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Approximation Properties of The Nonlinear Jain Operators

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Abstract
We define the nonlinear Jain operators of max-product type. We studied approximation properties of these operators.

Keywords: *Nonlinear max-product operators; max-product Jain operators; degree of approximation.*

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1. Introduction

The main topic in the classical approximation theory is approximating a continuous function $f:[a,b]\to R$ with more elementary functions such as polynomials, trigonometric functions, etc.. The well-known Korovkin's theorem, which gives a simple proof of Weierstrass theorem, is based on the approximation of functions by linear and positive operators. The underlying algebraic structure of these mentioned operators is linear over R and they are also linear operators. In 2006, Bede et.al [4] asked whether they could change the underlying algebraic structure to more general structures. In this sense they presented nonlinear Shepard-type operators by replacing the operations sum and product by max and product. They proved Weierstrass-type uniform approximation theorem and obtained error estimates in terms of the modulus of continuity. Following this paper Bede et. al. [5] defined and studied pseudo linear approximation operators. Based upon these studies, there appeared an open problem in the book of S.Gal [10] in which the max-product type Bernstein operators were introduced. Related to this open problem, a nonlinear modification of the classical Bernstein operators were first studied by Bede and Gal [3] (see also [2]). The idea behind these studies were also applied to other well-known approximating operators. Several authors introduced the nonlinear versions of the stated operators and studied order of approximation [3,4,12]. Also see [6] for the collected papers.

The nonlinear Favard-Szasz-Mirakjan operators of max-product kind is introduced in [2] as (here $\bigvee$ means

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maximum)

$$F_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n})}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}$$

whose order of pointwise approximation is obtained as $\omega_1(f; \sqrt{x}/\sqrt{n})$. In [7], the authors dealed with the same operator in order to obtain the same order of approximation but by a simpler method. They also presented some shape preserving properties of the operators.

In 1972, Jain [11] introduced the following operators to generalize classical Szász-Mirakyan operators : for $\lambda > 0$ and $0 \leq \beta < 1$,

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} \omega_\beta(k, nx) f\left(\frac{k}{n}\right), f \in C[0,\lambda], n \in \mathbb{N}$$

where the basis function is

$$\omega_\beta(k,x) = x(x+k\beta)^{k-1} \frac{e^{-(x+k\beta)}}{k!}; k = 0,1,2,\dots,$$

and

$$\sum_{k=0}^{\infty} \omega_\beta(k,x) = 1.$$

It is easy to see that for $\beta = 0$, the operator reduces to the classical Szász-Mirakyan operators. Farcas [9] proved a Voronovskaja type result for Jain's operators. Doğru et. al. [8] investigated a modification of the Jain operators preserving the linear functions. Recently, Özarslan [12] introduced the Stancu type generalization of Jain's operators and investigated the weighted approximation properties and Olgun et. al. [13] introduced a generalization of Jain's operators based on a function ρ. Also, Bernstein and generalizations of Jain operators were studied by many authors (see [14]-[21].) The aim of this study is to introduce the nonlinear Jain operators of max-product type and estimate the rate of pointwise convergence of the operators**.** The non-truncated Jain operators are defined by

$$T_{n,\beta}^{(M)}(f;x) = \frac{\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x) f(\frac{k}{n})}{\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x)}, n \in \mathbb{N} \tag{1.1}$$

where $W_{n,k,\beta}(x) = (nx+k\beta)^{k-1} \dfrac{e^{-(nx+k\beta)}}{k!}$ and $f : [0,\lambda] \to \mathbb{R}_+$ is considered as a bounded function on $[0,\lambda]$, $\lambda > 0$.

2. Preliminaries

Here, it is emphasized some general notations about the nonlinear operators of max-product kind. Over the set of positive reals, $\mathbb{R}_+$, we deal with the operations $\bigvee$ (maximum) and $\cdot$ (product). Then $(\mathbb{R}_+, \bigvee, \cdot)$ has a semiring structure and it is called as Max-Product algebra.

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

A discrete max-product type approximation operator $L_n : CB_+(I) \to CB_+(I)$, has a general form

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x,x_i) \cdot f(x_i),$$

or

$$L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x,x_i) \cdot f(x_i)$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(\cdot,x_i) \in CB_+(I)$ and $x_i \in I$, for all $i = \{0,1,2,\cdots\}$. These operators are nonlinear, positive operators and satisfy a a pseudo-linearity condition of the form

$$L_n(\alpha\cdot\ f\vee\beta\cdot\ g)(x)=\alpha\cdot\ L_n(f)(x)\vee\beta\cdot\ L_n(g)(x),\forall\alpha,\beta\in\mathbb{R}_+,f,g:I\to\mathbb{R}_+.$$

In order to give some properties of the operators L_n, we present the following auxiliary Lemma.

Lemma 2.1. ([2]) Let $I\subset\mathbb{R}$ be a bounded or unbounded interval,

$$CB_+(I)=\{f:I\to\mathbb{R}_+;\ f\text{ continuous and bounded on }I\},$$

and $L_n:CB_+(I)\to CB_+(I)$, $n\in\mathbb{N}$ be a sequence of operators satisfying the following properties :

(i) (Monotonicity)

$f,g\in CB_+(I)$ satisfy $f\le g$ then $L_n(f)\le L_n(g)$ for all $n\in\mathbb{N}$;

(ii) (Subadditivity)

$L_n(f+g)\le L_n(f)+L_n(g)$ for all $f,g\in CB_+(I)$.

Then for all $f,g\in CB_+(I)$, $n\in\mathbb{N}$ and $x\in I$ we have

$$|L_n(f)(x)-L_n(g)(x)|\le L_n(|f-g|)(x).$$

Remark 2.1. Max-product for Jain operators defined by (4) verify the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition

$$L_n(f\vee g)(x)=L_n(f)(x)\vee L_n(g)(x),f,g\in CB_+(I).$$

Indeed, taking in the above equality $f\le g,f,g\in CB_+(I)$, it easily follows $L_n(f)(x)\le L_n(g)(x)$.

Furthermore, the Jain operators of max-product type is positive homogenous, that is $L_n(\lambda f)=\lambda L_n(f)$ for all $\lambda\ge 0$.

Corollary 2.2. ([2]) Let $L_n:CB_+(I)\to CB_+(I)$, $n\in\mathbb{N}$ be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 1 and in addition be a positive homogenous operator. Then for all $f\in CB_+(I)$,$n\in\mathbb{N}$ and $x\in I$ we have

$$|f(x)-L_n(f)(x)|\le\left[\frac{1}{\delta}L_n(\varphi_x)(x)+L_n(e_0)(x)\right]\omega(f;\delta)+f(x)\cdot\ |L_n(e_0)(x)-1|\,,$$

where $\delta>0$, $e_0(t)=1$ for all $t\in I,\varphi_x(t)=|t-x|$ for all $t\in I,x\in I$.

$$\omega(f;\delta)=\max_{\substack{x,y\in I\\|x-y|\le\delta}}|f(x)-f(y)|$$

is the first modulus of continuity. If I is unbounded then we suppose that there exists $L_n(\varphi_x)(x)\in\mathbb{R}_+\bigcup\{+\infty\}$, for any $x\in I,n\in\mathbb{N}$.

Corollary 2.3. ([2]) Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0)=e_0$, for all $n\in\mathbb{N}$. Then for all $f\in CB_+(I)$, $n\in\mathbb{N}$ and $x\in I$ we have

$$|f(x)-L_n(f)(x)|\le\left[1+\frac{1}{\delta}L_n(\varphi_x)(x)\right]\omega(f;\delta).$$

3. Construction of the Operators and Auxiliary Results

Since $T^{(M)}_{n,\beta}(f)(0)-f(0)=0$ for all n, throughout the paper we may suppose that $x>0$. We need the following notations and Lemmas for the proof the main results.

For each $k,j\in\{1,2,...,\}$ and $x\in\left[\frac{aj+\beta}{n},\frac{a(j+1)+\beta}{n}\right]$,$j=0,x\in\left[0,\dfrac{a+\beta}{n}\right]=\left[0,\dfrac{e^\beta}{n}\right]$, $a=e^\beta-\beta,0\le\beta<1$, let us denote

$$M_{k,n,j}(x):=\frac{W_{n,k,\beta}\left(x\right)\left|\frac{k}{n}-x\right|}{W_{n,j,\beta}\left(x\right)},m_{k,n,j}(x):=\frac{W_{n,k,\beta}\left(x\right)}{W_{n,j,\beta}\left(x\right)}.$$

where $W_{n,k,\beta}$ is defined as in the operators (1.1). It is clear that if $k\ge j+1$ then

$$M_{k,n,j}(x)=\frac{W_{n,k,\beta}(x)\left(\frac{k}{n}-x\right)}{W_{n,j,\beta}(x)}$$

and if $k \leq j$ then

$$M_{k,n,j}(x) = \frac{W_{n,k,\beta}(x)\left(x - \frac{k}{n}\right)}{W_{n,j,\beta}(x)}.$$

Lemma 3.1. Denoting $W_{n,k,\beta}(x) = (nx + k\beta)^{k-1} \dfrac{e^{-(nx+k\beta)}}{k!}$, we have

$$\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x) = W_{n,j,\beta}(x), \text{ for all } x \in \left[\frac{aj + \beta}{n}, \frac{a(j+1) + \beta}{n}\right],$$

where $a = e^{\beta} - \beta,\ j = 1, 2, ..., x \in \left[0, \frac{a+\beta}{n}\right] = \left[0, \frac{e^{\beta}}{n}\right].$

Proof. Firstly, we show that for fixed $n \in \mathbb{N}$ and $0 \leq k$ we have

$$0 \leq W_{n,k+1,\beta}(x) \leq W_{n,k,\beta}(x) \text{ if and only if } x \in \left[0, \frac{a(k+1) + \beta}{n}\right].$$

Indeed, writing the the above inequality explicitly, we have

$$0 \leq (nx + (k+1)\beta)^k \frac{e^{-(nx+(k+1)\beta)}}{(k+1)!} \leq (nx + k\beta)^{k-1} \frac{e^{-(nx+k\beta)}}{k!}.$$

If $x = 0$, this inequality is true. For $x > 0$, after simplifications it becomes

$$\begin{aligned}
\left(\frac{nx + (k+1)\beta}{nx + k\beta}\right)^k &\leq \frac{e^{\beta}(k+1)}{nx + k\beta} \\
(nx + k\beta)\left(\frac{nx + (k+1)\beta}{nx + k\beta}\right)^k &\leq e^{\beta}(k+1) \\
(nx + k\beta)\left(1 + \frac{\beta}{nx + k\beta}\right)^k &\leq e^{\beta}(k+1) \\
nx &\leq \frac{1}{\left(1 + \frac{\beta}{nx+k\beta}\right)^k} e^{\beta}(k+1) - k\beta \\
x &\leq \frac{e^{\beta}(k+1)}{n} - \frac{k\beta}{n} \\
&= \frac{e^{\beta}(k+1) - k\beta}{n} = \frac{\left(e^{\beta} - \beta\right)(k+1) + \beta}{n} \\
&= \frac{a(k+1) + \beta}{n},
\end{aligned}$$

where $a = \left(e^{\beta} - \beta\right),\ 0 \leq \beta < 1$. Then

$$0 \leq x \leq \frac{a(k+1) + \beta}{n}, a = e^{\beta} - \beta.$$

By taking $k = 0, 1, 2, ...$ in the inequality just proved above, we get

$$\begin{aligned}
W_{n,1,\beta}(x) &\leq W_{n,0,\beta}(x), \text{ if and only if } x \in \left[0, \frac{a + \beta}{n}\right], \\
W_{n,2,\beta}(x) &\leq W_{n,1,\beta}(x), \text{ if and only if } x \in \left[0, \frac{2a + \beta}{n}\right], \\
&\vdots \\
W_{n,k+1,\beta}(x) &\leq W_{n,k,\beta}(x), \text{ if and only if } x \in \left[0, \frac{a(k+1) + \beta}{n}\right].
\end{aligned}$$

From the above inequalities, we obtain,

$$\begin{aligned} \text{if } x &\in \left[0,\frac{a+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x)\leq W_{n,0,\beta}(x),\ \text{for all } k=0,1,... \\ \text{if } x &\in \left[\frac{a+\beta}{n},\frac{2a+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x)\leq W_{n,1,\beta}(x),\ \text{for all } k=0,1,... \\ \text{if } x &\in \left[\frac{2a+\beta}{n},\frac{3a+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x)\leq W_{n,2,\beta}(x),\ \text{for all } k=0,1,... \end{aligned}$$

and proceeding in the same manner,

$$\text{if } x\in\left[\frac{aj+\beta}{n},\frac{a\left(j+1\right)+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x)\leq W_{n,j,\beta}(x), \text{for all }\ k=0,1,2,...$$

then we have

$$0\leq W_{n,k+1,\beta}(x)\leq W_{n,k,\beta}(x) \text{ if and only if } x\in\left[\frac{aj+\beta}{n},\frac{a\left(j+1\right)+\beta}{n}\right].$$

Lemma 3.2. For all $k,j\in\{1,2,...,\}$, and $x\in\left[\frac{aj+\beta}{n},\frac{a(j+1)+\beta}{n}\right]$, $j=0, x\in\left[0,\frac{a+\beta}{n}\right]=\left[0,\frac{e^{\beta}}{n}\right]$, we have

$$m_{k,n,j}(x)\leq 1.$$

Proof. We have two cases: 1) $k\geq j$ and 2) $k<j$.
Let $k\geq j$. Since the function $g(x)=\dfrac{1}{x}$ is nonincreasing on $\left[\frac{aj+\beta}{n},\frac{a(j+1)+\beta}{n}\right]$ it follows

$$\begin{aligned} \frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} &= \frac{W_{n,k,\beta}\left(x\right)}{W_{n,k+1,\beta}\left(x\right)}=\frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k+1)\beta)^{k}\dfrac{e^{-(nx+(k+1)\beta)}}{(k+1)!}} \\ &= \frac{(nx+k\beta)^{k}e^{\beta}\left(k+1\right)}{(nx+(k+1)\beta)^{k}(nx+k\beta)}, x\in\left[\frac{aj+\beta}{n},\frac{a\left(j+1\right)+\beta}{n}\right] \\ &\geq 1, \end{aligned}$$

which implies

$$m_{j,n,j}(x)\geq m_{j+1,n,j}(x)\geq m_{j+2,n,j}(x)\geq ...$$

We now turn to the case $k\leq j$

$$\begin{aligned} \frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} &= \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k-1)\beta)^{k-2}\dfrac{e^{-(nx+(k-1)\beta)}}{(k-1)!}} \\ &= \frac{(nx+k\beta)^{k-2}}{(nx+(k-1)\beta)^{k-2}}, x\in\left[\frac{aj+\beta}{n},\frac{a\left(j+1\right)+\beta}{n}\right] \\ &\geq 1, \\ \text{where } \frac{(nx+k\beta)^{k-2}}{(nx+(k-1)\beta)^{k-2}} &= \left(1+\frac{\beta}{nx+(k-1)\beta}\right)^{k-2}\geq 1 \text{ and } \frac{(nx+k\beta)}{e^{\beta}\left(k-1\right)}\geq 1. \end{aligned}$$

which implies

$$m_{j,n,j}(x)\geq m_{j-1,n,j}(x)\geq m_{j-2,n,j}(x)\geq ...$$

Since $m_{j,n,j}(x)=1$, the proof of the lemma is complete.

Lemma 3.3. Let $x\in\left[\frac{aj+\beta}{n},\frac{a(j+1)+\beta}{n}\right]$,

(i) If $k\geq(j+1)$ is such that

$$k-\sqrt{\beta k^{2}+a_{1}k+a_{2}j+a_{3}-a\beta kj}\geq aj,$$

then

$$M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$$

where $a_1 = -\beta^2 + 2e^\beta + 2\beta - 1, a_2 = -2a\beta - 2a - ae^\beta, a_3 = -\beta^2 + 2e^\beta + \beta - \beta e^\beta$.

(ii) If $k \leq j$ is such that

$$k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} \leq aj,$$

then

$$M_{k,n,j}(x) \geq M_{k-1,n,j}(x).$$

where $a_4 = 2\beta - \beta^2 + a + 1, a_5 = -2\beta a$.

Proof. (i) We observe that

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &= \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k+1)\beta)^{k}\dfrac{e^{-(nx+(k+1)\beta)}}{(k+1)!}}\frac{\left(\frac{k}{n}-x\right)}{\left(\frac{k+1}{n}-x\right)} \\
&= \left(1-\frac{\beta}{nx+(k+1)\beta}\right)^{k-1}\frac{e^{\beta}(k+1)}{nx+(k+1)\beta}\frac{(k-nx)}{(k+1-nx)} \\
&\geq \frac{(k+1)}{nx+(k+1)\beta}\frac{(k-nx)}{(k+1-nx)}\left(1-\frac{\beta}{nx+(k+1)\beta}\right)^{k-1}e^{\beta} \\
&\geq \frac{(k+1)}{(j+1)a+(k+1)\beta}\frac{(k-(j+1)a)}{(k+1-ja)},
\end{aligned}$$

$x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$. Then, since the condition

$$k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj} \geq aj,$$

where $a_1 = -\beta^2 + 2e^\beta + 2\beta - 1, a_2 = -2a\beta - 2a - ae^\beta, a_3 = -\beta^2 + 2e^\beta + \beta - \beta e^\beta$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq 1.$$

(ii) We observe that

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} &= \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k-1)\beta)^{k}\dfrac{e^{-(nx+(k-1)\beta)}}{(k-1)!}}\frac{\left(x-\frac{k}{n}\right)}{\left(x-\frac{k-1}{n}\right)} \\
&= \left(1+\frac{\beta}{nx+(k-1)\beta}\right)^{k}\frac{nx+k\beta}{e^{\beta}k}\frac{(nx-k)}{(nx-k+1)} \\
&\geq \frac{ja+\beta+k\beta}{k}\frac{ja+\beta-k}{ja+\beta-k+1}
\end{aligned}$$

Then, since the condition

$$k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} \leq aj,$$

where $a_4 = 2\beta - \beta^2 + a + 1, a_5 = -2\beta a$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq 1,$$

which proves the lemma.

4. Approximation Result

For the function $f \in CB_+(I)$, we obtain the degree of approximation by using the Shisha-Mond Theorem given in [1],[2].

Theorem 4.1. If $f : [0,\lambda] \to \mathbb{R}_+$ is a bounded and continuous function on $[0,\lambda]$, $\lambda > a+1, a = e^{\beta} - \beta, 0 \le \beta < 1$, then we get the following estimate

$$\left| T_{n,\beta}^{(M)}(f)(x) - f(x) \right| \le 6\lambda\omega_1\left(f, \frac{1}{\sqrt{n}}\right), \text{for all } n \in \mathbb{N}, x \in [0,\lambda],$$

where

$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,\lambda], |x-y| \le \delta\}.$$

Proof. Since $T_n^{(M)}(e_0)(x) = 1$ and using the Shisha-Mond Theorem, we have

$$\left| T_n^{(M)}(f)(x) - f(x) \right| \le \left(1 + \frac{1}{\delta_n} T_n^{(M)}(\varphi_x)(x)\right)\omega_1(f,\delta_n)$$

where $(\varphi_x)(t) = |t-x|$. Hence, it is sufficient to estimate the following term

$$E_n(x) := T_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x) \left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x)}$$

Let $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$ and $j \in \{1, 2, ..., \}$ is arbitrarily fixed. By Lemma 3.1 we get

$$E_n(x) = \max_{k=0,1,2,\ldots}\{M_{k,n,j}(x)\}, x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right].$$

For $j = 0$, we get

$$M_{k,n,0}(x) = nx(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{k!}\left|\frac{k}{n} - x\right|, k \ge 0$$

If $k = 0$, then we have

$$M_{0,n,0}(x) = x = \sqrt{x}\sqrt{x} \le \sqrt{x}\sqrt{\frac{a+\beta}{n}} = \sqrt{\frac{e^{\beta}x}{n}} \le \sqrt{\frac{e^{\beta}\lambda}{n}}$$

If $k = 1$ then

$$\begin{aligned} M_{k,n,0}(x) &= nx(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{k!}\left|\frac{k}{n} - x\right|, x \in \left[0, \frac{e^{\beta}}{n}\right] \\ &= nx(nx+\beta)^{0}\frac{e^{-\beta}}{1!}\left|\frac{1}{n} - x\right| \\ &\le xe^{-\beta} = \sqrt{x}\sqrt{x}e^{-\beta} \\ &\le \sqrt{\frac{xe^{\beta}}{n}} \le \sqrt{\frac{e^{\beta}\lambda}{n}}. \end{aligned}$$

If $k \ge 2$ then

$$\begin{aligned} M_{k,n,0}(x) &= nx(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{k!}\left|\frac{k}{n} - x\right|, x \in \left[0, \frac{e^{\beta}}{n}\right] \\ &\le x(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{(k-1)!} \\ &\le x \\ &\le \sqrt{\frac{e^{\beta}\lambda}{n}}. \end{aligned}$$

So, we obtain an upper estimate for each $M_{k,n,j}(x)$ where $j \in \{1,2,...,\}$ is fixed, $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$ and $k = 1,...,$. Actually, we will prove that

$$M_{k,n,j}(x) \le \max\left\{\frac{\sqrt{\max\{a_4,a_5\}}+2a}{\sqrt{n}}, \sqrt{\frac{e^{\beta}\lambda}{n}}, \frac{\sqrt{\max\{a_1,a_2\}}}{\sqrt{n}}\right\},$$

for all $x \in [0,\lambda], n \in \mathbb{N}$.

The proof of the inequality (2) will be investigated by the following cases:

1) $k \ge (j+1)$ and 2)$k \le j$.

Case 1) Subcase a) Initially, let take

$$k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj} < aj,$$

then we get

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)\left(\frac{k}{n} - x\right) \\ &\le \left(\frac{k}{n} - x\right) \le \left(\frac{k}{n} - \frac{ja+\beta}{n}\right) \\ &\le \frac{k}{n} - \frac{k}{n} + \frac{\sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj}}{n} \\ &\le \frac{\sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj}}{n} \\ &\le \frac{\sqrt{a_1 + a_2 j}}{n} \le \sqrt{\max\{a_1,a_2\}}\frac{1}{\sqrt{n}}. \end{aligned}$$

Subcase b) Now let $k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj} \ge aj$.

Since the function $g(x) = x - \sqrt{\beta x^2 + a_1 x + a_2 j + a_3 - a\beta xj}$ is nondecreasing, it follows that there exists $\overline{k} \in \{2,3,...,\}$,of maximum value, such that $\overline{k} - \sqrt{\beta \overline{k}^2 + a_1\overline{k} + a_2 j + a_3 - a\beta\overline{k}j} < aj$. Then for $k_1 = \overline{k} + a$ we get $k_1 - \sqrt{\beta k_1^2 + a_1 k_1 + a_2 j + a_3 - a\beta k_1 j} \ge aj$,

$$\begin{aligned} M_{\overline{k}+a,n,j}(x) &= m_{\overline{k}+a,n,j}(x)\left|\frac{\overline{k}+a}{n} - x\right| \\ &\le \left(\frac{\overline{k}+a}{n} - \frac{\overline{k} - \sqrt{\beta\overline{k}^2 + a_1\overline{k} + a_2 j + a_3 - a\beta\overline{k}j}}{n}\right) \\ &\le \sqrt{\max\{a_1,a_2\}}\frac{1}{\sqrt{n}}. \end{aligned}$$

The last above inequality follows from the fact that

$\overline{k} - \sqrt{\beta\overline{k}^2 + a_1\overline{k} + a_2 j + a_3 - a\beta\overline{k}j} < aj$ necessarily implies $k < 3aj$. Also, we have $k_1 \ge (j+1)$. Indeed, this is a consequence of the fact that g is nondecreasing and because is easy to see that $g(j) < j$. By Lemma 3.3, (i) it follows that $M_{\overline{k}+1,n,j}(x) \ge M_{\overline{k}+2,n,j}(x) \ge ...$

Hence, we get $M_{k,n,j}(x) \le \sqrt{\max\{a_1,a_2\}}\frac{1}{\sqrt{n}}$ for any $\overline{k} \in \{\overline{k}+1, \overline{k}+2,...,\}$.

Case 2) Subcase a) Firstly, let $k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} > aj$. Then we get,

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)\left(x - \frac{k}{n}\right) \\ &\le \frac{a(j+1)+\beta}{n} - \frac{k}{n} \\ &\le \frac{k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} + \beta}{n} - \frac{k}{n} \\ &\le \frac{\sqrt{\max\{a_4,a_5\}} + \beta}{\sqrt{n}}. \end{aligned}$$

Subcase b) Suppose now that $k+\sqrt{\beta k^2+a_4k+a_5j-\beta^2-a\beta kj}\le aj$. Let $\widetilde{k}\in\{1,2,...,\}$ be the minimum value such that

$$\widetilde{k}+\sqrt{\beta\widetilde{k}^2+a_4\widetilde{k}+a_5j-\beta^2-a\beta\widetilde{k}j}>aj.$$

Then $k_2=\widetilde{k}-a$ satisfies $k_2+\sqrt{\beta k_2^2+a_4k_2+a_5j-\beta^2-a\beta k_2j}\le aj$ and

$$\begin{aligned}M_{\widetilde{k}-a,n,j}(x) &= m_{\widetilde{k}-a,n,j}(x)\left(x-\frac{\widetilde{k}-a}{n}\right)\\ &\le \frac{a(j+1)+\beta}{n}-\frac{\widetilde{k}-a}{n}\\ &\le \frac{\widetilde{k}+\sqrt{\beta\widetilde{k}^2+a_4\widetilde{k}+a_5j-\beta^2-a\beta\widetilde{k}j}+a}{n}-\frac{\widetilde{k}-a}{n}\\ &\le \frac{\sqrt{\max\{a_4,a_5\}}+2a}{\sqrt{n}}.\end{aligned}$$

For the last inequality we used the obvious relationship $k_2=\widetilde{k}-a$,

$$k_2+\sqrt{\beta k_2^2+a_4k_2+a_5j-\beta^2-a\beta k_2j}\le aj$$

which implies $\widetilde{k}\le(j+1)$ and $k_2\le j$.

By Lemma 3.2, (ii) it follows that

$$M_{\widetilde{k}-a,n,j}(x)\ge M_{\widetilde{k}-2a,n,j}(x)\ge M_{\widetilde{k}-3a,n,j}(x)\ge ...\ge M_{0,n,j}(x).$$

We thus obtain $M_{k,n,j}(x)\le\dfrac{\sqrt{\max\{a_4,a_5\}}+2a}{\sqrt{n}}$ for any $k\le j$ and $x\in\left[\frac{aj+\beta}{n},\frac{a(j+1)+\beta}{n}\right]$.

Collecting all the above estimates we have the proof of case (2). Thus, the proof is completed.

5. Conclusion

In this study, we introduced the nonlinear Jain operators of max-product type. We also estimate the rate of pointwise convergence of these operators.

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Author's contributions

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