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Geometric Structure of the Set of Pairs of Matrices under Simultaneous Similarity

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Abstract

In this paper pairs of matrices under similarity are considered because of their scientific applications, especially pairs of matrices being simultaneously diagonalizable. For example, a problem in quantum mechanics is the position and momentum operators, because they do not have a shared base representing the system's states. They do not commute, and that is why switching operators form a crucial element in quantum physics.
A study of the set of linear operators consisting of pairs of simultaneously diagonalizable matrices is done using geometric constructions such as the principal bundles. The main goal of this work is to construct connections that allow us to establish a relationship between the local geometry around a point with the local geometry around another point. The connections give us a way to help distinguish bundle sections along tangent vectors.

1. Introduction

Let $\mathfrak{M}$ be the manifold of pairs of n-order real matrices $T = (X_1, X_2)$. A frequent question, in both mathematics and physics, is to ask if it is possible to find a base in the space $\mathbb{R}^n$ in which both matrices diagonalize, that is, to ask if they diagonalize simultaneously. Concretely, the simultaneous diagonalization of pairs of symmetric matrices has a particular interest, (see [5], [7] and [8], for example), due to its applications in sciences. For example, they appear when we must give the "instanton"solution of Yang-Mills field presented in an octonion form, and it can be represented by triples of traceless matrices [1], [6], [13]. Another application of simultaneous diagonalization is found when studying, for example, thermal transmissivity, whose study is different depending on whether the interaction matrices diagonalize simultaneously [12].
In order to formalize the simultaneous diagonalization problem, it is necessary to start by defining an equivalence relation called similarity, which allows establishing criteria for simultaneous diagonalization.
It is well known that, in the space of n-square real matrices, the subset of diagonalizable matrices is generic. Then, any non-diagonalizable matrix is arbitrarily close to a diagonalizable matrix and reduced to a diagonal form by a small perturbation of its entries. This property cannot be generalized to the case of simultaneous diagonalization of a pair (or m-tuple) of n-order real square matrices. Necessary or sufficient conditions for simultaneous diagonalization have been studied. These studies have been realized under different points of view, for example, analysing the spectra of families of pairs of matrices [8] computing versal deformations [2].
A good tool for distinguishing one subset from another within a differentiable variety could be by trying to identify it from the zeros of bundle sections built on the variety, then, the characteristic classes allow to identify its obstructions. In this particular setup, the interest is about the set of the m-tuples of simultaneously diagonalizable real matrices. Some results about families of pairs of matrices that are simultaneously diagonalizable can be found in [7], [8].
Principal bundles [10] have significant applications in different mathematical areas as topology and differential geometry, in special bundles given by a Lie group action. The first attempts to apply the theory of fiber bundles in the field of physics were made by E. Lubkin [11], who pointed out that the caliber fields had a fiber bundle structure. Further, they form part of the basic framework of gauge theories describing the interaction of forces by differentiating connections [14], and quantum theory [3].
An important object in principal bundles theory is that of connection. Visually, a connection gives us a way to move through the fibers of a principal bundle through isomorphisms between them, which leads us to curvature invariants. In this article, a connection on a specific main bundle is defined as well as the curvature derived from the connection.

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2. Preliminaries

2.1. Simultaneous equivalence of pairs of matrices

The purpose of this section is to give necessary and sufficient conditions that for two pairs of matrices,$T=(X_1,X_2),T'=(Y_1,Y_2)$ are simultaneously diagonalizable. First of all, we define the simultaneous similarity equivalence relating the elements of $\mathfrak{M}$.

Definition 2.1. *Let $T=(X_1,X_2),T'=(Y_1,Y_2)$ be two pairs of matrices in $\mathfrak{M}$. Then, T is simultaneously similar to T' if and only if there exists $P\in\mathfrak{G}=Gl(n;\mathbb{R})$ such that*

$$T'=(Y_1,Y_2)=(PX_1P^{-1},PX_2P^{-1})=PTP^{-1}. \tag{2.1}$$

A particular case of pairs of matrices is that those that are similar to a pair of matrices which are both diagonal, that is, they diagonalize simultaneously.

Definition 2.2. *The pair of matrices $T=(X_1,X_2)$ is simultaneously diagonalizable if and only if there exists an equivalent pair formed by diagonal matrices.*

Necessary conditions for simultaneous diagonalizable pairs can be found in the following propositions(see [7], [9]):

Proposition 2.3. *Let $T=(X_1,X_2)$ be a simultaneously diagonalizable pair. Then both matrices X_i must be diagonalizable. (The converse is false).*

Proposition 2.4. *Let $T=(X_1,X_2)$ be a simultaneously diagonalizable pair. Then, the Lie bracket $[X_1,X_2]=0$.*

Regarding sufficient conditions, we have the following results.

Theorem 2.5. *Let $T=(X_1,X_2)$ be a pair of commuting n-order square matrices and suppose that the matrix X_i, for some $i=1,2$, is diagonalizable with simple eigenvalues ($\lambda_j\neq\lambda_k$ for all $j\neq k$; $k,j=1,\dots n$). Then T is a pair of simultaneously diagonalizable matrices.*

Proof. Without loss of generality, we can assume that X_1 is diagonalizable.
Let $v_1,\dots,v_n$ be a basis such that $X_1(v_i)=\lambda_iv_i$ for $i=1,\dots,n$.
Then, $X_1(X_2v_i)=X_2(X_1v_i)=\lambda_iX_2v_i$.
So, if $X_2v_i\neq 0$, it is an eigenvector of X_1 of eigenvalue λ_i, but condition $\lambda_k\neq\lambda_\ell$ implies that $\dim\operatorname{Ker}(X_1-\lambda_jI)=1$, then, $X_2v_i=\mu_2v_i$, that is to say v_i is an eigenvector for X_2 of eigenvalue μ_2. If $X_2v_i=0$ the vector v_i is an eigenvector of X_2 of eigenvalue zero. That is to say, $\{v_1,\dots,v_n\}$ is a basis of eigenvectors for X_2, and T is a pair of simultaneous diagonalizable matrices with $P=\begin{pmatrix}v_1^t & \dots & v_n^t\end{pmatrix}^{-1}$. □

Remark 2.6. *The matrix X_2 does not necessarily have simple eigenvalues.*

Theorem 2.7. *Let X_1,X_2 be two n-order square matrices. If X_1 and X_2 are commuting and diagonalizable matrices, then, $T=(X_1,X_2)$ is simultaneously diagonalizable.*

Proof. Let P_1 be an invertible matrix such that $D_1=P_1X_1P_1^{-1}=\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}$ with $D_i^1=\lambda_i^1I\in M_{n_i}(\mathbb{C})$, $1\leq i\leq r$ and $n_1+\dots+n_r=n$.
Let us consider $v_{1_1},\dots,v_{n_1},\dots,v_{1_r},\dots,v_{n_r}$ the vector columns of P_1^{-1}, then

$$\begin{aligned}X_2X_1v_{i_\ell}&=X_2\lambda_\ell v_{i_\ell}=\lambda_\ell X_2v_{i_\ell}\\ X_2X_1v_{i_\ell}&=X_1X_2v_{i_\ell}\end{aligned}$$

Consequently $X_2v_{i_\ell}$ is an eigenvector of X_1 of eigenvalue λ_ℓ or $X_2v_{i_\ell}=0$, in any case we have that $X_2v_{i_\ell}\in[v_{1_\ell},\dots,v_{n_\ell}]=F_\ell$, consequently, $X_2F_\ell\subset F_\ell$. That is, the subspace F_ℓ is X_2 invariant for all $1\leq\ell\leq r$, and $P_1X_2P_1^{-1}$ is block-diagonal matrix

$$P_1X_2P_1^{-1}=\begin{pmatrix}Y_1^2 & & \\ & \ddots & \\ & & Y_r^2\end{pmatrix}$$

where the size of each block Y_j^2 is the same of the corresponding block D_j^1 in the matrix $P_1X_aP_1^{-1}$.
If all matrices Y_k^2 are diagonal the proof is concluded. If any submatrix Y_k^2 is not diagonal, then taking into account that the matrix X_2 diagonalizes, all submatrices Y_k^2 diagonalize.

Consider $P_2=\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}$ where P_2^j diagonalizes Y_2^2 for $1\leq j\leq r$.
Clearly P_2 diagonalizes D_1:

$$\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}\begin{pmatrix}P_2^1 & & \\ & \ddots & \\ & & P_2^r\end{pmatrix}^{-1}=\begin{pmatrix}P_2^1D_1^1(P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^rD_r^1(P_2^r)^{-1}\end{pmatrix}=\begin{pmatrix}P_2^1\lambda_1^1I(P_2^1)^{-1} & & \\ & \ddots & \\ & & P_2^r\lambda_r^1I(P_2^r)^{-1}\end{pmatrix}=\begin{pmatrix}D_1^1 & & \\ & \ddots & \\ & & D_r^1\end{pmatrix}$$

Then P_2P_1 diagonalizes X_1 and X_2.

□

2.1.1. Invariant polynomials associated to a pair of matrices under similarity

We are going to construct polynomials $\mathscr{P}(T)$ with $2n^2$ variables x^1_{ij}, x^2_{ij} $1 \leq i, j \leq n$, corresponding to the elements of the pair of matrices $T = (X_1, X_2) = ((x^1_{ij}), (x^2_{ij}))$.

Example 2.8. *Let $T = ((x^1_{ij}), (x^2_{ij}))$ be a pair of matrices. We can define the polynomial*

$$\mathscr{P}(T) = trace\, X_1 + trace\, X_2 = x^1_{11} + \ldots + x^1_{nn} + x^2_{11} + \ldots + x^2_{nn}.$$

We are interested in those which will be invariant under simultaneous similarity in the following sense.

Definition 2.9. *Let $T \in \mathfrak{M}$. A polynomial $\mathscr{P}(T)$ is called invariant under similarity, if $\mathscr{P}(T) = \mathscr{P}(PTP^{-1})$ for all $P \in GL(n;\mathbb{R})$.*

For this study, we will use the characteristic polynomials associated with each matrix of the pair.
Given the pair of matrices $T = (X_1, X_2)$, we can associate it with the following polynomial:

$$\sigma_T(t) = \det(tI - X_1) \cdot \det(tI - X_2) = \prod_{j=1}^{2} \det(tI - X_j) \tag{2.2}$$

Proposition 2.10. *The polynomial (2.2) is invariant under simultaneous similarity.*

Proof.

$$\sigma_{PTP^{-1}}(t) = \prod_{j=1}^{2} \det(tI - PX_jP^{-1}) = (\det P)^2 (\det P^{-1})^2 \prod_{j=1}^{2} \det(tI - X_j) = \sigma_T(t)$$

□

The polynomial $\sigma_T(t)$ can be written in the following manner

$$\sigma_T(t) = \prod_{j=1}^{2} \left(\Sigma_{i=0}^{n} \sigma_i^j(X_j) t^i \right) = \sum_{i=0}^{2n} \sigma_i(T) t^i$$

where, clearly, $\sigma_0(T) = \prod_{j=1}^{2} \det X_j$ and $\sigma_{2n}(T) = 1$.
For the set of variables x^1_{ij}, x^2_{ij} $1 \leq i, j \leq n$, we consider the corresponding pair $T = ((x^1_{ij}), (x^2_{ij}))$. Then, the polynomials $\sigma_i(T)$ are homogeneous polynomials in the given variables.

Proposition 2.11. *Each polynomial $\sigma_i(T)$ is an invariant polynomial.*

Proof. It suffices to note that $\sigma_i^j(X_j)$ is invariant. □

Let be now $T = (X_1, X_2)$, a simultaneously diagonalizable pair, then and taking into account the invariance of the characteristic polynomial we have that $\sigma_i(T)$ are expressed in terms of the eigenvalues of X_i, $i = 1, 2$:

$$\prod_{j=1}^{2} \det(tI - X_j) = \prod_{j=1}^{2} \prod_{k=1}^{n} (t - \lambda_k^j) = \Sigma_{i=0}^{2n} \sigma_i(\lambda_1^1, \ldots, \lambda_n^1, \lambda_1^2, \ldots, \lambda_n^2) t^i$$

Then, in this class of pairs of matrices, these polynomials can be expressed with $2n$ variables instead of the $2n^2$ variables intervening in the general case.

2.2. Fiber Bundles

Following Husmoller [10], a fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces called the total space, base space of the bundle, and the fiber, respectively, and $\pi : E \to B$ is a continuous surjection called the bundle projection, satisfying the following local triviality condition: for every $x \in E$, there is an open neighborhood $U \subset B$ of $\pi(x)$ (called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd}$$

where $\pi_1 : U \times F \to U$ is the natural projection. The set of all $\{(U_i, \varphi_i)\}$ is called a local trivialization of the bundle.
Thus for any $p \in B$, $\pi^{-1}(\{p\})$ is homeomorphic to F and is called the fiber over p.
A trivial example of a bundle is the one given by

$$(B \times F, \pi, B, F)$$

where $\pi : B \times F \longrightarrow B$ is the projection on the first factor, in this case, the fibers are $\{p\} \times F$ for all $p \in B$.
A fiber bundle (E', B', π', F') is a subbundle of (E, B, π, F) provided E' is a subspace of E, B' is a subspace of B, and π' is the restriction of π to E', $\pi' = \pi_{E'} : E' \longrightarrow B'$,

In the special case where the fiber is a group G, the fiber bundle is called the principal bundle. In this case, any fiber $\pi^{-1}(b)$ is a space isomorphic to G. More specifically, G acts freely without fixed points on the fibers.
In the case where E, B and F are smooth manifolds and all the functions above are smooth maps, the fiber bundle is called a smooth fiber bundle.
It is possible to induce bundles in the following manner.
Let $\pi : E \longrightarrow B$ be a fiber bundle with fiber F and let $f : B' \longrightarrow B$ be a continuous map. Then, a fiber bundle over B' can be deduced as follows:

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \subseteq B' \times E$$

and equip it with the subspace topology and the projection map $\pi' : f^*E \longrightarrow B'$ defined as the projection onto the first factor:

$$\pi'(b', e) = b'$$

Defining f' so that the following diagram is commutative

$$\begin{tikzcd} f^*E \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"'] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

we have that (f^*E, B', π') is a fiber bundle so that the fibers on $b \in B$ correspond to the fibers on $f^{-1}(b)$.
An important concept on fiber bundles is the cross-section notion.

Definition 2.12. *A cross section of a bundle (E, B, π, F) is a map $s : B \longrightarrow E$ such that $\pi s = I_B$. In other words, a cross section is a map $s : B \longrightarrow E$ such that $s(b) \in \pi^{-1}(b)$, the fibre over b, for each $b \in B$.*

Let (E', B, π', F') be a subbundle of (E, B, π, F), and let s be a cross section of (E, B, π, F). Then s is a cross section of (E', B, π', F') if and only if $s(b) \in E'$ for each $b \in B$.
One of the main goals of studying cross sections is to account for the existence or non-existence of global sections. When there are some problems with constructing a global section, one says that there are some obstructions.

3. Bundle of pairs of matrices

3.1. Lie group actions

The simultaneous equivalence relation defined in (2.1), can be seen as the action of a Lie group $\mathfrak{G}$ over $\mathfrak{M}$ in the following manner:
Let us consider the following map:

$$\begin{aligned} \alpha : \mathfrak{G} \times \mathfrak{M} &\longrightarrow \mathfrak{M} \\ (P, T) &\longrightarrow PTP^{-1} = (PX_1P^{-1}, PX_2P^{-1}) \end{aligned}$$

that verifies

i) If $I \in \mathfrak{G}$ is the identity element, then $\alpha(I, T) = T$ for all $T \in \mathfrak{M}$.
ii) If P_1 and P_2 are in $\mathfrak{G}$, then $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1P_2, T)$ for all $T \in \mathfrak{M}$.
Indeed: $\alpha(P_1, \alpha(P_2, T)) = \alpha(P_1, P_2TP_2^{-1}) = P_1P_2TP_2^{-1}P_1^{-1} = (P_1P_2)T(P_1P_2)^{-1} = \alpha(P_1P_2, T)$

So, the map α defines an action of $\mathfrak{G}$ over the differentiable manifold $\mathfrak{M}$ that allows seeing the equivalent classes as differentiable manifolds providing a hard link between geometry and algebra.
Analogously we can define an action of $\mathfrak{G}$ over $\mathfrak{G} \times \mathfrak{M}$ in the following manner:

$$\begin{aligned} \beta : \mathfrak{G} \times (\mathfrak{G} \times \mathfrak{M}) &\longrightarrow \mathfrak{G} \times \mathfrak{M} \\ (Q, (P, T)) &\longrightarrow (PQ^{-1}, \alpha(Q^{-1}, T)). \end{aligned}$$

It will be denoted by β_Q the restriction of β to the set $\{Q\} \times (\mathfrak{G} \times \mathfrak{M})$ and by $\beta_{(P,T)}$ the restriction of β to the set $\mathfrak{G} \times \{(P, T)\}$.

Proposition 3.1. *The $\mathfrak{G}$-action β is free, transitive and its orbits are diffeomorphic to $\mathfrak{G}$*

Proof. Suppose that $\beta(Q, (P, T)) = (P, T)$, so

$$\begin{aligned} \beta(Q, (P, T)) &= (PQ^{-1}, \alpha(Q^{-1}, T)) \\ &= (PQ^{-1}, Q^{-1}TQ) \\ &= (P, T) \end{aligned}$$

then, $PQ^{-1} = P$ and $Q^{-1}TQ = T$ and $Q = I$.

$$\begin{aligned}\beta(R,\beta(Q,(P,T))) &= \beta(R,(PQ^{-1},\alpha(Q,T)))\\ &= \beta(R,(PQ^{-1},QTQ^{-1}))\\ &= (PQ^{-1}R^{-1},\alpha(R,QTQ^{-1}))\\ &= (PQ^{-1}R^{-1},RQTQ^{-1}R^{-1})\\ &= (P(RQ)^{-1},\alpha(RQ,T))\\ &= \beta(RQ,(P,T)),\end{aligned}$$

Let us denote by $\mathscr{O}(P,T)$ the orbit of T under $\mathfrak{G}$-action $\mathscr{O}(P,T) = \{(\overline{P},\overline{T}) = \beta(Q,(P,T)), \forall Q \in \mathfrak{G}\}$

$$\begin{aligned}\varphi : \mathfrak{G} &\longrightarrow \mathscr{O}(P,T)\\ Q &\longrightarrow (\overline{P},\overline{T}) = \beta(Q,(P,T))\end{aligned}$$

φ is a diffeomorphism:
If $\varphi(Q) = \varphi(\overline{Q})$, then $PQ = P\overline{Q}$ consequently $Q = \overline{Q}$
And, for $(\overline{P},\overline{T}) \in \mathscr{O}(P,T)$, there exists $Q \in \mathfrak{G}$ with $(\overline{P},\overline{T}) = (PQ^{-1},QTQ^{-1})$, so $\varphi(Q) = (\overline{P},\overline{T})$. □

The set $\mathfrak{M}$ is identified as the set of orbits class $\mathfrak{G}\times\mathfrak{M}/\beta$.

Proposition 3.2. *There exists a bijection between* $\mathfrak{M}$ *and* $\mathfrak{G}\times\mathfrak{M}/\beta$.

Proof. We define f as

$$\begin{aligned}\mathfrak{G}\times\mathfrak{M}/\beta &\longrightarrow \mathfrak{M}\\ (P,T)\circ\mathfrak{G} &\longrightarrow T'\end{aligned}$$

where T' is in such a way that there exists $Q \in \mathfrak{G}$ such that $\beta(Q,(P,T)) = (I,T')$

1) It suffices to take $Q = P$ to obtain $T' = P^{-1}TP$.
2) f is well-defined because of unicity of T':
   Let $(I,T') \sim (I,T'')$, then, there exists Q such that

   $$\beta(Q,(I,T')) = (IQ^{-1},\alpha(Q^{-1},T')) = (I,T'').$$

   So, $IQ^{-1} = I$ and $Q^{-1} = I = Q$ and $IQ^{-1}\alpha(Q^{-1},T') = \alpha(I,T') = T'$.
3) f is bijective:
   If $f(I,T')\circ\mathfrak{G} = f(I,T'')\circ\mathfrak{G}$, then $T' = T''$ and $f(I,T')\circ\mathfrak{G} = f(I,T'')\circ\mathfrak{G}$, so f is injective.
   And, clearly, for all $T \in \mathfrak{M}$, $f(I,T)\circ\mathfrak{G} = T$ and f is surjective.

□

Proposition 3.3. *The* $\mathfrak{G}$*-action preserves the fibers* $F_T = \alpha^{-1}(T)$ *of* $\alpha : \mathfrak{G}\times\mathfrak{M} \longrightarrow \mathfrak{M}$.

Proof. Let $(P,\overline{T}) \in \alpha^{-1}(T)$, then

$$\alpha(Q,(P,\overline{T})) = (PQ^{-1},QTQ^{-1}) = PQ^{-1}Q\overline{T}Q^{-1}QP^{-1} = P\overline{T}P^{-1} = T.$$

So, $(PQ^{-1},QTQ^{-1}) \in \alpha^{-1}(T)$. □

From propositions 3.1 and 3.3 we can deduce the following result.

Proposition 3.4. $(\mathfrak{G}\times\mathfrak{M},\mathfrak{M},\alpha,\mathfrak{G})$ *is a principal fiber bundle.*

Clearly, we observe that F_T is diffeomorphic to $\mathfrak{G}$:

$$\begin{aligned}\psi : F_T &\longrightarrow \mathfrak{G}\\ (Q,\overline{T}) &\longrightarrow Q\end{aligned}$$

If $\psi(Q,\overline{T}) = (\overline{Q},\overline{\overline{Q}})$, then $Q = \overline{Q}$ and $Q\overline{T}Q^{-1} = \overline{Q}\overline{\overline{T}}\overline{Q}^{-1} = Q\overline{\overline{T}}Q^{-1}$, so $\overline{T} = \overline{\overline{T}}$ and the map ψ is injective.
On the other hand, for all $Q \in \mathfrak{G}$, there exists $(Q,Q^{-1}TQ) \in F_T$ such that $\psi(Q,Q^{-1}TQ) = T$, so, the map ψ is surjective.

4. Connections and curvature

A connection is a mathematical object defined over a differentiable manifold that allows the local geometry around a point to be related to local geometry around another point in the manifold. The connection is an object that shows us how to derive local sections and thus compare the fibers on different points of the base space [4].
Curvature is useful to obtain characteristic classes that are global invariants that measure the deviation of the local product structure from a global product structure. The theory of characteristic classes generalizes the idea of obstructions to construct cross-sections of fiber bundles.
Let us use the notation $T_I\mathfrak{G}$ for the tangent space to the manifold $\mathfrak{G}$ at the unit element I. Since $\mathfrak{G}$ is an open subset of $M_n(\mathbb{R})$, we have that $T_I(\mathfrak{G}) = M_n(\mathbb{R})$, and, since $\mathfrak{M}$ is a linear space, $T_T(\mathfrak{M}) = \mathfrak{M}$, then $T_{(I,T)}(\mathfrak{G}\times\mathfrak{M}) = M_n \times \mathfrak{M}$.
The action β of $\mathfrak{G}$ over $\mathfrak{G}\times\mathfrak{M}$ permits us to construct a vertical subspace $T_{(P,T)}\mathscr{O}(P,T) \subset T(\mathfrak{G}\times\mathfrak{M})$.

$$T_{(P,T)}\mathscr{O}(P,T) = \mathscr{I}mg\, d\beta_{(P,T)} = \{(-PQ,[T,Q]) \mid \forall Q \in T\mathfrak{G}\}$$

where $[T,Q] = ([X_1,Q],[X_2,Q])$.
(To describe $\mathscr{I}mg\, d\beta_{(P,T)}$ it suffices to compute the linear approximation of $\beta_{(P,T)}(I+\varepsilon Q)^{-1} \sim \beta_{(P,T)}(I-\varepsilon Q)$).
The subspace $T_{(P,T)}\mathscr{O}(P,T)$ is generated by $d\beta(A_{ij})$ with $\{A_{ij}\}$ a basis for $T\mathfrak{G} = M_n(\mathbb{R})$.
Consider the Euclidean scalar product in the space $T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ defined as:

$$\langle (P,T_1),(Q,T_2\rangle = \langle (P,(X_1,X_2)),(Q,(Y_1,Y_2))\rangle = \operatorname{tr} P\overline{Q}^t + \operatorname{tr} X_1\overline{Y}_1^t + \operatorname{tr} X_2\overline{Y}_2^t = \operatorname{tr} P\overline{Q}^t + \operatorname{tr} T_1\overline{T}_2^t.$$

An orthogonal element $(X,Y) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ to $T_{(P,T)}\mathscr{O}(P,T)$ is a solution of the equation:

$$\langle (-PQ,[T,Q]),(X,Y)\rangle = \operatorname{tr}(-PQ\overline{X}^t) + \operatorname{tr}([T,Q]\overline{Y}^t) = 0.$$

It is possible to construct a horizontal subspace

$$\begin{aligned} T_{(P,T)}\mathscr{O}(P,T)^{\perp} &= \{(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \mid -\overline{A}^t P + \overline{B}^t T - T\overline{B}^t = 0\} \\ &= \{(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \mid -\overline{A}^t P + [\overline{B}^t, T] = 0\}, \end{aligned}$$

where $[\overline{B}^t, T]$ denotes $[\overline{B}_1^t, X_1] + [\overline{B}_2^t, X_2]$.

Definition 4.1. *Given a principal bundle $(\mathfrak{G}\times\mathfrak{M},\mathfrak{M},\alpha,\mathfrak{G})$, a differentiable distribution $\mathscr{H}$ of fields over $\mathfrak{G}\times\mathfrak{M}$ such that for each point $(P,T)\in\mathfrak{G}\times\mathfrak{M}$, the subspace $H_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \subset T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ is called connection if it verifies:*

a) $T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) = V_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \oplus H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$
b) For each $(P,T)\in\mathfrak{G}\times\mathfrak{M}$ and for each $Q\in\mathfrak{G}$, for the translation $\beta_Q(P,T) = (PQ^{-1}, Q^{-1}TQ)$, the space $H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ fulfills $H_{\beta_Q(P,T)}(\mathfrak{G}\times\mathfrak{M}) = H_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$

So, the connections allow us to decompose the vectors into a vertical part $V_u(E)$ and a horizontal part $H_u(E)$, which we will call the vertical and horizontal subspace, respectively, of $T_u(E)$ concerning the connection $\mathscr{H}$.

Proposition 4.2. *The subgroup $T_{(P,(X_1,X_2))}\mathscr{O}(P,(X_1,X_2))^{\perp}$ verifies the conditions of definition 4.1.*

Proof. Let $(C,D) \in T_{\beta_Q(P,T)}\mathscr{O}(\beta_Q(P,T))^{\perp}$, then,

$$-\overline{C}^t PQ + \overline{D}_1^t Q^{-1}X_1Q - Q^{-1}X_1Q\overline{D}_1^t + \overline{D}_2^t Q^{-1}X_2Q - Q^{-1}X_2Q\overline{D}_2^t = 0$$

or, equivalently:

$$-Q\overline{C}^t P + Q\overline{D}_1^t Q^{-1}X_1 - X_1Q\overline{D}_1^t Q^{-1} + Q\overline{D}_2^t Q^{-1}X_2 - X_2Q\overline{D}_2^t Q^{-1} = 0$$

Then, setting $\overline{E}^t = Q\overline{C}^t$, $\overline{F}_1^t = Q\overline{D}_1^t Q^{-1}$ and $\overline{F}_2^t = Q\overline{D}_2^t Q^{-1}$, we have that equivalently $(E,F) \in T_{(P,T)}\mathscr{O}(P,T)^{\perp}$. □

The bijectivity of the map

$$\begin{aligned} d\beta_{(P,T)}(\mathfrak{G}) &\longrightarrow T_{(P,T)}\mathscr{O}(P,T) \\ Q &\longrightarrow (-PQ, X_1Q - QX_1, X_2Q - QX_2) \end{aligned}$$

permit us to define a 1-form ω over $\mathfrak{G}\times\mathfrak{M}$ with values in the Lie algebra $\mathfrak{G}$ as follows: Given $(A,B) \in T_{(P,T)}(\mathfrak{G}\times\mathfrak{M})$ there exists a unique element $Q \in T\mathfrak{G}$ such that the vertical component of (A,B) is $A_v = d\beta_{(P,T)}Q$, thus we define $\omega(A,B) = Q$. It is clear that $\omega(A,B) = 0$ if and only if (A,B) is horizontal.
From this 1-form, it is possible to build a 2-form Ω in the following maner:

$$\begin{aligned} \Omega : T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) \times T_{(P,T)}(\mathfrak{G}\times\mathfrak{M}) &\longrightarrow T_I\mathfrak{G} \\ (X,Y) = ((P_1,T_1),(P_2,T_2)) &\longrightarrow \Omega((P_1,T_1),(P_2,T_2)) = \Omega(X,Y) \end{aligned}$$

verifying:

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X),\omega(Y)] \tag{4.1}$$

and (4.1) is called curvature of the connection.

5. Conclusion

In this work, we build bundles to obtain algebraic objects providing more geometric information about the space of pairs of matrices. With them, we consider an operator called connection, and we define the curvature associated with it. These are ingredients to obtain invariants that measure in a certain way how the local product structure of the bundle separates from a global product structure.

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Author's contributions

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