Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2022, 15(2), 442-447
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2022, 15(2), 442-447
DOI: 10.18185/erzifbed.982509
Research Article

On The Fano Planes in (7n-1)-dimensional Projective Spaces

Ziya Akça [1*], Abdilkadir Altıntaş [1]

[1] Eskişehir Osmangazi Üniversitesi, Fen Edebiyat Fakültesi, Matematik-Bilgisayar Bölümü
Eskişehir- Türkiye

Geliş / Received: 13/08/2021, Kabul / Accepted: 14/05/2022

Abstract
In this study, we obtain Fano planes whose points are (n-1)-dimensional subspaces and lines are (3n-1)-dimensional subspaces of P in (7n-1)-dimensional projective space P , using (n;k)-SCID. We give examples of Fano planes whose the lines of Fano configuration are planes and the points are points of $PG(6,2)$ and the lines of Fano configuration are 5-spaces and the points are lines of $PG(13,2)$.

Keywords: Klein mapping, embedding, Latin and Greek Planes, (n;k)-SCID.

(7n-1)-Boyutlu Projektif Uzayda Fano Düzlemleri Üzerine

Öz
Bu çalışmada, (n:k)-SCID kullanarak (7n-1)-boyutlu P projektif uzayında noktaları P nin (n-1)-boyutlu alt uzayları ve doğruları P nin (3n-1)-boyutlu alt uzayları olan Fano düzlemleri elde ediyoruz. $PG(6,2)$ de noktaları noktalar ve doğruları düzlemler olan ve $PG(13,2)$ de noktaları doğrular ve doğruları 5-uzaylar olan Fano düzlemlerinin örneklerini veriyoruz.

Anahtar Kelimeler: Klein dönüşümü, Gömme, Latin ve Greek Düzlemleri, (n;k)-SCID

1. Introduction

We consider the (n+1)-dimensional vector space V over Galois field $GF(q)$ where q is prime. The set of equivalence classes are the 1-dimensional subspaces of V with the origin deleted is called n-dimensional projective space $PG(n,q)$. Any m-space of $PG(n,q)$ is a set of points all of whose corresponding to vectors determine a (m+1)-dimensional subspace of V. For $m=0,1,...,n-1$, m-spaces are called a point, a line, a plane,…, a hyperplane, respectively. Hence an (n+1)-dimensional vector space gives rise to an n-dimensional projective space, [1,2,4,6,7].

Given a 4-dimensional vector space V over a field $GF(q)$, the projective space $PG(3,q)$ is the geometry of 1, 2, and 3-dimensional subspace of V. Given a 6-dimensional vector space V over a field $GF(q)$, the projective space $PG(5,q)$ is the geometry of 1, 2, 3, 4 and 5-dimensional subspace of V.

* **Corresponding Author:** zakca@ogu.edu.tr
Ziya Akça, https://orcid.org/0000-0001-6379-0546
Abdilkadir Altıntaş, https://orcid.org/0000-0002-7012-352X

The geometry on quadrics has been studied both within classical algebraic and Galois geometry. Veroneseans and Klein quadrics in finite projective spaces play an important role in finite geometry.

In this study, we give examples of Fano planes embedded in $PG(6,2)$ and $PG(13,2)$ using (2,0)-SCID and (5,1)-SCID, respectively, and a main result.

2. Material and Methods

Definition 2.1. A projective plane $PG(2,q)$ is an incidence structure (P,L,I) where P is a set whose elements are called points, L is a set whose elements are called lines and $I \subset P \times L$ is an incidence relation such that the following axioms are satisfied:

P1) Every pair of distinct points are incident with a unique common line,

P2) Every pair of distinct lines are incident with a unique common point,

P3) $PG(2,q)$ contains a set of four points with the property that no three of them are incident with a common line, [3,8].

Theorem 2.2. For finite projective plane $PG(2,q)$, there is a positive integer q that has the following properties. The number q is called the order of the projective plane.

i) Every line in contains $q+1$ points

ii) Every point in has $q+1$ lines on it

iii) There exist q^2+q+1 points in $PG(2,q)$,

iv) There exist q^2+q+1 lines in $PG(2,q)$, [3, 8].

Definition 2.3. The Klein mapping,

$$\gamma : \Im \rightarrow PG(5,q)$$

assigns to a line of L of $PG(3,q)$ the point $(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$ of $PG(5,q)$ where $(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$ are the line's Plücker coordinates, [9,10].

Definition 2.4. The lines of projective three-space are, via the Klein mapping, in one-to-one correspondence with points of a hyperbolic quadric of the projective 5-space.

The quadric of $PG(5,q)$ defined by equation,

$$X_0X_3 + X_1X_4 + X_2X_5 = 0$$

is called the Klein quadric and is denoted by the symbol H^5, [9].

The two equivalence classes are known as the *Latin planes* and the *Greek planes*. The points of $PG(3,2)$ are mapped to the Latin Planes, whereas the planes are mapped to the Greek planes, [5,9].

In $PG(3,2)$, there are 15 points, 35 lines and 15 planes.

For the points of $PG(3,2)$, one can check with the next table:

$$N_1=(0,0,0,1),\ N_6=(0,1,1,0),\ N_{11}=(1,0,1,1)$$
$$N_2=(0,1,0,0),\ N_7=(0,1,1,1),\ N_{12}=(1,1,0,0)$$
$$N_3=(0,1,0,1),\ N_8=(1,1,1,0),\ N_{13}=(1,1,0,1)$$
$$N_4=(0,0,1,0),\ N_9=(1,1,1,1),\ N_{14}=(1,0,0,0)$$
$$N_5=(0,0,1,1),\ N_{10}=(1,0,1,0),\ N_{15}=(1,0,0,1).$$

Latin planes of Klein Quadric have Grassmanian coordinates in the projective space $PG(19,2)$, [2].

$$N_1=(0,0,0,1)\rightarrow P_{\alpha 1}=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0)$$
$$N_2=(0,1,0,0)\rightarrow P_{\alpha 2}=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0)$$
$$N_3=(0,1,0,1)\rightarrow P_{\alpha 3}=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,1,1,0,0)$$
$$N_4=(0,0,1,0)\rightarrow P_{\alpha 4}=(0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0)$$
$$N_5=(0,0,1,1)\rightarrow P_{\alpha 5}=(0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,1,0,1,0)$$
$$N_6=(0,1,1,0)\rightarrow P_{\alpha 6}=(0,0,0,0,0,0,0,0,1,1,0,0,0,0,1,1,0,0,0,0)$$
$$N_7=(0,1,1,1)\rightarrow P_{\alpha 7}=(0,0,0,0,0,0,0,1,1,1,0,0,0,1,1,1,1,1,1,0)$$
$$N_8=(1,1,1,0)\rightarrow P_{\alpha 8}=(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$$
$$N_9=(1,1,1,1)\rightarrow P_{\alpha 9}=(1,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0)$$
$$N_{10}=(1,0,1,0)\rightarrow P_{\alpha 10}=(1,1,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0)$$
$$N_{11}=(1,0,1,1)\rightarrow P_{\alpha 11}=(1,1,0,0,1,0,0,0,0,0,0,1,1,1,1,0,1,1,0,0)$$
$$N_{12}=(1,1,0,0)\rightarrow P_{\alpha 12}=(1,0,1,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0)$$
$$N_{13}=(1,1,0,1)\rightarrow P_{\alpha 13}=(1,0,1,0,1,0,1,1,0,1,0,1,0,0,0,0,1,0,0,0)$$
$$N_{14}=(1,0,0,0)\rightarrow P_{\alpha 14}=(1,1,1,0,0,0,1,0,1,1,0,0,1,0,1,1,0,0,1,1)$$
$$N_{15}=(1,0,0,1)\rightarrow P_{\alpha 15}=(1,1,1,0,1,0,1,1,1,1,0,1,1,1,1,1,1,1,1,0)$$

and Greek planes of Klein Quadric have Grassmanian coordinates in the projective space $PG(19,2)$, [2].

$$D_1=[0,0,0,1]\rightarrow P_{\beta 1}=(0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$$
$$D_2=[0,0,1,0]\rightarrow P_{\beta 2}=(0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)\ .$$

$$D_3 = [0,0,1,1] \rightarrow P_{\beta 3} = (0,0,1,1,0,1,1,0,0,0,0,1,1,0,0,0,0,0,0,0,0)$$
$$D_4 = [0,1,0,0] \rightarrow P_{\beta 4} = (0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0)$$
$$D_5 = [0,1,0,1] \rightarrow P_{\beta 5} = (0,1,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0)$$
$$D_6 = [0,1,1,0] \rightarrow P_{\beta 6} = (0,0,0,0,1,1,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0)$$
$$D_7 = [0,1,1,1] \rightarrow P_{\beta 7} = (0,1,1,1,1,1,1,0,0,0,1,1,1,0,0,0,0,0,0,0,0)$$
$$D_8 = [1,0,0,0] \rightarrow P_{\beta 8} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)$$
$$D_9 = [1,0,0,1] \rightarrow P_{\beta 9} = (0,0,0,1,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0)$$
$$D_{10} = [1,0,1,0] \rightarrow P_{\beta 10} = (0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,1,1)$$
$$D_{11} = [1,0,1,1] \rightarrow P_{\beta 11} = (0,0,1,1,0,1,1,1,1,0,0,0,0,0,0,1,0,0,0,0,0)$$
$$D_{12} = [1,1,0,0] \rightarrow P_{\beta 12} = (0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,1,0,1)$$
$$D_{13} = [1,1,0,1] \rightarrow P_{\beta 13} = (0,1,0,1,0,0,0,0,1,0,1,0,1,1,0,1,0,1,1,1)$$
$$D_{14} = [1,1,1,0] \rightarrow P_{\beta 14} = (0,0,0,0,1,1,0,1,0,0,1,1,0,1,0,0,0,1,1,1)$$
$$D_{15} = [1,1,1,1] \rightarrow P_{\beta 15} = (0,1,1,1,1,1,1,1,1,0,1,1,1,1,0,1,0,1,0,1)$$

3. Main Theorem and Proof

Let $PG(n,q)$ be a projective space, and let -1 $\leq k < n$. A set E of n-dimensional subspaces of $PG(n,q)$ with the property that any two elements of E intersect precisely in a k-dimensional subspace is called an (n;k)-SCID (set of subspaces with constant intersection dimension).

Let E be any SCID. Let $E_1,\ E_2 \in E$. Then each k-dimensional subspace of $E_1 \cap E_2$ is called an intersection k-space of E. (for k = 0,1 these are called intersection points, intersection lines, resp.).

For example, a (2,0)-SCID is a set of planes intersecting mutually in exactly one point. Latin planes of Klein Quadric are example of (2,0)-SCID and similarly Greek planes of Klein Quadric are example of (2,0)-SCID. Any two planes in the same class have only one intersection point.

We give an example of Fano plane embedded in $PG(6,2)$ using (2,0)-SCID.

Example3.1. $P_0 = (1,0,0,0,0,0,0)$, $P_{12} = (0,1,0,0,0,0,0)$, $P_{13} = (0,0,1,0,0,0,0)$ $P_{14} = (0,0,0,1,0,0,0)$, $P_{23} = (0,0,0,0,1,0,0)$, $P_{24} = (0,0,0,0,0,1,0)$, $P_{34} = (0,0,0,0,0,0,1)$ be seven independent points of a projective spce $PG(6,2)$. We define the planes $E_1, E_2, ..., E_7$ as follows;

$$E_1 = \langle P_{12},\ P_{13},\ P_{14} \rangle$$
$$E_2 = \langle P_{12},\ P_{23}, P_{24} \rangle$$
$$E_3 = \langle P_{13},\ P_{23}, P_{34} \rangle$$
$$E_4 = \langle P_{14},\ P_{24}, P_{34} \rangle$$
$$E_5 = \langle P_0, P_{12},\ P_{34} \rangle$$
$$E_6 = \langle P_0,\ P_{13}, P_{24} \rangle$$
$$E_7 = \langle P_0,\ P_{14}, P_{23} \rangle.$$

Any two of these planes intersect on one of the points $P_0, P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$. Any point $P_0, P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$ lie on exactly three planes. If we consider the planes $E_1, E_2, ..., E_7$ as lines and the points $P_0, P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$ as points, this configuration is a Fano plane $PG(2,2)$ in $PG(6,2)$. Hence we have seven lines and seven points. Any two lines (planes $E_1, E_2, ..., E_7$) intersects on a point, any two point defines a line.

In Example 3.2, we use (5,1)-SCID to get Fano plane in $PG(13,2)$.

Example 3.2. Let $P_0, P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}$, $Q_0, Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}$ be fourteen independent points of a projective space $PG(13,2)$. We define the projective 5-spaces $S_1, S_2, ..., S_7$ as follows;

$$S_1 = \langle P_{12}, Q_{12}, P_{13}, Q_{13}, P_{14}, Q_{14} \rangle$$
$$S_2 = \langle P_{12}, Q_{12}, P_{23}, Q_{23}, P_{24}, Q_{24} \rangle$$
$$S_3 = \langle P_{13}, Q_{13}, P_{23}, Q_{23}, P_{34}, Q_{34} \rangle$$
$$S_4 = \langle P_{14}, Q_{14}, P_{24}, Q_{24}, P_{34}, Q_{34} \rangle$$
$$S_5 = \langle P_0, Q_0, P_{12}, Q_{12}, P_{34}, Q_{34} \rangle$$
$$S_6 = \langle P_0, Q_0, P_{13}, Q_{13}, P_{24}, Q_{24} \rangle$$
$$S_7 = \langle P_0, Q_0, P_{14}, Q_{14}, P_{23}, Q_{23} \rangle.$$

Any two of these spaces intersect on one of the lines

$$P_0Q_0, P_{12}Q_{12}, P_{13}Q_{13}, P_{14}Q_{14}, P_{23}Q_{23}, P_{24}Q_{24}, P_{34}Q_{34}$$

and any line

$$P_0Q_0, P_{12}Q_{12}, P_{13}Q_{13}, P_{14}Q_{14}, P_{23}Q_{23}, P_{24}Q_{24}, P_{34}Q_{34}$$

lie on exactly three 5-spaces. If we consider the planes $S_1, S_2, ..., S_7$ as lines and line set

$$L = \{P_0Q_0, P_{12}Q_{12}, P_{13}Q_{13}, P_{14}Q_{14}, P_{23}Q_{23}, P_{24}Q_{24}, P_{34}Q_{34}\}$$

as points, this configuration is a Fano plane $PG(2,2)$ in $PG(13,2)$.

4. Conclusion

In this study, if we add points $R_0, R_1, ..., R_{34}$ to point set, similar arquments works. Generally, let P be a (7n-1)- dimesional projective space $n \geq 1$, we can construct Fano configuration (point and line sets defined similarly in Example 3.1 and Example 3.2).

The lines are (3n-1)-dimesional subspaces of P and points are (n-1)-dimensional subspaces of P. Example 3.1 is special case for $n = 1$, the lines of Fano configuration are planes and the points are points. Example 3.2 is special case for $n=2$, the lines of Fano configuration are 5-spaces and the points are lines of $PG(13,2)$.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Acknowledgment

This work was supported by the Scientific Research Projects Commission of Eskişehir Osmangazi University under Project Number 2019-2542.

References

[1] Akça, Z., Altıntaş, A., (2021) A Note on Fano Configurations in the Projective Space PG(5,2) , Konuralp Journal of Mathematics,. 9, 1, 190–192.

[2] Akça, Z., Altıntaş, A., (2022) Some Results on the Klein Quadric Representation for a General Quantum Spacetime, Prespacetime Journal,. 13, 5, 1–6.

[3] Akça, Z., Ekmekçi S., Bayar A., (2016) On Fano Configurations of the Left Hall Plane of order 9, Konuralp Journal of Mathematics,. 4, 2, 124–131.

[4] Artin, E., (1964) Galois theory. 82. University of Notre Dame Press, Notre Dame.

[5] Cullinane, H., (2007) The Klein Correspondence, Penrose Space-Time and a Finite Model, http://finitegeometry.org/sc/64/KleinCorr.html.

[6] Ekmekçi S., Akça, Z., Bayar A., (2009) On the classification of fuzzy projective planes of fuzzy 3-dimensional projective space, Chaos Solitons & Fractals,. 40, 5, 2146-2151.

[7] Hirschfeld, J.W.P., Thas, J.A., (2016) General Galois Geometries. 409. Springer Monongraphs in Mathematics.

[8] Kaya, R., (2004) Projektif Geometri. 391. Osmangazi Üniversitesi Yayınları, Eskişehir.

[9] Klein, F., (1868) Über die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Koordinaten auf eine kanonische, Form Math,. 539-578.

[10] Plücker, J., (1865) On a New Geometry of Space, Philosophical Transactions of the Royal Society of London,. 155, 725-791.