e-ISSN: 2564-7954 CUJSE 19(1): 022-028 (2022) Research Article

Çankaya University

Journal of Science and Engineering

https://dergipark.org.tr/cankujse

CUJSE

Minimization of Quadratic Functionals Through Γ-Hilbert Space

Sahin Injamamul Islam *[1], Nirmal Sarkar [2], and Ashoke Das [3]

[1, 2, 3]Department of Mathematics, Raiganj University , India

Keywords

Γ-Hilbert space,
Gateaux Γ-derivative,
Frechet Γ-derivative,
Relative extremum,
Stationary point,
Quadratic functionals.

Abstract

In this article we introduce the Gateaux differential and Frechet differential in Γ-Hilbert space. We show the examples and related theorems in this space. We have noticed that two differentials mentioned above will be equal for certain condition. Also, we discuss the relative extremum and the stationary point of a functional in Γ-Hilbert space. We already investigated the characteristics of both bounded and unbounded operators of Γ-Hilbert space. Now, by using previous concept we elaborate optimization problems and extremum of quadratic functionals in Γ-Hilbert space. Here we observe that how the function of the solution of a operator equation minimizes the quadratic functionals. Finally we describe the Minimization of quadratic functionals and its related theorem via Γ-Hilbert space.

1. Introduction

After the introduce of Γ-Hilbert Space [1] in 2008, further study was also found in the paper of A. Ghosh, A. Das and T E Aman in 2017 [2]. Also S. I. Islam and Ashoke Das have discussed the characteristics of bounded operators in their paper [3]. After that we get the the concept of Γ-Differential function and Γ-Differential operator in the paper [4]. Now we try to elaborate Minimization of Quadratic Functionals via Γ-Hilbert Space in this paper.To discuss this problem , it is essential to present a few ideas of the calculas of operators in Banach space. For that we use the definition of Gateaux Γ-differential, Frechet Γ-differential, stationary point , related examples and theorem. This paper more concerned with the optimization and their applications. In this paper, after consulting the main author, we have made some changes in the main definition of Γ-Hilbert space [1]. so at to begin with we remind the definition of Γ-Hilbert space and related definitions.

Definition 1.1. *Let E be a linear space over the field F and Γ be a semi group with respect to addition.Then the mapping $\langle\cdot,\cdot,\cdot\rangle : E\times\Gamma\times E\to F$ is called a Γ - Inner Product on (E,Γ) if*

1. *$\langle\cdot,\cdot,\cdot\rangle$ is linear in first variable and additive in second variable.*
2. *$\langle x,\gamma,y\rangle = \langle y,\gamma,x\rangle\ \forall\ x,y\in E$ and $\gamma\in\Gamma$.*
3. *$\langle x,\gamma,x\rangle > 0\ \forall\ x\neq 0$.*
4. *$\langle x,\gamma,x\rangle > 0$ if and only if $x = 0$.*

$[(E,\Gamma),\langle .,.,.\rangle]$ is called a Γ-inner product space over F.
A complete Γ-inner product space is called Γ-Hilbert space.

*Corresponding author: sahincool92@gmail.com
Received: August 13, 2021, Accepted: February 23, 2022

Definition 1.2. *If we write* $\| u \|_{\gamma}^{2} = \langle u, \gamma, u \rangle$ *, for* $u \in H$ *and* $\gamma \in \Gamma$ *then* $\| u \|_{\gamma}^{2}$ *satisfy all the conditions of norm.*

Definition 1.3. *Again if* $\| u \|_{\Gamma}$ *={* $\| u \|_{\gamma}$*:* $\gamma \in \Gamma$*} , then this norm is called the* Γ*-norm of the* Γ*-Hilbert space.*

Definition 1.4. *The function* $f : H_{\Gamma} \to R$ *(from a* Γ*-Hilbert space* H_{Γ} *to a real number set* R*) is said to be* γ*-differentiable at a point* $h \in H_{\Gamma}$ *for fixed* $\gamma \in \Gamma$ *if there exist* $f_{\gamma}^{'} : H_{\Gamma} \to R$ *such that for each* $\epsilon > 0$*,* $\exists$ *a* $\delta > 0$ *such that*
$|f(x) - f(h) - f_{\gamma}^{'}(x - h)| \leqslant \epsilon \| x - h \|_{\gamma}$ *whenever* $\| x - h \|_{\gamma} < \delta$*. Here* $f_{\gamma'}$ *is called the* γ*-derivative of* f *at* $h \in H_{\Gamma}$*.*

Definition 1.5. *The function* $f : H_{\Gamma} \to R$ *(from a* Γ*-Hilbert space* H_{Γ} *to a real number set* R*) is said to be* Γ*-differentiable at a point* $h \in H_{\Gamma}$ *if there exist* $f_{\Gamma}^{'} : H_{\Gamma} \to R$ *such that for each* $\epsilon > 0$*,* $\exists$ *a* $\delta > 0$ *such that*
$|f(x) - f(h) - f_{\Gamma}^{'}(x - h)| \leqslant \epsilon \| x - h \|_{\Gamma}$ *whenever* $\| x - h \|_{\Gamma} < \delta$*.*

Definition 1.6. *The operator* $T : H_{\Gamma} \to H_{\Gamma_1}$ *(from a* Γ*-Hilbert space* H_{Γ} *to another* Γ*-Hilbert space* H_{Γ_1}*)is said to be* γ*-differentiable at a point* $h \in H_{\Gamma}$ *for fixed* $\gamma \in \Gamma$ *if there exist* $T_{\gamma}^{'} : H_{\Gamma} \to H_{\Gamma_1}$ *such that for each* $\epsilon > 0$*,* $\exists$ *a* $\delta > 0$ *such that*
$|T(x) - T(h) - T_{\gamma}^{'}(x - h)| \leqslant \epsilon \| x - h \|_{\gamma}$ *whenever* $\| x - h \|_{\gamma} < \delta$*.*
$T_{\gamma}^{'}$ *be the linear transformation from* H_{Γ} *to* H_{Γ_1} *is called the* γ*-derivative of* T*.*

Note: Similarly we can define Γ-derivative of an operator from H_{Γ} to H_{Γ_1}.

2. Basic Results

In this area, we consider two Banach spaces namely B and B_1 over a field F, which can be either the real numbers R or the complex numbers C. Also, we suppose an operator $T : B \to B_1$ which need not to be linear.

Definition 2.1. *Let* x *be a fixed point of* B *. Then the operator* $T : B \to B_1$ *is called Gateaux* Γ*-differential at the point* x *if there exist a continuous linear operator* A *such that*

$$\lim_{t \to 0} \| \frac{T(x + sh) - T(x)}{t} - Ah \|_{\gamma} = 0,$$

for all $h \in B_1$ *and* $\gamma \in \Gamma$ *, where* $t \to 0$ *in* F*. Then the operator* A *is said to be the Gateaux* Γ*-differential of the operator* T *at* x *and is denoted by* $A(h) = dT_{\gamma}(x, h)$ *at* h*.*

Note: In case, the operator T is linear , then $dT_{\gamma}(x, h) = T(h)$, it implies $dT_{\gamma}(x) = T \; \forall \; x \in B_1$.

Theorem 2.2. *If the Gateaux* Γ*-differential exists then it is unique.*

Proof. Let that two operators A and A_1 satisfy Gateaux Γ-differential. Then, for each $h \in B$ and for each $t > 0$, we have
$\| A(h) - A_1(h) \|_{\gamma} =\| (\frac{T(x+th)-T(x)}{t} - A_1(h)) - (\frac{T(x+th)-T(x)}{t} - A(h)) \|_{\gamma}$

$\leq \| \frac{T(x+th)-T(x)}{t} - A_1(h) \|_{\gamma} + \| \frac{T(x+th)-T(x)}{t} - A(h) \|_{\gamma} \to 0,$

as $t \to 0$. So $\| A(h) - A_1(h) \|_{\gamma} = 0$ for all $h \in B$.

This proves the theorem. □

Definition 2.3. *Suppose in a Banach space* B*,* x *be a fixed point. A operator* $A : B \to B_1$ *which is continuous and linear is known as the Frechet* Γ*-differential of the operator* $T : B \to B_1$ *at the point* x *if* $T(x+h) - T(x) = Ah + \psi(x, h)$ *and*

$$\lim_{\|h\|_\gamma \to 0} \frac{\| (x,h) \|_\gamma}{\| h \|_\gamma} = 0.$$

or equivalently,

$$\lim_{\|h\|_\gamma \to 0} \frac{\| T(x,h) - T(x) - Ah \|_\gamma}{\| h \|_\gamma} = 0.$$

The Frechet Γ-differential at the point x is denoted by $T_\gamma'(x)$.

Theorem 2.4. *If there is a Frechet Γ-differential at a point in a mapping , then there is the Gateaux Γ-differential at the same point and both the differentials are similar.*

Proof. Let T_γ be an operator and $T_\gamma : A_1 \to A_2$, also let $x \in A_1$ and $\gamma \in \Gamma$. If T_γ has the Frechet Γ-differential at x , then

$$\lim_{\|h\|_\gamma \to 0} \frac{\| T_\gamma(x,h) - T_\gamma(x) - Ah \|_\gamma}{\| h \|_\gamma} = 0,$$

where A is continuous linear operator $A : A_1 \to A_2$.

Now for any nonzero constant $h \to A_1$, we get

$$\lim_{t \to 0} \| \frac{T_\gamma(x+th) - T_\gamma(x)}{t} - Ah \|_\gamma$$

$$= \lim_{t \to 0} \frac{\| T_\gamma(x+th) - T_\gamma(x) - A(th) \|_\gamma}{\| th \|_\gamma} \| h \|_\gamma = 0.$$

It shows that A is the Gateaux Γ-differential of T_γ at the point x. □

Example 2.5. *Suppose A be a linear operator and also bounded on a real Γ-Hilbert space H_Γ and let f_γ be the functional of H_Γ defined by $f_\gamma(x)$ =$\langle x, \gamma, Ax \rangle$. We shall prove that the Frechet Γ-differential of f_γ will be $f_\gamma'(x)(h)$= $\langle x, \gamma, (A + A^*)x \rangle$ where $\gamma \in \Gamma$.*

Suppose h be any arbitrary element of H_Γ. Now for each $\gamma \in \Gamma$, we have

$f_\gamma(x+h) - f_\gamma(x)$ - $\langle x,\gamma,(A + A^*)x \rangle$

=$\langle x+h, \gamma, A(x+h) \rangle$ - $\langle x, \gamma, Ax \rangle$ - $\langle h, \gamma, Ax \rangle$ - $\langle h, \gamma, A^*x \rangle$

=$\langle x, \gamma, Ax \rangle$+$\langle x, \gamma, Ah \rangle$+$\langle h, \gamma, Ax \rangle$ + $\langle h, \gamma, Ah \rangle$ - $\langle x, \gamma, Ax \rangle$ - $\langle h, \gamma, Ax \rangle$ - $\langle h, \gamma, A^*x \rangle$

= $\langle h, \gamma, Ah \rangle$.

Consequently,

$$\lim_{\|h\|_\gamma \to 0} \frac{\| f_\gamma(x,h) - f_\gamma(x) - \langle h, \gamma, A + A^* \rangle x \|_\gamma}{\| h \|_\gamma} = \lim_{\|h\|_\gamma \to 0} \frac{\| \langle h, \gamma, Ah \rangle \|_\gamma}{\| h \|_\gamma} = 0.$$

Hence the result follows .

Definition 2.6. *If $T: B \to B_1$ is Frechet Γ-differentiable in an open set $S \subset B$ and T_γ' is the Frechet Γ-differential at the point $x \in S$, then T is said to be twice Frechet Γ-differential at x. The Frechet Γ-differential of T_γ' at the point x is called the second Frechet Γ-differential of T and is denoted by $T_\gamma''(x)$.*

Example 2.7. *Let us assume a real Γ-Hilbert space $L^2([a,b])$. Define a functional $f_\gamma : L^2([a,b]) \to R$ by the double integral*

$$f_\gamma(x) = \int_a^b \int_a^b x(u)K(u,v)x(v)\gamma dv du, \qquad (i)$$

where K is a continuous function and $\gamma \in \Gamma$.
Now we characterize a linear operator T on $L^2([a,b])$ by

$$(Tx)(v) = \int_a^b K(v,u)x(u)\gamma du. \qquad (ii)$$

Now we rewrite (i) in the form :

$$f_\gamma(x) = \langle x, \gamma, Tx \rangle. \qquad (iii)$$

Hence,

$$\begin{aligned} f_\gamma(x+h) &= \langle x+h, \gamma, T(x+h) \rangle \\ &= \langle x+h, \gamma, Tx + Th \rangle \\ &= f_\gamma(x) + \langle h, \gamma, Tx \rangle + \langle x, \gamma, Th \rangle + \langle h, \gamma, Th \rangle. \qquad (iv) \end{aligned}$$

Therefore we can get the second Frechet Γ-differential f_γ'' by (iv) , so that

$$f_\gamma''(x)(h,h) = \langle h, \gamma, Th \rangle. \qquad (v)$$

Now we have

$$\begin{aligned} f_\gamma''(x)(h,k) &= \tfrac{1}{2}\left[\langle h, \gamma, Tk \rangle + \langle k, \gamma, Th \rangle\right] \\ &= \langle h, \gamma, \tfrac{1}{2}(T + T^*)K \rangle, \qquad (vi) \end{aligned}$$

where the adjoint of T is T^*. If $K(u,v) = K(v,u)$, then the operator T will be self-adjoint, and (vi) becomes

$$f_\gamma''(x)(h,k) = \langle h, \gamma, Tk \rangle.$$

Above expression is symmetric in h and k.
On a real Γ-Hilbert space , above conclusion can be generalized to any functional f_γ.

Optimization Problems: In general calculas, the maximum and minimum problems refer to the values of the independent variables for which a function reaches its maximum value or minimum value. If a differentiable function includes a maximum or a minimum value at a certain point, at this point its derivative vanishes. It turns out that this characteristic can be extended to both the maximum or minimum value to the functional in the Γ-normed space. We will prove that a function which is real defined in a subset of the Γ-normed space has a maximum value or minimum value at a certain point, then both Gateaux Γ-differential or Frechet Γ-differential wii be zero at the same point. we use the term extremum to mention both maximum or minimum in the following discussion.

Definition 2.8. *Suppose f be a real valued functional defined on a subset S of a Γ-Normed space E is said to be a relative extremum at a point $x_0 \in S$ if $\exists$ an open ball $\Gamma - B(x_0, r) \subset E$ such that $f(x_0) \leqslant f(x)$ (or $f(x_0) \geqslant f(x)$) holds $\forall\, x \in \Gamma - B(x_0, r) \cap S$ where x_0 is the centre and r is the radius of the open ball.*

Theorem 2.9. *Suppose a functional $f_\gamma : E \to R$ is Gateaux Γ-differentiable at the point $x_0 \in E$ and hold a realtive extremum at x_0, then $df_\gamma(x_0, h) = 0$ for all $h \in E$ and $\gamma \in \Gamma$.*

Corollary 2.10. *Consider the functional $f_\gamma : E \to R$ is Frechet Γ-differentiable at $x_0 \in E$ and holds a realtive extremum at x_0, then $f_\gamma(x_0) = 0 \; \forall \; \gamma \in \Gamma$.*

Definition 2.11. *A point x is said to be stationary point at which $df_\gamma(x, h)$ = 0 or $f^{'}_\gamma(x)$ = 0 for all $h \in E$.*

Example 2.12. *Let us assume a functional*

$$J(u) = \langle Au, \gamma, u\rangle \text{ - } 2\langle u, \gamma, f\rangle,$$

where f belonging to a real Γ-Hilbert space H_Γ, $\gamma \in \Gamma$ and A is a linear operator in H_Γ which is self-adjoint. Now,

$$J(u + h) - J(u) = 2\langle Au, \gamma, h\rangle \text{ - } 2\langle f, \gamma, h\rangle + \langle Ah, \gamma, h\rangle.$$

By fixing $\langle J^{'}(u), \gamma, h\rangle = 2\langle Au, \gamma, h\rangle$ - $2\langle f, \gamma, h\rangle + \langle (Au - f), \gamma, h\rangle$, we can find

$$\| J(u + h) - J(u) - \langle J^{'}(u), \gamma, h\rangle \| = \| \langle Ah, \gamma, h\rangle \| \leqslant M \| h \|^2,$$

where M is constant. So, it succeed that the Frechet Γ-differential of $J(u)$ is

$$J^{'}(u) = 2(Au - f).$$

Moreover, it succeed that

$$\langle J^{'}(u + h), \gamma, k\rangle \text{ - } \langle J^{'}(u), \gamma, k\rangle = \langle 2Ah, \gamma, k\rangle.$$

Thus, the second Frechet Γ-differential is

$$J^{''}(u)(h, k) = \langle 2Ah, \gamma, k\rangle.$$

this shows that it is independent of $u \in H_\Gamma$.
It find out that $J(u)$ has a local minimum if A is a positive operator the operator equation $Au = f$ satisfies by u.

3. Main Result

Minimization of Quadratic Fuctionals

Let T be a positive definite operator which is also real and symmetric defined on Γ-Hilbert space H_Γ. We study the operator equation

$$Tu = f, \qquad (vii)$$

where we assume f be an element of H_Γ. since T is positive definite operator so the solution of (vii) exists. Furthermore, it can be shown that the solution of (vii) is a function which minimizes the quadratic functional

$$J(u) = \langle Tu, \gamma, u\rangle \text{ - } 2\langle f, \gamma, u\rangle \text{ for all } \gamma \in \Gamma. \qquad (viii)$$

On the other hand, if we getting a solution u which minimizes $J(u)$ on H_Γ, then the desirable solution of (vii) is u. That is the fundamental result which is expressed in the following theorem.

Theorem 3.1. *Let us suppose $T : H_\Gamma \to H_\Gamma$ is a linear and symmetric positive definite operator on a real Γ-Hilbert space H_Γ and f is an element of H_Γ. Then the quadratic functional $J(u)$= $\langle Tu, \gamma, u\rangle$ - $2\langle f, \gamma, u\rangle \; \forall \; \gamma \in \Gamma$, reaches its minimum value for some $u_0 \in H_\Gamma$ if and only if the solution of that operator equation $Tu = f$ is u_0.*

Proof. Let us assume that the solution of the operator equation is u_0. Suppose u be an element of H_Γ. Then for each $\gamma \in \Gamma$,

$$J(u) - J(u_o) = \langle Tu, \gamma, u\rangle - 2\langle f, \gamma, u\rangle - \langle Tu_o, \gamma, u_o\rangle + 2\langle f, \gamma, u_o\rangle$$

$$= \langle Tu, \gamma, u\rangle - 2\langle Tu_o, \gamma, u\rangle + \langle Tu_o, \gamma, u_o\rangle$$

$$= \langle T(u - u_o), \gamma, u - u_o\rangle.$$

Since T is a positive definite operator, $\langle Tu_o, \gamma, u_o\rangle \geqslant 0$, and also the equality holds only when $u = u_o$. So,

$$J(u) \geqslant J(u_o).$$

We can clearly see that $J(u)$ attains the minimum of $Tu = f$ at the solution u_0.
Conversely, let u_0 minimizes $J(u)$ which is an element of H_Γ, i.e.$J(u) \geqslant J(u_o) \forall u \in H_\Gamma$. In particular,

$$J(u_o + sv) \geqslant J(u_o),$$

for some real numbers s and $v \in H_\Gamma$.

Now, $J(u_o + sv) = \langle T(u_o + sv), \gamma, u_o + sv\rangle$ - $2\langle f, \gamma, u_o + sv\rangle$.

$$= \langle Tu_o, \gamma, u_o\rangle + 2s\langle Tu_o, \gamma, v\rangle + s^2\langle Tv, \gamma, v\rangle - 2\langle f, \gamma, u_o\rangle - 2t\langle f, \gamma, v\rangle.$$

or,

$$\frac{J(u_o+sv)-J(u_o)}{s} = 2\langle Tu_o - f, \gamma, v\rangle + s\langle Tv, \gamma, v\rangle \; where \; \gamma \in \Gamma.$$

If the limit $s \to 0$, then the above expression leading to the Gateaux Γ-differential

$$dJ(u_0, v) = 2\langle Tu_0, \gamma, v\rangle.$$

$\forall\, v \in H_\Gamma$ and $\gamma \in \Gamma$. Since at $u = u_0$, $J(u)$ has a local minimum then $dJ(u_0, v) = 0$ for any $v \in H_\Gamma$.

This imply that $Tu_0 - f = 0$, which conclude that u_0 is the solution of the operator equation. □

Corollary 3.2. *Above mentioned problem of Minimizationcan be interpret as the Maximization problem for $-J(u)$. Theorem 3.1 can be generalize to a operator T which is symmetric and positive definite on a complex Γ-Hilbert space H_Γ. The functional then change as*

$$J(u) = \langle Au, \gamma, u\rangle - \langle f, \gamma, u\rangle - \langle u, \gamma, f\rangle\,.$$

Example 3.3. *Let $T : H_\Gamma \to K_\Gamma$ is a linear and symmetric bounded operator, where H_Γ and K_Γ are the real Γ-Hilbert spaces. We will minimize*

$$J(u) = \| \, Tu - a \, \|_\gamma^2,$$

where $u \in H_\Gamma$ and $a \in K_\Gamma$. Now, we have $J(u) = \langle Tu - a, \gamma, Tu - a\rangle = \langle Tu, \gamma, Tu\rangle - 2\langle a, \gamma, Tu\rangle + \langle a, \gamma, a\rangle$

$$= \langle T^*Tu, \gamma, u\rangle - 2\langle T^*a, \gamma, u\rangle + \langle a, \gamma, a\rangle,$$

and hence,

$$J(u + h) - J(u) = \langle 2(T^*Tu - T^*a), \gamma, h\rangle + \langle T^*Th, \gamma, h\rangle.$$

Clearly, the first differential is given by

$$J^{'}(u) = 2T^*Tu - 2T^*a\,.$$

So,

$$\langle J^{'}(u+h), \gamma, k\rangle - \langle J^{'}(u), \gamma, k\rangle = \langle 2T^{*}Th, \gamma, k\rangle.$$

Therefore,

$$J^{''}(u)(h,k) = \langle 2T^{*}Th, \gamma, k\rangle,$$

which is not dependent of u*. Thus at* $u = u_0$*,* J *has an extremum if* $J^{'}(u_0) = 0$*, that is* $T^{*}Tu_0 = T^{*}a$*. If we suppose that the operator* $T^{*}T$ *is positive, then* $J^{''}(u) \geqslant 0$ *and the local minimum of* $J(u)$ *is* u_0*.*

4. Conclusions

In the present paper, we presented some important and interesting definitions as well as some results which are playing a key role to expand the ideas of optimization problems through Γ-Hilbert space. Furthermore, this idea can be used to solve optimization problems. After that, using such concept as Relative extremum, Gateaux Γ-differential, we proof the most important theorem of minimization of quadratic functionals on generalized Hilbert space that is Γ-Hilbert space. Also, finally get the fundamental result. In physics, this fundamental result can be expressed as the "minimization of some energy function". Furthermore, it may be a fundamental principle of mechanics that nature is acting here so as to minimize the energy. We notice that, problem of minimization may be illustrate as a maximization problem for a certain quadratic functional.

Acknowledgements

The researchers would like to thank Dr.T.E. Aman for his important recommendations to progress the paper. The second author would like admit the financial support form University Grant Commission (UGC-NET JRF).

Declaration of Competing Interest

The researchers pronounce that there's no competing budgetary interface or individual connections that impacts the work in this paper.

Authorship Contribution Statement

Sahin Injamamul Islam: Data creation, Draft preparation, Writing, Reviewing.
Nirmal Sarkar: Methodology, Writing, Editing, Investigation.
Ashoke Das: Reviewing, Supervision, Investigation.

References

[1] T. E. Aman and D. K. Bhattacharya, "Γ-Hilbert Space and linear quadratic control problem," Revista de la Academia Canaria de Ciencias, vol. 15, no. 1-2, pp. 107-114, 2004.

[2] A. Gosh, A. Das and T. E. Aman, "Representation Theorem on Γ-Hilbert Space," International Journal of Mathematics Trends and Technology, vol. 52, no. 9, pp. 608-615, 2017.

[3] S. Islam and A. Das, "On Some bounded Operators and their characterizations in Γ-Hilbert Space," Cumhuriyet Science Journal, vol. 41, no. 4, pp. 854-861, 2020.

[4] A. Das, A. Ghosh and T. E. Aman, "Calculas on Γ-Hilbert Space," Journal of Interdisciplinary Cycle Research. vol. 12, no. 7, pp. 254-268, 2020.