Results in Nonlinear Analysis **4** (2021) No. 4, 235–243.
https://doi.org/10.53006/rna.986088
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Notes on judgment criteria of convex functions of several variables

Dedicated to the memory of Professor Bo-Ying Wang

Huan-Nan Shi[a], *Pei Wang*[a], *Jian Zhang*[a], *Wei-Shih Du*[b,c]

[a]*Basic Teaching Department, Teacher's College of Beijing Union University, Beijing 100011, P. R. China*
[b]*Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan*
[c]*Corresponding author*

Abstract

By transferring the judgment of convex functions of several variables into the judgment of convex functions of one variable, the authors discuss the convexity of some convex functions of several variables.

Keywords: convex function of several variables, convex function of one variable, judgment criteria
2010 MSC: 26B25, 26D15

1. Introduction

The theory of convex analysis plays an important role in almost all branches of mathematics, physics, dynamic systems theory, optimization, and so forth. Convex function theory is an important part of the general topic of convexity with a long history and full of application value.

Definition 1. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

i) A set $\Omega \subset \mathbb{R}^n$ is said to be convex if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $0 \le \alpha \le 1$ implies

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega.$$

Email addresses: shihuannan2014@qq.com, sfthuannan@buu.edu.cn (Huan-Nan Shi), ldwangpei@buu.edu.cn (Pei Wang), sftzhangjian@buu.edu.cn (Jian Zhang), wsdu@mail.nknu.edu.tw (Wei-Shih Du)

Received August 24, 2021; Accepted: September 26, 2021; Online: September 29, 2021

ii) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y}) \tag{1}$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and for all $\alpha \in [0,1]$. If the strict inequality in (1) holds whenever $\boldsymbol{x} \neq \boldsymbol{y}$ and $\alpha \in [0,1]$, then φ is said to be strictly convex. If $-\varphi$ is convex, then φ is said to be concave; if $-\varphi$ is strictly convex, then φ is said to be strictly concave.

There have been a number of literature on convex functions of one variable. But the literature on convex functions of several variables is relatively scarce. A primary problem is the judgment of convex functions.

The following proposition is a well-known criteria for the judgment of convex functions of one variable.

Proposition 1. *Let $I \subset \mathbb{R}$ be an open convex set (that is, an interval) and let $g : I \to \mathbb{R}$ be twice differentiable. Then*

1) a function g is convex on I if and only if $g''(t) \ge 0$ for all $t \in I$;

2) if $g''(t) > 0$ on I for all $t \in I$, then g is strictly convex on I.

As for the judgment criteria of convex functions of several variables, we have the following theorems.

Theorem 1 ([4, p. 644, B.3.d] and [10, p. 38, Proposition 4.3])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open convex set and let $\varphi : \Omega \to \mathbb{R}$ be twice differentiable. Then φ is convex on Ω if and only if the Hessian matrix*

$$H(\boldsymbol{x}) = \left(\frac{\partial^2 \varphi(\boldsymbol{x})}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n}$$

is nonnegative definite on Ω. If $H(\boldsymbol{x})$ is positive definite on Ω, then φ is strictly convex on Ω.

In [12], Wu and Zhu proved the following judgment theorem for convex functions of several variables.

Theorem 2 ([12, p. 80, Theorem 1.3.1])**.** *Let φ be a continuously differentiable function on the convex set $\Omega \subset \mathbb{R}^n$. Then φ is a convex function on Ω if and only if*

$$\varphi(\boldsymbol{x}) \ge \varphi(\boldsymbol{y}) + [\nabla\varphi(\boldsymbol{y})]^T(\boldsymbol{x} - \boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \Omega,$$

where $\nabla\varphi(\boldsymbol{y}) = \left(\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial y_2}, \ldots, \frac{\partial\varphi}{\partial y_n}\right)$.

The following judgment theorem allows us to convert the convexity of functions of several variables into the convexity of functions of one variable to judge. We translate and recite its proof from [10] as follows.

Theorem 3 ([10, p. 38, Proposition 4.4])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open convex set and $\varphi : \Omega \to \mathbb{R}$. For $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, let $g(t) = \varphi(t\boldsymbol{x} + (1-t)\boldsymbol{y})$ on $(0,1)$. Then*

(a) the function φ is convex on Ω if and only if g is convex on $(0,1)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$;

(b) the function φ is strictly convex on Ω if and only if g is strictly convex on $(0,1)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\boldsymbol{x} \neq \boldsymbol{y}$.

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $r, s, \alpha \in (0,1)$. If the function φ is convex on Ω, then

$$\begin{aligned} g(\alpha r + (1-\alpha)s) &= \varphi((\alpha r + (1-\alpha)s)\boldsymbol{x} + (1 - \alpha r - (1-\alpha)s)\boldsymbol{y}) \\ &= \varphi(\alpha(r\boldsymbol{x} + (1-r)\boldsymbol{y}) + (1-\alpha)(s\boldsymbol{x} + (1-s)\boldsymbol{y})) \\ &\le \alpha\varphi(r\boldsymbol{x} + (1-r)\boldsymbol{y}) + (1-\alpha)\varphi(s\boldsymbol{x} + (1-s)\boldsymbol{y}) \\ &= \alpha g(r) + (1-\alpha)g(s). \end{aligned}$$

Hence, the function g is convex on $(0, 1)$.

For any $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}} \in \Omega$, since Ω is an open convex set, for a sufficiently small ε, we have

$$\boldsymbol{x} = \frac{\tilde{\boldsymbol{x}} - \varepsilon(\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{y}})}{1 - 2\varepsilon} \in \Omega \quad \text{and} \quad \boldsymbol{y} = \frac{\tilde{\boldsymbol{y}} - \varepsilon(\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{y}})}{1 - 2\varepsilon} \in \Omega.$$

Let $r = 1 - \varepsilon$ and $s = \varepsilon$. Then $r, s \in (0, 1)$, $\tilde{\boldsymbol{x}} = r\boldsymbol{x} + (1 - r)\boldsymbol{y}$, and $\tilde{\boldsymbol{y}} = s\boldsymbol{x} + (1 - s)\boldsymbol{y}$. Since $g(t) = \varphi(t\boldsymbol{x} + (1 - t)\boldsymbol{y})$ is convex on $(0, 1)$, for $\alpha \in (0, 1)$, we have

$$\begin{aligned} &\varphi(\alpha\tilde{\boldsymbol{x}} + (1 - \alpha)\tilde{\boldsymbol{y}} = \varphi(\alpha r + (1 - \alpha)s)\boldsymbol{x} + (1 - \alpha r - (1 - \alpha)s)\boldsymbol{y}) \\ &= g(\alpha r + (1 - \alpha)s) \leq \alpha g(r) + (1 - \alpha)g(s) = \alpha\varphi(\tilde{\boldsymbol{x}}) + (1 - \alpha)\varphi(\tilde{\boldsymbol{y}}). \end{aligned}$$

Hence, the function φ is convex on Ω.

Following a similar argument as in the proof above and taking $\boldsymbol{x} \neq \boldsymbol{y}$ and $r \neq s$, one can verify the "strictly convex" case. □

In general literature, Theorem 1 is usually applied to judge the convexity of functions of several variables. Such applications often increase the difficulty and complexity of proofs. Using Theorem 3 as a judgement criteria is sometimes more concise. In this paper, we will give some important examples to demonstrate the conciseness.

2. Applications of Theorem 3

In this section, we use the notation $\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n\}$.

Theorem 4 ([15, pp. 1177–1179, Section 6, Theorem 6.1])**.** *Let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$*,* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$*, and* $\sum_{i=1}^n x_i = 1$*. Then the function* $l(\boldsymbol{a}) = \prod_{i=1}^n a_i^{x_i}$ *is concave in* $\boldsymbol{a} \in \mathbb{R}^n_+$*.*

Sketch of the proof of Theorem 6.1 in [15]. Directly computing shows that the Hessian matrix of $l(\boldsymbol{a})$ is

$$H = \begin{pmatrix} \frac{x_1(x_1-1)l(\boldsymbol{a})}{a_1^2} & \frac{x_1x_2l(\boldsymbol{a})}{a_1a_2} & \cdots & \frac{x_1x_nl(\boldsymbol{a})}{a_1a_n} \\ \frac{x_1x_2l(\boldsymbol{a})}{a_1a_2} & \frac{x_2(x_2-1)l(\boldsymbol{a})}{a_2^2} & \cdots & \frac{x_2x_nl(\boldsymbol{a})}{a_1a_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1x_nl(\boldsymbol{a})}{a_1a_n} & \frac{x_2x_nl(\boldsymbol{a})}{a_2a_n} & \cdots & \frac{x_n(x_n-1)l(\boldsymbol{a})}{a_n^2} \end{pmatrix}. \tag{2}$$

Then, for $1 \leq i \leq n$, the i-th leading principal minor determinant of the matrix (2) is

$$l^i(\boldsymbol{a})\prod_{j=1}^{i}\left(\frac{x_i}{a_j}\right)^2 \begin{vmatrix} 1 - \frac{1}{x_1} & 1 & \cdots & 1 \\ 1 & 1 - \frac{1}{x_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \frac{1}{x_i} \end{vmatrix} = (-1)^i l^i(\boldsymbol{a})(1 - x_1 - x_2 - \cdots - x_i)\prod_{j=1}^{i}\frac{x_j}{a_j^2}.$$

The signs of these determinants are determined by $(-1)^i$. From the discriminant method of the semi-negative definite matrix (see [11], pp. 8–9), it follows that the Hessian matrix in (2) is nonnegative definite. As a result, from Theorem 1, we conclude that the function $l(\boldsymbol{a})$ is concave in $\boldsymbol{a} \in \mathbb{R}^n_+$. □

Alternative proof of Theorem 6.1 in [15] by Theorem 3. For $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n_+$, and $t \in [0, 1]$, let

$$\varphi(t) = f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) = \prod_{i=1}^{n}[ta_i + (1 - t)b_i]^{x_i}.$$

Then

$$\varphi'(t) = \varphi(t) \sum_{i=1}^{n} x_i p_i$$

and

$$\varphi''(t) = \varphi'(t) \sum_{i=1}^{n} x_i p_i - \varphi(t) \sum_{i=1}^{n} x_i p_i^2 = \varphi(t) \left[\left(\sum_{i=1}^{n} x_i p_i \right)^2 - \sum_{i=1}^{n} x_i p_i^2 \right],$$

where

$$p_i = \frac{a_i - b_i}{t a_i + (1-t) b_i}, \quad 1 \le i \le n.$$

By virtue of monotonicity of the weighted power mean, we acquire

$$\sum_{i=1}^{n} x_i p_i \le \left(\sum_{i=1}^{n} x_i p_i^2 \right)^{1/2}.$$

Therefore, we see that $\varphi''(t) \le 0$. According to Theorem 3, we obtain that the function $f(\boldsymbol{a}) = l(\boldsymbol{a})$ is concave on $\mathbb{R}_+^n$. The proof of Theorem 4 is complete. □

Remark 1. After carefully comparing the above two proofs, we think that the second proof of Theorem 4 by using Theorem 3 is technically simpler.

Theorem 5. *Let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$*,* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$*, and* $\sum_{i=1}^{n} x_i = 1$*. Then the weighted harmonic mean of* n *variables*

$$h(\boldsymbol{a}) = f(a_1, a_2, \ldots, a_n) = \left(\sum_{i=1}^{n} \frac{x_i}{a_i} \right)^{-1}$$

is a convex function in $\boldsymbol{a}$ *on* $\mathbb{R}_+^n$*.*

Proof. For $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$, and $t \in [0, 1]$, let

$$\phi(t) = h(t\boldsymbol{a} + (1-t)\boldsymbol{b}) = \left[\sum_{i=1}^{n} \frac{x_i}{t a_i + (1-t) b_i} \right]^{-1}.$$

Then

$$\phi'(t) = - \left[\sum_{i=1}^{n} \frac{x_i}{t a_i + (1-t) b_i} \right]^{-2} \left[\sum_{i=1}^{n} \frac{-x_i (a_i - b_i)}{(t a_i + (1-t) b_i)^2} \right]$$

and

$$\phi''(t) \left[\sum_{i=1}^{n} \frac{x_i}{t a_i + (1-t) b_i} \right]^2 = 2 \left[\sum_{i=1}^{n} \frac{x_i}{t a_i + (1-t) b_i} \right]^{-1} \left[\sum_{i=1}^{n} \frac{-x_i (a_i - b_i)}{(t a_i + (1-t) b_i)^2} \right]^2 - \sum_{i=1}^{n} \frac{2 x_i (a_i - b_i)^2}{(t a_i + (1-t) b_i)^3}.$$

To prove $\phi''(t) \le 0$, we only need to prove

$$\left[\sum_{i=1}^{n} \frac{-x_i (a_i - b_i)}{(t a_i + (1-t) b_i)^2} \right]^2 \le \left[\sum_{i=1}^{n} \frac{x_i}{t a_i + (1-t) b_i} \right] \left[\sum_{i=1}^{n} \frac{x_i (a_i - b_i)^2}{(t a_i + (1-t) b_i)^3} \right],$$

this is,

$$\left[\sum_{i=1}^{n} \frac{x_i (a_i - b_i)}{(t a_i + (1-t) b_i)^2} \right]^2 \le \sum_{i=1}^{n} \left[\frac{\sqrt{x_i}}{\sqrt{t a_i + (1-t) b_i}} \right]^2 \sum_{i=1}^{n} \left[\frac{\sqrt{x_i}\,(a_i - b_i)}{(t a_i + (1-t) b_i)^{3/2}} \right]^2.$$

From the Cauchy inequality, we see that this inequality holds. The proof of of Theorem 5 is complete. □

Theorem 6. *Let* $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+$, *and* $\sum_{i=1}^n x_i = 1$. *Then the function*

$$g(\boldsymbol{a}) = f(a_1, a_2, \dots, a_n) = \ln\left(\sum_{i=1}^n x_i a_i\right)$$

is concave on $\mathbb{R}^n_+$.

Proof. For $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n_+$, and $t \in [0, 1]$, let

$$r(t) = g(t\boldsymbol{a} + (1-t)\boldsymbol{b}) = \ln\left[\sum_{i=1}^n (ta_i + (1-t)b_i)x_i\right].$$

Then

$$r'(t) = \frac{\sum_{i=1}^n (a_i - b_i)x_i}{\sum_{i=1}^n [ta_i + (1-t)b_i]x_i}$$

and

$$r''(t) = -\frac{\left[\sum_{i=1}^n (a_i - b_i)x_i\right]^2}{\left(\sum_{i=1}^n [ta_i + (1-t)b_i]x_i\right)^2} \le 0.$$

By Theorem 3, we see that $g(\boldsymbol{a})$ is a concave function on $\mathbb{R}^n_+$. The proof of Theorem 6 is complete. □

Proposition 2 ([12, p. 83])**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be defined by*

$$f(\boldsymbol{x}) = f(x_1, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2 + x_1x_2 + \dots + x_1x_n + x_2x_3 + \dots + x_2x_n + \dots + x_{n-1}x_n.$$

Then f *is a convex function on* $\mathbb{R}^n$.

Proof. We write

$$f(\boldsymbol{x}) = x_1 \sum_{i=1}^n x_i + x_2 \sum_{i=2}^n x_i + \dots + x_{n-1} \sum_{i=n-1}^n x_i + x_n \sum_{i=n}^n x_i = \sum_{j=1}^n \left(x_j \sum_{i=j}^n x_i\right).$$

For $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$, and $t \in [0, 1]$, let

$$r(t) = g(t\boldsymbol{a} + (1-t)\boldsymbol{b}) = \sum_{j=1}^n \left[\big(ta_j + (1-t)b_j\big) \sum_{i=j}^n (ta_i + (1-t)b_i)\right].$$

Then

$$r'(t) = \sum_{j=1}^n \left[(a_j - b_j) \sum_{i=j}^n (ta_i + (1-t)b_i) + \big(ta_j + (1-t)b_j\big) \sum_{i=j}^n (a_i - b_i)\right]$$

and

$$r''(t) = 2\sum_{j=1}^n \left[(a_j - b_j) \sum_{i=j}^n (a_i - b_i)\right] = 2\sum_{j=1}^n \left(y_j \sum_{i=j}^n y_i\right) = \left(\sum_{j=1}^n y_j\right)^2 + \sum_{j=1}^n y_j^2 \ge 0,$$

where $y_j = a_j - b_j$. Applying Theorem 3 reveals that $f(\boldsymbol{x})$ is a convex function on $\mathbb{R}^n$. □

Remark 2. In [12, p. 83], making use of Theorem 2, Wu and Zhu discussed Proposition 2 as follows. For $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, it is valid that

$$\begin{gathered} f(\boldsymbol{x}) \ge f(\boldsymbol{y}) + [\nabla f(\boldsymbol{y})]^T(\boldsymbol{x} - \boldsymbol{y}) \iff \\ x_1^2 + x_2^2 + \dots + x_n^2 + x_1x_2 + \dots + x_1x_n + x_2x_3 + \dots + x_2x_n + \dots + x_{n-1}x_n \\ \ge y_1^2 + y_2^2 + \dots + y_n^2 + y_1y_2 + \dots + y_1y_n + y_2y_3 + \dots + y_2y_n + \dots + y_{n-1}y_n \end{gathered}$$

$$+\begin{pmatrix} 2y_1+y_2+\cdots+y_n \\ 2y_2+y_3+\cdots+y_n \\ \vdots \\ 2y_n+y_{n-1} \end{pmatrix}^T (x_1-y_1, x_2-y_2, \ldots, x_n-y_n)^T \Longleftrightarrow$$

$$x_1^2+x_2^2+\cdots+x_n^2+x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n$$
$$\geq -y_1^2-y_2^2-\cdots-y_n^2+2x_1y_1+x_1y_2+\cdots+x_1y_n+2x_2y_2+x_2y_3+\cdots+x_2y_n+\cdots+2x_ny_n+x_ny_{n-1}.$$

Wu and Zhu claimed that, if taking $x_1=-1$, $y_1=0$, $x_2=0$, $y_2=0$, $\ldots$, $x_{n-1}=0$, $y_{n-1}=0$, $x_n=3$, and $y_n=2$, then the above inequalities do not hold, and then f is a non-convex function. In fact, the above inequality does hold. Hence, it is wrong to assert that f is a non-convex function in [12, p. 83].

Proposition 3 ([1, p. 39]). *The function*

$$q(x,y)=\frac{1}{(x+y)^2}$$

is convex on $\mathbb{R}^2$.

Proof. For $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$ and $t\in[0,1]$, let

$$\psi(t)=q(tx_1+(1-t)y_1, tx_2+(1-t)y_2)=\frac{1}{[tx_1+(1-t)y_1+tx_2+(1-t)y_2]^2}.$$

Then

$$\psi''(t)=\frac{6[(x_1+x_2)-(y_1+y_2)]^2}{[t(x_1+x_2)+(1-t)(y_1+y_2)]^4}\geq 0.$$

Employing Theorem 3 reveals that $q(x,y)$ is a convex function on $\mathbb{R}^2$. The required proof is complete. □

Remark 3. In [1, p. 39], with the help of Theorem 1, Chen considered the convexity of $q(x,y)$ only in the square region $D=\{(x,y)\in\mathbb{R}^2 : 1\leq x\leq 2, 1\leq y\leq 2\}$. In fact, as showed by Proposition 3, the function $q(x,y)$ is convex on the whole plane $\mathbb{R}^2$.

3. Convexity of arithmetic mean of integral form

In this section, we discuss the convexity of the arithmetic mean of integral form.

Theorem 7 ([3, p. 854, Theorem 1]). *Let I be an interval with nonempty interior on $\mathbb{R}$ and f be a continuous function on I. Then*

$$F(x,y)=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y f(s)\,\mathrm{d}\,s, & x,y\in I, x\neq y\\ f(x), & x=y\end{cases} \tag{3}$$

is Schur-convex (or Schur-concave, respectively) on $I\times I$ if and only if f is convex (or concave, respectively) on I.

By virtue of Theorem 1, Zhang and Chu proved in [14, p. 1063, Theorem] the following proposition.

Proposition 4. *Under the condition of Theorem 7, if f is a convex function, then the function $F(x,y)$ in* (3) *is a convex function.*

Wulbert [13, Lemma 2.4] gave an alternative proof of Proposition 4.

Applying Theorem 3, we now provide a new proof of Proposition 4.

Proof. In order to apply Theorem 3, we need to show that the one-variable function

$$g(t) = F(t\boldsymbol{\alpha} + (1-t)\boldsymbol{\beta}) = F\big(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2\big)$$

is convex on $(0,1)$, where $\boldsymbol{\alpha} = (x_1, y_1)$ and $\boldsymbol{\beta} = (x_2, y_2) \in \Omega$.

If letting $z_1 = ty_1 + (1-t)y_2$ and $z_2 = tx_1 + (1-t)x_2$, then

$$g(t) = F(z_1, z_2) = \begin{cases} \dfrac{1}{z_1 - z_2} \displaystyle\int_{z_2}^{z_1} f(s)\,\mathrm{d}\,s, & z_1 \neq z_2; \\ f(z_1), & z_1 = z_2. \end{cases}$$

Consequently, it is sufficient to show that the twice derivative $g''(t) > 0$.

By assumption, the case $z_1 = z_2$ holds trivially. For the case of $z_1 \neq z_2$, letting $a = y_1 - y_2$ and $b = x_1 - x_2$, then we have

$$\begin{aligned} g''(t) = {} & \frac{\frac{2(a-b)^2}{z_1 - z_2} \int_{z_2}^{z_1} f(s)\,\mathrm{d}\,s - (a-b)^2[f(z_1) - f(z_2)]}{(z_1 - z_2)^2} \\ & + \frac{a^2[f(z_2) - f(z_1) - f'(z_1)(z_2 - z_1)] + b^2[f(z_1) - f(z_2) - f'(z_2)(z_1 - z_2)]}{(z_1 - z_2)^2}. \end{aligned} \tag{4}$$

Due to the Hermite–Hadamard integral inequality, see the paper [9], the first term in (4) is negative. On the other hand, by Taylor's formula, the second term in (4) is equal to

$$\frac{a^2 f''(z_1 + \theta_1(z_2 - z_1)) + b^2 f''(z_2 + \theta_2(z_1 - z_2))}{2}$$

with $\theta_1, \theta_2 \in (0,1)$, which, due to $f''(x) > 0$, is positive. Accordingly, to show $g''(t) > 0$, it is sufficient to prove the second term is bigger than the absolute value of the first term in (4).

Corollary 15 in [2, p. 49] reads that, if f is a twice differentiable convex function on an open interval I and $[a,b] \subset I$, then

$$0 \le \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}\,x \le \frac{1}{2} \frac{[(b-a)f'(b) - (f(b) - f(a))][f(b) - f(a) - (b-a)f'(a)]}{(b-a)[f'(b) - f'(a)]} \tag{5}$$

provided that $f'(b) \neq f'(a)$. Utilizing the inequality (5), we arrive at

$$[f(z_1) - f(z_2)] - \frac{2}{z_1 - z_2} \int_{z_2}^{z_1} f(s)\,\mathrm{d}\,s \le \frac{[f(z_2) - f(z_1) - f'(z_1)(z_2 - z_1)][f(z_1) - f(z_2) - f'(z_2)(z_1 - z_2)]}{(z_1 - z_2)[f'(z_1) - f'(z_2)]}.$$

Further setting

$$X = f(z_2) - f(z_1) - f'(z_1)(z_2 - z_1)$$

and

$$Y = f(z_1) - f(z_2) - f'(z_2)(z_1 - z_2)$$

and noticing that

$$X + Y = (z_1 - z_2)[f'(z_1) - f'(z_2)],$$

we figure out

$$[f(z_1) - f(z_2))] - \frac{2}{z_1 - z_2} \int_{z_2}^{z_1} f(s)\,\mathrm{d}\,s \le \frac{XY}{X+Y}.$$

Consequently, the second term in (4) is equal to $\frac{a^2 X + b^2 Y}{(z_1 - z_2)^2}$. From the inequality

$$(a-b)^2 \frac{XY}{X+Y} \le a^2 X + b^2 Y,$$

we derive $g''(t) > 0$. This finishes the proof. $\square$

Remark 4. In [7, Theorem 1.1] and [8, Lemma 3], Theorem 7 was generalized as follows.

For a continuous function f and a positive continuous weight on I, the weighted arithmetic mean

$$Q(x,y)=\begin{cases}\dfrac{\int_x^y p(t)f(t)\,\mathrm{d}\,t}{\int_x^y p(t)\,\mathrm{d}\,t}, & x\neq y\\ f(x), & x=y\end{cases} \tag{6}$$

of f with the weight p is Schur-convex (or Schur-concave, respectively) on I^2 if and only if the inequality

$$\frac{\int_x^y p(t)f(t)\,\mathrm{d}\,t}{\int_x^y p(t)\,\mathrm{d}\,t}\le\frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)} \tag{7}$$

holds (or reverses, respectively) for $(x,y)\in I^2$.

Under the condition (7), is the function $Q(x,y)$ defined in (6) convex on I^2?

Remark 5. Proposition 2 in [5, p. 377] states that, if $f(x)$ is not a constant in $[a,b]$, is continuous in $[a,b]$, and satisfies $m\le f'(x)\le M$ for $a<x<b$, then

$$\begin{aligned}\frac{mM(b-a)^2+2(b-a)[mf(a)-Mf(b)]+[f(a)-f(b)]^2}{2(M-m)}&\le\int_a^b f(x)\,\mathrm{d}\,x\\ &\le-\frac{mM(b-a)^2+2(b-a)[mf(a)-Mf(b)]+[f(a)-f(b)]^2}{2(M-m)}.\end{aligned} \tag{8}$$

The double inequality (8) was reformulated in [6, pp. 28–29, Theorem 5.2] as

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)(b-a)}. \tag{9}$$

When $f(x)$ is a twice differentiable convex function on $[a,b]$, we obtain $m=f'(a)$ and $M=f'(b)$. As a result, the inequality (9) becomes the inequality (5) in [2, p. 49, Corollary 15]. Consequently, the inequality (5) in [2, p. 49, Corollary 15] is a special case of [5, p. 377, Proposition 2] and [6, pp. 28–29, Theorem 5.2].

Acknowledgements

The authors express their heartily thanks to anonymous referees for their valuable suggestions and careful comments on the original version of this paper.

References

[1] X.-D. Chen, Remarks on convex functions, Journal of Western Chongqing University Natural Science Edition **2** (2003), no. 2, 37–40; available online at `http://dx.chinadoi.cn/10.3969/j.issn.1673-8012.2003.04.012`. (Chinese)

[2] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite–Hadamard Inequalities and Applications, Amended version, RGMIA Monographs, Victoria University, 2002; available online at `http://rgmia.org/monographs/hermite_hadamard.html`.

[3] N. Elezović and J. Pečarić, A note on Schur-convex functions, Rocky Mountain J. Math. **30** (2000), no. 3, 853–856; available online at `https://doi.org/10.1216/rmjm/1021477248`.

[4] A.W. Marshall, I. Olkin, and B.C. Arnold, Inequalities: Theory of Majorization and its Applications, 2nd Ed., Springer Verlag, New York-Dordrecht-Heidelberg-London, 2011; available online at `http://dx.doi.org/10.1007/978-0-387-68276-1`.

[5] F. Qi, Inequalities for an integral, Math. Gaz. **80** (1996), no. 488, 376–377; available online at `https://doi.org/10.2307/3619581`.

[6] F. Qi, Pólya type integral inequalities: origin, variants, proofs, refinements, generalizations, equivalences, and applications, Math. Inequal. Appl. **18** (2015), no. 1, 1–38; available online at `https://doi.org/10.7153/mia-18-01`.

[7] F. Qi, J. Sándor, S.S. Dragomir, and A. Sofo, Notes on the Schur-convexity of the extended mean values, Taiwanese J. Math. **9** (2005), no. 3, 411–420; available online at `https://doi.org/10.11650/twjm/1500407849`.

[8] H.-N. Shi, S.-H. Wu, and F. Qi, An alternative note on the Schur-convexity of the extended mean values, Math. Inequal. Appl. **9** (2006), no. 2, 219–224; available online at `http://dx.doi.org/10.7153/mia-09-22`.

[9] Y. Shuang and F. Qi, Integral inequalities of Hermite–Hadamard type for extended s-convex functions and applications, Mathematics **6** (2018), no. 11, Article 223, 12 pages; available online at `https://doi.org/10.3390/math6110223`.

[10] B.-Y. Wang, Foundations of Majorization Inequalities, Beijing Normal University Press, Beijing, 1990. (Chinese)

[11] S.-G. Wang, M.-X. Wu, and Z.-Z. Jia, Inequalities in Matrix Theory, Second Ed., Science Press, Beijing, 2006.

[12] J.-J. Wu and Y.-G. Zhu, Several methods for determining the convexity of a function, Journal of Communication University of China Science and Technology, **27** (2002), no. 6, 79–83. (Chinese)

[13] D.E. Wulbert, Favard's inequality on average values of convex functions, Math. Comput. Modelling **37** (2003), no. 12-13, 1383–1391; available online at `https://doi.org/10.1016/S0895-7177(03)90048-3`.

[14] X.M. Zhang and Y.M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math. **40** (2010), no. 3, 1061–1068; available online at `https://doi.org/10.1216/RMJ-2010-40-3-1061`.

[15] N. G. Zheng, X.M. Zhang, and Y.M. Chu, Convexity and geometrical convexity of exponential and logarithmic means in N variables, Acta Math. Sci. Ser. A (Chin. Ed.) **28** (2008), no. 6, 1173–1180. (Chinese)