the sequence $\{ka_k - (k+1)a_{k+1}\}$ is also non-increasing. For this, we define $b_k = ka_k - (k+1)a_{k+1}$. Using (2.1), we find that

$$\begin{aligned}
b_k - b_{k+1} &= ka_k - 2(k+1)a_{k+1} + (k+2)a_{k+2} \\
&= \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} - \frac{2(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)(k+1))} + \frac{(k+2)\beta^{k+1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)(k+2))} \\
&\geq \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} - \frac{2(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)(k+1))} \\
&\geq \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} - \frac{2(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} \\
&= \frac{k^2(1+\alpha)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} - \frac{2(k+1)\beta^k\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} \\
&= \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)}Y(k)
\end{aligned}$$

where $Y(k) = k^2(1+\alpha) - 2(k+1)\beta$. We want to show that $Y(k)$ is non-negative for all $k \geq 1$. Using the fact that $k^2 \geq 2k-1$, for all $k \geq 1$ we obtain

$$Y(k) \geq 2(\alpha - \beta + 1)k - (\alpha + 2\beta + 1).$$

By hypotheses $2(\alpha - \beta + 1)$ is non-negative and

$$Y(k) \geq Y(1) = \alpha - 4\beta + 1 \geq 0.$$

This observation shows that the sequence b_k, namely the sequence $\{ka_k - (k+1)a_{k+1}\}$ is non-increasing. This proves the theorem. □

Example 2.4. If we take $\beta = 1/4$ and $\alpha = 1/2$ in Theorem 2.3, then

$$\mathbb{R}_{\frac{1}{2},\frac{1}{4}}(z) = z + \sum_{k=2}^{\infty} \frac{(\frac{1}{4})^{k-1}\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}k)} z^k$$

is starlike in $\mathbb{U}$.

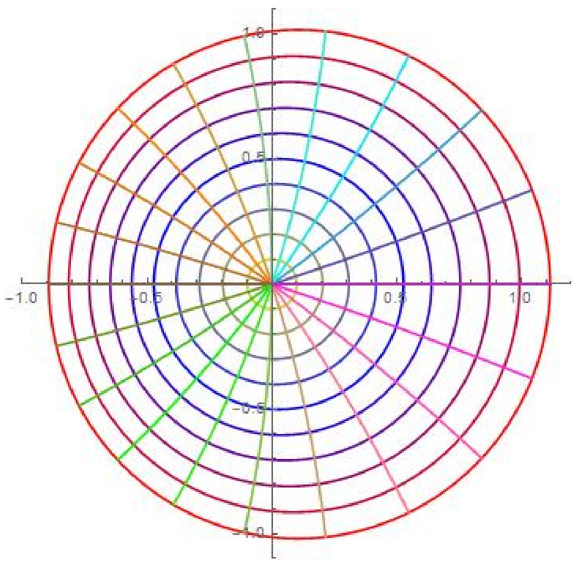

Figure 2. Mapping of $\mathbb{R}_{\frac{1}{2},\frac{1}{4}}(z)$ over $\mathbb{U}$

The following lemma allows us to prove our next theorem.

Lemma 2.5. *If $k \in \mathbb{N}$ and $\alpha \geq 0$, then*

$$(1+\alpha)^{k-1}(k-1)!\Gamma(1+\alpha) \leq \Gamma\left((1+\alpha)k\right).$$

Proof. We will prove by induction that for all integers $k \in \mathbb{N} = \{1, 2, ...\}$. The case $k = 1$ is trivial. Now we assume that the inequality holds for $k = n$. Hence by the induction hypothesis we get

$$\begin{aligned}(1+\alpha)^n n!\Gamma(1+\alpha) &= (1+\alpha)n(1+\alpha)^{n-1}(n-1)!\Gamma(1+\alpha) \\ &\leq (1+\alpha)n\Gamma((1+\alpha)n) \\ &= \Gamma((1+\alpha)n+1) \\ &\leq \Gamma((1+\alpha)(n+1)).\end{aligned}$$

This completes the proof. □

From Lemma 2.5, for $k \in \mathbb{N}$ and $\alpha \geq 0$ we can write

$$\frac{\Gamma(1+\alpha)}{\Gamma\left((1+\alpha)k\right)} \leq \frac{1}{(1+\alpha)^{k-1}(k-1)!}. \tag{2.2}$$

Theorem 2.6. *Let $\alpha \geq 0$ and $\beta > 0$. If $\alpha > \frac{\beta}{W(2e)-1} - 1$, where W is the Lambert W function, then normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is starlike in $\mathbb{U}$.*

Proof. Let $p(z)$ be the function defined by

$$p(z) = \frac{z\mathbb{R}'_{\alpha,\beta}(z)}{\mathbb{R}_{\alpha,\beta}(z)}, \ \ (z \in \mathbb{U}).$$

Since

$$\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} \neq 0, \ \ (z \in \mathbb{U}),$$

the function p is analytic in $\mathbb{U}$ and $p(0) = 1$. To prove our theorem, we need to show that $Re\,(p(z)) > 0$, $z \in \mathbb{U}$. It is easy to show that, if $|p(z) - 1| < 1$, $z \in \mathbb{U}$, then $Re\,(p(z)) > 0$. For $z \in \mathbb{U}$, using (1.2) and (2.2) we obtain

$$\begin{aligned}\left|\mathbb{R}'_{\alpha,\beta}(z) - \frac{\mathbb{R}_{\alpha,\beta}(z)}{z}\right| &= \left|\sum_{k=2}^{\infty}\frac{(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}z^{k-1}\right| \\ &< \sum_{k=2}^{\infty}\frac{(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\ &\leq \sum_{k=2}^{\infty}\frac{(k-1)\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\ &= \frac{\beta}{(1+\alpha)}e^{\frac{\beta}{1+\alpha}}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\left|\frac{\mathbb{R}_{\alpha,\beta}(z)}{z}\right| &= \left|1 + \sum_{k=2}^{\infty}\frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}z^{k-1}\right| \\ &> 1 - \sum_{k=2}^{\infty}\frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\ &\geq 1 - \sum_{k=2}^{\infty}\frac{\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\ &= 2 - e^{\frac{\beta}{1+\alpha}}.\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we get

$$\begin{aligned}|p(z)-1| &= \left|\frac{z\mathbb{R}'_{\alpha,\beta}(z)}{\mathbb{R}_{\alpha,\beta}(z)}-1\right| \\ &= \left|\frac{\mathbb{R}'_{\alpha,\beta}(z)-\frac{\mathbb{R}_{\alpha,\beta}(z)}{z}}{\frac{\mathbb{R}_{\alpha,\beta}(z)}{z}}\right| \\ &< \frac{\frac{\beta}{1+\alpha}e^{\frac{\beta}{1+\alpha}}}{2-e^{\frac{\beta}{1+\alpha}}}.\end{aligned}$$

Thus $\mathbb{R}_{\alpha,\beta}(z)\in\mathcal{S}^*$ if $\frac{\beta}{1+\alpha}e^{\frac{\beta}{1+\alpha}}<2-e^{\frac{\beta}{1+\alpha}}$ or equivalently $\alpha>\frac{\beta}{W(2e)-1}-1$, where W is the Lambert W function. This completes the proof of the theorem. □

Example 2.7. If we take $\beta=1$, then from Theorem 2.6 it should be $\alpha>\frac{1}{W(2e)-1}-1\approx 1,67$. Thus the function $\mathbb{R}_{2,1}(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(3)}{\Gamma(3k)}z^k$ is starlike in $\mathbb{U}$.

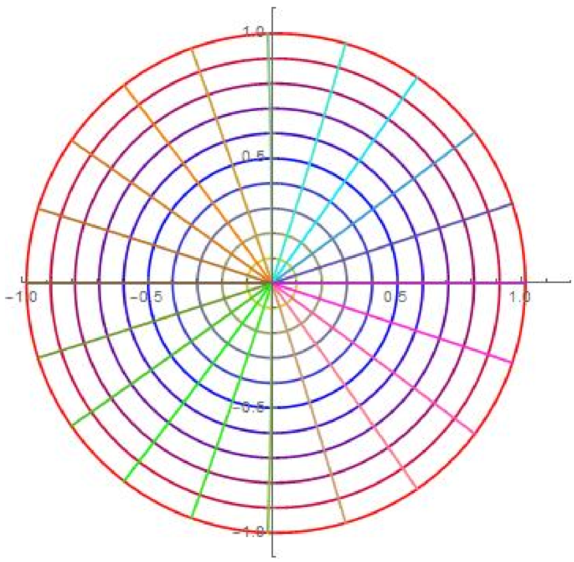

Figure 3. Mapping of $\mathbb{R}_{2,1}(z)$ over $\mathbb{U}$

Theorem 2.8. *Let $\alpha\geq 0$ and $\beta>0$. If $\frac{\beta}{1+\alpha}<0.199496$ then normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is convex in $\mathbb{U}$.*

Proof. Let $p(z)$ be the function defined by

$$p(z)=1+\frac{z\mathbb{R}''_{\alpha,\beta}(z)}{\mathbb{R}'_{\alpha,\beta}(z)},\quad (z\in\mathbb{U}).$$

Then $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. To prove $\mathbb{R}_{\alpha,\beta}(z)$ is convex in $\mathbb{U}$, we need to show that $|p(z)-1|<1$, $z\in\mathbb{U}$. For $z\in\mathbb{U}$, using (1.2) and (2.2), we get

$$\begin{aligned}\left|z\mathbb{R}''_{\alpha,\beta}(z)\right| &= \left|\sum_{k=2}^{\infty}\frac{k(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}z^{k-1}\right| \\ &< \sum_{k=2}^{\infty}\frac{k(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\ &\leq \sum_{k=2}^{\infty}\frac{k(k-1)\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\ &= \frac{\beta(2\alpha+\beta+2)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)^2},\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\left|\mathbb{R}'_{\alpha,\beta}(z)\right| &= \left|1+\sum_{k=2}^{\infty}\frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}z^{k-1}\right| \\ &> 1-\sum_{k=2}^{\infty}\frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\ &\geq 1-\sum_{k=2}^{\infty}\frac{k\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\ &= 2-\frac{(\alpha+\beta+1)e^{\frac{\beta}{1+\alpha}}}{1+\alpha}.\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get

$$\left|\frac{z\mathbb{R}''_{\alpha,\beta}(z)}{\mathbb{R}'_{\alpha,\beta}(z)}\right| < \frac{\frac{\beta(2\alpha+\beta+2)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)^2}}{2-\frac{(\alpha+\beta+1)e^{\frac{\beta}{1+\alpha}}}{1+\alpha}}.$$

Thus $\mathbb{R}_{\alpha,\beta}(z) \in \mathcal{C}$ if

$$\frac{\beta}{1+\alpha}\left(\frac{\left(2+\frac{\beta}{1+\alpha}\right)e^{\frac{\beta}{1+\alpha}}}{2-\left(1+\frac{\beta}{1+\alpha}\right)e^{\frac{\beta}{1+\alpha}}}\right) < 1$$

or equivalently $\frac{\beta}{1+\alpha} < 0.199496$. This completes the proof of the theorem. □

Example 2.9. The function $\mathbb{R}_{1,\frac{1}{3}}(z) = z + \sum_{k=2}^{\infty}\frac{1}{3^{k-1}\Gamma(2k)}z^k$ is convex in $\mathbb{U}$.

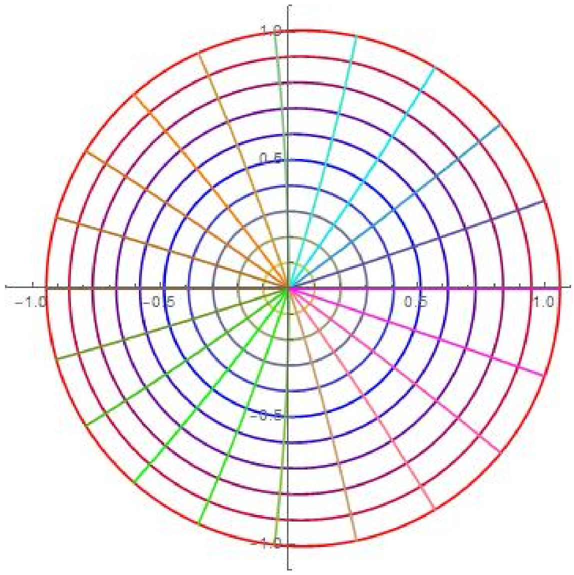

Figure 4. Mapping of $\mathbb{R}_{1,\frac{1}{3}}(z)$ over $\mathbb{U}$

Theorem 2.10. *Let $\alpha \geq 0$ and $\beta > 0$. If $\alpha \geq 3\beta - 1$, then $\mathbb{R}_{\alpha,\beta}(z^2)/z$ is close-to-convex with respect to convex function $2^{-1}log(\frac{1+z}{1-z})$.*

Proof. It is easy to see that

$$\frac{\mathbb{R}_{\alpha,\beta}(z^2)}{z} = z + \sum_{k=2}^{\infty} a_{2k-1} z^{2k-1}$$

where

$$a_{2k-1} = \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)}, \quad \text{for} \quad k \geq 2 \quad \text{and} \quad a_1 = 1.$$

We note that under the stated conditions $a_{2k-1} \geq 0$ for all $k \geq 2$ and $3a_3 = \frac{3\beta\Gamma(1+\alpha)}{\Gamma(2(1+\alpha))} \leq 1$. In view of Lemma 1.3 we have to prove that $\{(2k-1)a_{2k-1}\}_{k\geq 2}$ is a decreasing sequence. Basic computations gives

$$\begin{aligned}(2k-1)a_{2k-1} - (2k+1)a_{2k+1} &= \frac{(2k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} - \frac{(2k+1)\beta^{k}\Gamma(1+\alpha)}{\Gamma((1+\alpha)(k+1))} \\ &\geq \frac{(2k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} - \frac{(2k+1)\beta^{k}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} \\ &= \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k+1)} T(k)\end{aligned}$$

where $T(k) = 2(1+\alpha)k^2 - (1+\alpha+2\beta)k - \beta$. Using the fact that $k^2 \geq 2k-1$, for all $k \geq 1$ we obtain

$$T(k) \geq (3\alpha - 2\beta + 3)k - (2\alpha + \beta + 2).$$

By the hypotesis we can write $3\alpha - 2\beta + 3 \geq 0$. So we obtain

$$T(k) \geq T(1) = \alpha - 3\beta + 1 \geq 0.$$

Thus, $\{(2k-1)a_{2k-1}\}$ is a decreasing sequence. This completes the proof of the theorem. □

Example 2.11. The function $\mathbb{R}_{0,\frac{1}{3}}(z^2)/z = z + \sum_{k=2}^{\infty} \frac{1}{3^{k-1}\Gamma(k)} z^{2k-1}$ is close-to-convex with respect to convex function $2^{-1}log(\frac{1+z}{1-z})$.

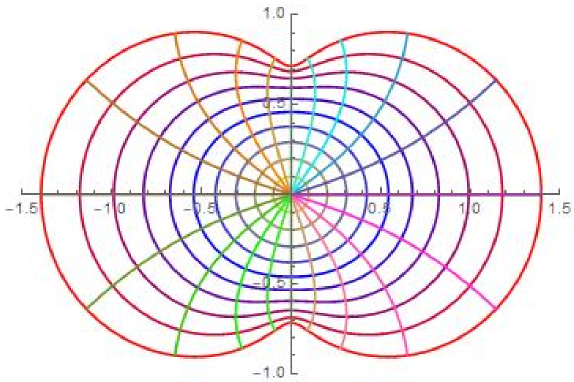

Figure 5. Mapping of $\mathbb{R}_{0,\frac{1}{3}}(z^2)/z$ over $\mathbb{U}$

Theorem 2.12. *Let* $\alpha \geq 0$ *and* $\beta > 0$*. If* $\alpha > \beta \log_2 e - 1$*, then normalized Rabotnov function* $\mathbb{R}_{\alpha,\beta}(z)$ *is univalent and starlike in* $\mathbb{U}_{1/2}$*.*

Proof. From (1.2) and (2.2) we can write

$$\begin{aligned}\left|\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} - 1\right| &= \left|\sum_{k=2}^{\infty} \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^{k-1}\right| \\ &< \sum_{k=2}^{\infty} \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\ &\leq \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\ &= e^{\frac{\beta}{1+\alpha}} - 1.\end{aligned}$$

In view of Lemma 1.4, normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is starlike in $\mathbb{U}_{1/2}$, if $e^{\frac{\beta}{1+\alpha}} - 1 < 1$. This is equivalent to hypothesis of theorem. This completes the proof of the theorem. □

Example 2.13. The function $\mathbb{R}_{\frac{1}{2},1}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}k)} z^k$ is starlike in $\mathbb{U}_{1/2}$.

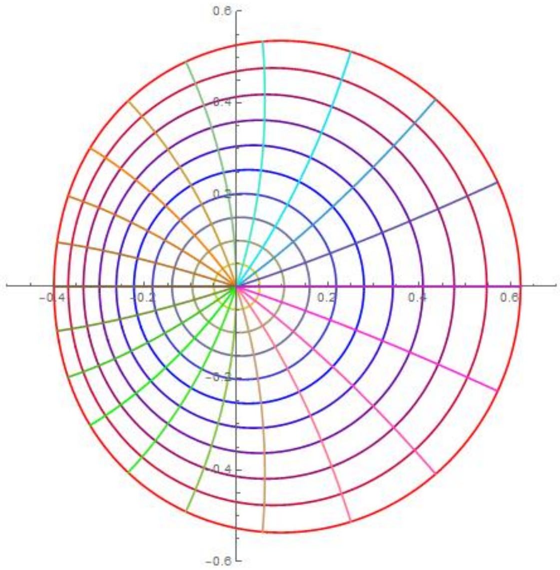

Figure 6. Mapping of $\mathbb{R}_{\frac{1}{2},1}(z)$ is starlike in $\mathbb{U}_{1/2}$

Theorem 2.14. *Let $\alpha \geq 0$ and $\beta > 0$. If $\alpha > \frac{\beta}{W(2e)-1} - 1$, where W is the Lambert W function, then the normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is convex in $\mathbb{U}_{1/2}$.*

Proof. Straightforward calculation would yield

$$\begin{aligned}\left|\mathbb{R}'_{\alpha,\beta}(z) - 1\right| &= \left|\sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^{k-1}\right| \\ &< \sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\ &\leq \sum_{k=2}^{\infty} \frac{k\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\ &= \left(1+\frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}} - 1.\end{aligned}$$

Under the given hypotheses, $\left(1+\frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}} - 1 < 1$. Using Lemma 1.5, we obtain $\mathbb{R}_{\alpha,\beta}(z)$ is convex in $\mathbb{U}_{1/2}$.

□

Example 2.15. The function $\mathbb{R}_{0,\frac{1}{3}}(z) = z + \sum_{k=2}^{\infty} \frac{1}{3^{k-1}\Gamma(k)} z^k$ is convex in $\mathbb{U}_{1/2}$.

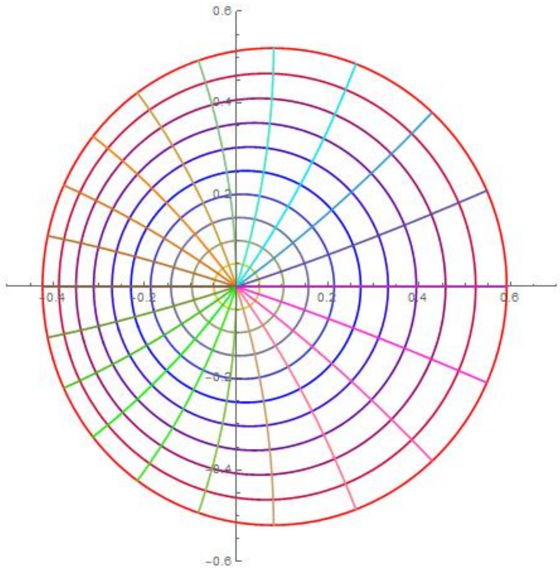

Figure 7. Mapping of $\mathbb{R}_{0,\frac{1}{3}}(z)$ is convex in $\mathbb{U}_{1/2}$

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