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NOTES ON THE SECOND-ORDER TANGENT BUNDLES WITH THE DEFORMED SASAKI METRIC

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Abstract. The paper deals with the second-order tangent bundle T^2M with the deformed Sasaki metric $\overline{g}$ over an $n-$dimensional Riemannian manifold (M,g). We calculate all Riemannian curvature tensor fields of the deformed Sasaki metric $\overline{g}$ and search Einstein property of T^2M. Also the weakly symmetry properties of the deformed Sasaki metric are presented.

1. Introduction

Let (M,g) be an n-dimensional Riemannian manifold and T^2M be its second-order tangent bundle. Second-order tangent bundles are of importance in differantial geometry. The geometry of the second-order tangent bundle T^2M over an $n-$dimensional manifold M which is the equivalent classes of curves with the same aceleration vector fields on M was studied in [9–12]. Dodson and Radivoiovici proved that a second-order tangent bundle T^2M of finite $n-$dimensional M becomes a vector bundle over M if and only if M has a linear connection in [6]. The lifts of tensor fields and connections given on M to its second-order tangent bundle T^2M were developped in [12]. In [7], Ishikawa defined a Sasaki-type lift metric in T^2M of a Riemanian manifold and investigated some of its properties. Moreover, in [3], the geometry of a second-order tangent bundle with a Sasaki-type metric was studied in detail. All forms of Riemannian curvature tensor of Sasaki metric on T^2M were computed and some curvature properties were examined in [8].

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In the present paper, motivated by the above works we study some geometric properties of the second-order tangent bundle T^2M equipped with the deformed Sasaki metric $\overline{g}$. We introduce the deformed Sasaki metric on T^2M over M and obtain the global results. Throughout this paper, all geometric objects assumed to be differentiable of class C^∞.

2. Preliminaries

Let M be an $n-$dimensional Riemannian manifold endowed with a linear connection ∇ and T^2M second-order tangent bundle be $3n-$dimensional manifold. T^2M has a natural bundle structure over M, $\pi_2 : T^2M \longrightarrow M$ denoting the canonical projection. If the canonical projection is denoted by $\pi_{12} : T^2M \longrightarrow TM$, then T^2M has a bundle structure over the tangent bundle TM with projection π_{12}. Let $\left(U, x^i\right)$ be a coordinate neighborhood of M and f be a curve in U which locally expressed as $x^i = f^i(t)$. If we take a $2-$jet j^2f belonging to $\pi_2^{-1}(U)$ and put

$$x^i = f^i(0), \; y^i = \frac{df^i}{dt}(0), \; z^i = \frac{d^2f^i}{dt^2}(0),$$

then the $2-$jet j^2f is expressed in a unique by the set $\left(x^i, y^i, z^i\right)$. Thus a system of coordinates $\left(x^i, y^i, z^i\right)$ is introduced in the open set ${\pi_2}^{-1}(U)$ of T^2M from $\left(U, x^i\right)$. The coordinates $\left(x^i, y^i, z^i\right)$ in ${\pi_2}^{-1}(U)$ are called the induced coordinates. On putting

$$\xi^i = x^i, \xi^{\overline{i}} = y^i, \xi^{\overline{\overline{i}}} = z^i,$$

the induced coordinates $\left(x^i, y^i, z^i\right)$ are denoted by $\{\xi^A\}$. The indices $A, B, C, \ldots$ take values $\{1,\ 2, \ldots,\ n;\ n+1,\ n+2, \ldots, 2n;\ 2n+1,\ 2n+2,\ \ldots, 3n\}$.

Let $X = X^i \frac{\partial}{\partial X^i}$ be the local expression in U of a vector field X on M. Then the vector fields X^{H_0}, X^{H_1} and X^{H_2} on T^2M are given, with respect to the induced coordinates $\{\xi^A\}$, by [4]

$$X^{H_0} = X^j \partial_j - u^s \Gamma^j_{sh} X^h \partial_{\overline{j}} - C^j_h X^h \partial_{\overline{\overline{j}}}, \tag{1}$$

$$X^{H_1} = X^j \partial_{\overline{j}} - 2u^s \Gamma^j_{sh} X^h \partial_{\overline{\overline{j}}} \tag{2}$$

ve

$$X^{H_2} = X^j \partial_{\overline{\overline{j}}} \tag{3}$$

with respect to the natural frame $\{\partial_A\} = \left\{ \partial_i,\ \partial_{\overline{i}},\ \partial_{\overline{\overline{i}}} \right\}$ in T^2M, where $C^j_h = z^m \Gamma^j_{hm} + u^s u^r (\partial_h \Gamma^j_{sr} + \Gamma^j_{hm} \Gamma^m_{sr} - 2\Gamma^j_{sm} \Gamma^m_{hr})$, Γ^j_{sr} are the coefficients of the Levi-Civita connection ∇ on M and $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\overline{i}} = \frac{\partial}{\partial y^i}$, $\partial_{\overline{\overline{i}}} = \frac{\partial}{\partial z^i}$. For the Lie bracket on T^2M in terms of the $\lambda-$lifts of vector fields X, Y on M, we have the following formulas:

$$\begin{aligned} \left[X^{H_0}, Y^{H_0}\right] &= [X, Y]^{H_0} - (R(X, Y)u)^{H_1} - (R(X, Y)\omega)^{H_2}, \\ \left[X^{H_0}, Y^{H_\mu}\right] &= (\nabla_X Y)^{H_\mu}, \quad \mu = 1, 2 \ \left[X^{H_\mu}, Y^{H_\alpha}\right] = 0, \quad \mu, \alpha = 1, 2 \end{aligned}$$

where R is the curvature tensor field of the connection ∇ on M defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ [5].

With the connection ∇ of g on M, we can introduce a frame field in each induced coordinate neighborhood ${\pi_2}^{-1}(U)$ of T^2M. In each coordinate neighborhood (U, x^i), using $X_i = \frac{\partial}{\partial x^i}$, from (1)-(3) we have

$$E_i = (X_i)^{H_0} = \left(\frac{\partial}{\partial x^i}\right)^{H_0} = \partial_i - u^s\Gamma^k_{is}\partial_{\overline{k}} - C^k_i\partial_{\overline{\overline{k}}},$$

$$E_{\overline{i}} = (X_i)^{H_1} = \left(\frac{\partial}{\partial x^i}\right)^{H_1} = \partial_{\overline{i}} - 2u^s\Gamma^k_{is}\partial_{\overline{\overline{k}}},$$

$$E_{\overline{\overline{i}}} = (X_i)^{H_2} = \left(\frac{\partial}{\partial x^i}\right)^{H_2} = \partial_{\overline{\overline{i}}}$$

with respect to the natural frame $\{\partial_A\}$ in T^2M [4].

3. The Deformed Sasaki Metric and Its Levi-Civita Connection

Definition 1. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle. The deformed Sasaki metric on the second-order tangent bundle over (M,g) is defined by the identities:*

$$\begin{cases} \overline{g}\left(X^{H_0}, Y^{H_0}\right) = fg(X,Y), \\ \overline{g}\left(X^{H_a}, Y^{H_b}\right) = 0, \quad a \neq b \\ \overline{g}\left(X^{H_1}, Y^{H_1}\right) = \overline{g}\left(X^{H_2}, Y^{H_2}\right) = g(X,Y), \end{cases}$$

for all vector fields X and Y on M, where f is a positive smooth function on (M,g).

The deformed Sasaki metric $\overline{g}$ and its inverse have components

$$\overline{g}_{\beta\gamma} = \begin{pmatrix} fg_{ij} & 0 & 0 \\ 0 & g_{ij} & 0 \\ 0 & 0 & g_{ij} \end{pmatrix} \text{ and } \overline{g}^{\gamma\alpha} = \begin{pmatrix} \frac{1}{f}g^{jk} & 0 & 0 \\ 0 & g^{jk} & 0 \\ 0 & 0 & g^{jk} \end{pmatrix}$$

with recpect to the adepted frame $\{E_\beta\}$. In adapted frame, the followings satisfy

$$[E_\beta, E_\gamma] = {\Omega_{\beta\gamma}}^{\varepsilon} E_\varepsilon,$$

$${\Omega_{ij}}^{\overline{k}} = u^p {R_{jip}}^k, \quad {\Omega_{i\overline{j}}}^{\overline{k}} = \Gamma^k_{ij},$$

$${\Omega_{ij}}^{\overline{\overline{k}}} = \omega^s {R_{jis}}^k, \quad {\Omega_{i\overline{\overline{j}}}}^{\overline{\overline{k}}} = \Gamma^k_{ij}.$$

Using the formula

$$\begin{aligned} \overline{\Gamma}^{\varepsilon}_{\beta\gamma} &= \frac{1}{2}\overline{g}^{\varepsilon\alpha}\left(E_\beta \overline{g}_{\alpha\gamma} + E_\gamma \overline{g}_{\alpha\beta} - E_\alpha \overline{g}_{\beta\gamma}\right) \\ &\quad + \frac{1}{2}\left({\Omega_{\beta\gamma}}^{\varepsilon} + {\Omega_{\cdot\beta\gamma}}^{\varepsilon} + {\Omega_{\cdot\gamma\beta}}^{\varepsilon}\right), \end{aligned}$$

where $\Omega_{.\ \beta\gamma}^{\ \varepsilon} = \overline{g}^{\alpha\varepsilon}\overline{g}_{\delta\gamma}\Omega_{\alpha\beta}{}^{\delta}$, the non-zero components $\overline{\Gamma}^{\varepsilon}_{\beta\gamma}$ are given by

$$\begin{aligned}\overline{\Gamma}^{k}_{ij} &= \Gamma^{k}_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{.}{}^{k}g_{ij}\right),\\ \overline{\Gamma}^{\overline{k}}_{ij} &= \frac{1}{2}u^p R_{jip}{}^{k},\ \overline{\Gamma}^{\overline{\overline{k}}}_{ij} = \frac{1}{2}\omega^s R_{jis}{}^{k},\end{aligned}$$

$$\overline{\Gamma}^{k}_{\overline{i}j} = \frac{1}{2f}u^p R_{pij}{}^{k},\ \overline{\Gamma}^{k}_{i\overline{j}} = \frac{1}{2f}u^p R_{pji}{}^{k},\ \overline{\Gamma}^{\overline{k}}_{i\overline{j}} = \Gamma^{k}_{ij},$$

$$\overline{\Gamma}^{k}_{\overline{\overline{i}}j} = \frac{1}{2f}\omega^s R_{sij}{}^{k},\ \overline{\Gamma}^{k}_{i\overline{\overline{j}}} = \frac{1}{2f}\omega^s R_{sji}{}^{k},\ \overline{\Gamma}^{\overline{\overline{k}}}_{i\overline{\overline{j}}} = \Gamma^{k}_{ij},$$

with respect to the adapted frame $\{E_\beta\}$.

Proposition 1. *Let (M,g) be a Riemannian manifold and $\overline{\nabla}$ be a Levi-Civita connection of $\left(T^2M,\overline{g}\right)$. Then we have*

1) $\overline{\nabla}_{E_i}E_j = \left\{\Gamma^k_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{.}{}^{k}g_{ij}\right)\right\}E_k + \frac{1}{2}u^p R^k_{jip}E_{\overline{k}} + \frac{1}{2}\omega^s R^k_{jis}E_{\overline{\overline{k}}}$,

2) $\overline{\nabla}_{E_{\overline{i}}}E_j = \frac{1}{2f}u^p R^k_{pij}E_k$,

3) $\overline{\nabla}_{E_i}E_{\overline{j}} = \frac{1}{2f}u^p R^k_{pji}E_k + \Gamma^k_{ij}E_{\overline{k}}$,

4) $\overline{\nabla}_{E_{\overline{\overline{i}}}}E_j = \frac{1}{2f}\omega^s R_{sij}{}^{k}E_k$,

5) $\overline{\nabla}_{E_i}E_{\overline{\overline{j}}} = \frac{1}{2f}\omega^s R^k_{sji}E_k + \Gamma^k_{ij}E_{\overline{\overline{k}}}$,

6) $\overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}} = 0$, $\overline{\nabla}_{E_{\overline{i}}}E_{\overline{\overline{j}}} = 0$, $\overline{\nabla}_{E_{\overline{\overline{i}}}}E_{\overline{j}} = 0$, $\overline{\nabla}_{E_{\overline{\overline{i}}}}E_{\overline{\overline{j}}} = 0$

with respect to the adapted frame $\{E_\beta\}$.

Proof. 1) By applying

$$\begin{aligned}\overline{\Gamma}^{\varepsilon}_{\beta\gamma} &= \frac{1}{2}\overline{g}^{\varepsilon\alpha}\left(E_\beta\overline{g}_{\alpha\gamma} + E_\gamma\overline{g}_{\alpha\beta} - E_\alpha\overline{g}_{\beta\gamma}\right)\\ &\quad + \frac{1}{2}\left(\Omega_{\beta\gamma}{}^{\varepsilon} + \Omega_{.\ \beta\gamma}^{\ \varepsilon} + \Omega_{.\ \gamma\beta}^{\ \varepsilon}\right)\end{aligned}$$

and direct calculation we get

$$\overline{\nabla}_{E_i}E_j = \overline{\Gamma}^{K}_{ij}E_K = \overline{\Gamma}^{k}_{ij}E_k + \overline{\Gamma}^{\overline{k}}_{ij}E_{\overline{k}} + \overline{\Gamma}^{\overline{\overline{k}}}_{ij}E_{\overline{\overline{k}}}$$

$$\begin{aligned}\overline{\Gamma}^{k}_{ij} &= \frac{1}{2}\overline{g}^{sk}\left(E_i\overline{g}_{sj} + E_j\overline{g}_{si} - E_s\overline{g}_{ij}\right) + \frac{1}{2}\left(\Omega_{ij}{}^{k} + \Omega_{.\ ij}^{\ k} + \Omega_{.\ ji}^{\ k}\right)\\ &= \frac{1}{2}\overline{g}^{sk}\left(\partial_i\left(fg_{sj}\right) + \partial_j\left(fg_{si}\right) - \partial_s\left(fg_{ij}\right)\right)\\ &= \frac{1}{2}g^{sk}\left(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\right) + \frac{1}{2f}g^{sk}\left(f_i g_{sj} + f_j g_{si} - f_s g_{ij}\right)\\ &= \Gamma^k_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{.}{}^{k}g_{ij}\right)\overline{\Gamma}^{\overline{k}}_{ij}\\ &= \frac{1}{2}\overline{g}^{\overline{s}\overline{k}}\left(E_i\overline{g}_{\overline{s}j} + E_j\overline{g}_{\overline{s}i} - E_{\overline{s}}\overline{g}_{ij}\right) + \frac{1}{2}\left(\Omega_{ij}{}^{\overline{k}} + \Omega_{.\ ij}^{\ \overline{k}} + \Omega_{.\ ji}^{\ \overline{k}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}u^p R^k_{jip}\ \overline{\Gamma}^{\overline{\overline{k}}}_{ij} = \frac{1}{2}\overline{g}^{\overline{\overline{s}}\overline{\overline{k}}}\left(E_i\overline{g}_{\overline{\overline{s}}j} + E_j\overline{g}_{\overline{\overline{s}}i} - E_{\overline{\overline{s}}}\overline{g}_{ij}\right) \\
&\quad + \frac{1}{2}\left(\Omega_{ij}{}^{\overline{\overline{k}}} + \Omega_{\cdot}{}^{\overline{\overline{k}}}{}_{ij} + \Omega_{\cdot}{}^{\overline{\overline{k}}}{}_{ji}\right) \\
&= \frac{1}{2}\omega^s R^k_{jis}.
\end{aligned}
$$

Thus we obtain

$$
\overline{\nabla}_{E_i}E_j = \left\{\Gamma^k_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{\cdot}{}^k g_{ij}\right)\right\}E_k + \frac{1}{2}u^p R^k_{jip}E_{\overline{k}} + \frac{1}{2}\omega^s R^k_{jis}E_{\overline{\overline{k}}}.
$$

The rest can be proven by following the same method in the proof of *1)*. We omit them to avoid repeat. □

Proposition 2. *The Levi-Civita connection $\overline{\nabla}$ of the deformed Sasaki metric $\overline{g}$ on T^2M is given as following*

$$
\overline{\nabla}_{X^{H_0}}Y^{H_0} = \left(\nabla_X Y + A_f\left(X,Y\right)\right)^{H_0} + \frac{1}{2}\left(R\left(Y,X\right)u\right)^{H_1} + \frac{1}{2}\left(R\left(Y,X\right)\omega\right)^{H_2},
$$

$$
\begin{aligned}
\overline{\nabla}_{X^{H_1}}Y^{H_0} &= \frac{1}{2f}\left(R\left(u,X\right)Y\right)^{H_0}, \\
\overline{\nabla}_{X^{H_0}}Y^{H_1} &= \frac{1}{2f}\left(R\left(u,Y\right)X\right)^{H_0} + \left(\nabla_X Y\right)^{H_1}, \\
\overline{\nabla}_{X^{H_2}}Y^{H_0} &= \frac{1}{2f}\left(R\left(\omega,X\right)Y\right)^{H_0}, \\
\overline{\nabla}_{X^{H_0}}Y^{H_2} &= \frac{1}{2f}\left(R\left(\omega,Y\right)X\right)^{H_0} + \left(\nabla_X Y\right)^{H_2},
\end{aligned}
$$

$$
\overline{\nabla}_{X^{H_1}}Y^{H_1} = 0, \quad \overline{\nabla}_{X^{H_2}}Y^{H_1} = 0, \quad \overline{\nabla}_{X^{H_1}}Y^{H_2} = 0, \quad \overline{\nabla}_{X^{H_2}}Y^{H_2} = 0
$$

for all vector fields X, Y on M, where

$$
A_f\left(X,Y\right) = \frac{1}{2f}\left(X\left(f\right)Y - Y\left(f\right)X - g\left(X,Y\right)\circ\left(df\right)^*\right).
$$

4. Riemannian Curvature Tensors of the Deformed Sasaki Metric

Let F be a smooth bundle endomorphism of T^2M. Then we have the lifts of F:

$$
\begin{aligned}
F^{H_0}\left(u\right) &= \sum u^i F(\partial_i)^{H_0}, \quad F^{H_1}\left(u\right) = \sum u^i F(\partial_i)^{H_1}, \\
F^{H_2}\left(u\right) &= \sum u^i F(\partial_i)^{H_2}, F^{H_0}\left(\omega\right) = \sum \omega^i F(\partial_i)^{H_0},
\end{aligned}
$$

$$
F^{H_1}\left(\omega\right) = \sum \omega^i F(\partial_i)^{H_1}, \quad F^{H_2}\left(\omega\right) = \sum \omega^i F(\partial_i)^{H_2}.
$$

Morever, the following expressions are obtained by direct standard calculations

$$
\overline{\nabla}_{X^{H_0}}u^i = X^{H_0}\left(u^i\right) = -u^s\Gamma^i_{sh}X^h,
$$

$$
\overline{\nabla}_{X^{H_1}}u^i = X^i, \quad \overline{\nabla}_{X^{H_2}}u^i = 0, \quad \overline{\nabla}_{X^{H_2}}\omega^i = X^i,
$$

$$\overline{\nabla}_{X^{H_0}}\omega^i = -C^i_h X^h, \quad \overline{\nabla}_{X^{H_1}}\omega^i = -2u^s\Gamma^i_{sh}X^h.$$

Lemma 1. *Let (M,g) be a Riemannian manifold and $\overline{\nabla}$ be a Levi-Civita connection of $\left(T^2M,\overline{g}\right)$. Let $F: T^2M \longrightarrow T^2M$ be a smooth endomorphism, then*

$$\begin{aligned}
\overline{\nabla}_{X^{H_0}}F^{H_0}(u) &= \left(\nabla_X F(u) + A_f(X,F(u))\right)^{H_0} \\
&\quad + \frac{1}{2}\left(R\left(F(u),X\right)u\right)^{H_1} + \frac{1}{2}\left(R\left(F(u),X\right)\omega\right)^{H_2}, \\
\overline{\nabla}_{X^{H_0}}F^{H_1}(u) &= \frac{1}{2f}\left(R\left(u,F(u)\right)X\right)^{H_0} + \left(\nabla_X F(u)\right)^{H_1}, \\
\overline{\nabla}_{X^{H_0}}F^{H_2}(u) &= \frac{1}{2f}\left(R\left(\omega,F(u)\right)X\right)^{H_0} + \left(\nabla_X F(u)\right)^{H_2}, \\
\overline{\nabla}_{X^{H_1}}F^{H_0}(u) &= \left(F(X)\right)^{H_0} + \frac{1}{2f}\left(R\left(u,X\right)F(u)\right)^{H_0}, \\
\overline{\nabla}_{X^{H_1}}F^{H_1}(u) &= \left(F(X)\right)^{H_1}, \quad \overline{\nabla}_{X^{H_1}}F^{H_2}(u) = \left(F(X)\right)^{H_2}, \\
\overline{\nabla}_{X^{H_2}}F^{H_0}(u) &= 0, \quad \overline{\nabla}_{X^{H_2}}F^{H_1}(u) = 0, \quad \overline{\nabla}_{X^{H_2}}F^{H_2}(u) = 0, \\
\overline{\nabla}_{X^{H_0}}F^{H_0}(\omega) &= \left(\nabla_X F(\omega) + A_f(X,F(u))\right)^{H_0} \\
&\quad + \frac{1}{2}\left(R\left(F(\omega),X\right)u\right)^{H_1} + \frac{1}{2}\left(R\left(F(\omega),X\right)\omega\right)^{H_2}, \\
\overline{\nabla}_{X^{H_0}}F^{H_1}(\omega) &= \frac{1}{2f}\left(R\left(u,F(\omega)\,X\right)^{H_0} + \left(\nabla_X F(\omega)\right)^{H_1}\right., \\
\overline{\nabla}_{X^{H_0}}F^{H_2}(\omega) &= \frac{1}{2f}\left(R\left(\omega,F(\omega)\,X\right)^{H_0} + \left(\nabla_X F(\omega)\right)^{H_2}\right., \\
\overline{\nabla}_{X^{H_1}}F^{H_0}(\omega) &= \frac{1}{2f}\left(R\left(u,X\right)F(\omega)\right)^{H_0}, \\
\overline{\nabla}_{X^{H_1}}F^{H_1}(\omega) &= 0, \quad \overline{\nabla}_{X^{H_1}}F^{H_2}(\omega) = 0, \\
\overline{\nabla}_{X^{H_2}}F^{H_0}(\omega) &= \left(F(X)\right)^{H_0} + \frac{1}{2f}\left(R\left(\omega,X\right)F(\omega)\right)^{H_0}, \\
\overline{\nabla}_{X^{H_2}}F^{H_1}(\omega) &= \left(F(X)\right)^{H_1}, \quad \overline{\nabla}_{X^{H_2}}F^{H_2}(\omega) = \left(F(X)\right)^{H_2}
\end{aligned}$$

for any vector field X on M and $u,\omega \in T^2M$.

Proposition 3. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki metric $\overline{g}$. The curvature tensor $\overline{R}$ of the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ on T^2M is given by the following formulas:*

$$\begin{aligned}
&1)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0} \\
= \;&\{R(X,Y)Z + \left(\nabla_X A_f\right)(Y,Z) - \left(\nabla_Y A_f\right)(X,Z) + A_f\left(X,A_f(Y,Z)\right) \\
&-A_f\left(Y,A_f(X,Z)\right) - \frac{1}{2f}R\left(u,R(X,Y)u\right)Z - \frac{1}{2f}R\left(\omega,R(X,Y)\omega\right)Z
\end{aligned}$$

$$+\frac{1}{4f}R\left(u,R\left(Z,Y\right)u\right)X+\frac{1}{4f}R\left(u,R\left(Z,X\right)u\right)Y$$
$$\left.+\frac{1}{4f}R\left(\omega,R\left(Z,Y\right)\omega\right)X+\frac{1}{4f}R\left(\omega,R\left(Z,X\right)\omega\right)Y\right\}^{H_0}$$
$$+\left\{\frac{1}{2}\nabla_Z R\left(X,Y\right)u+\frac{1}{2}R\left(Y,A_f\left(X,Z\right)\right)u-\frac{1}{2}R\left(X,A_f\left(Y,Z\right)\right)u\right\}^{H_1}$$
$$+\left\{\frac{1}{2}\nabla_Z R\left(X,Y\right)\omega+\frac{1}{2}R\left(Y,A_f\left(X,Z\right)\right)\omega-\frac{1}{2}R\left(X,A_f\left(Y,Z\right)\right)\omega\right\}^{H_2},$$

$$2)\overline{R}\left(X^{H_1},Y^{H_0}\right)Z^{H_0}$$
$$=\left\{-\left(\nabla_Y\frac{1}{2f}\right)R(u,X)Z-\frac{1}{2f}(\nabla_Y R)(u,X)Z\right.$$
$$\left.+\frac{1}{2f}R(u,X)(A_f(Y,Z))-\frac{1}{2f}A_f(Y,R(u,X)Z)\right\}^{H_0}$$
$$+\left\{-\frac{1}{2}R\left(Y,Z\right)X+\frac{1}{4f}R\left(Y,R\left(u,X\right)Z\right)u\right\}^{H_1}+\left\{\frac{1}{4f}R\left(Y,R\left(u,X\right)Z\right)\omega\right\}^{H_2},$$

$$3)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_1}$$
$$=\left\{(\nabla_X\frac{1}{2f})R\left(u,Z\right)Y+\frac{1}{2f}(\nabla_X R)\left(u,Z\right)Y-(\nabla_Y\frac{1}{2f})R\left(u,Z\right)X\right.$$
$$\left.-\frac{1}{2f}(\nabla_Y R)\left(u,Z\right)X+\frac{1}{2f}A_f\left(X,R\left(u,Z\right)Y\right)-\frac{1}{2f}A_f\left(Y,R\left(u,Z\right)X\right)\right\}^{H_0}$$
$$+\left\{R\left(X,Y\right)Z+\frac{1}{4f}\left[R\left(R\left(u,Z\right)Y,X\right)u-R\left(R\left(u,Z\right)X,Y\right)u\right]\right\}^{H_1}$$
$$+\frac{1}{4f}\{[R\left(R\left(u,Z\right)Y,X\right)\omega-R\left(R\left(u,Z\right)X,Y\right)\omega]\}^{H_2},$$

$$4)\overline{R}\left(X^{H_1},Y^{H_1}\right)Z^{H_0}=\left\{\frac{1}{f}R\left(X,Y\right)Z+\frac{1}{4f^2}\left[R\left(u,X\right)R\left(u,Y\right)Z-R\left(u,Y\right)R\left(u,X\right)Z\right]\right\}^{H_0},$$

$$5)\overline{R}\left(X^{H_0},Y^{H_1}\right)Z^{H_1}=\left\{-\frac{1}{2f}R\left(Y,Z\right)X-\frac{1}{4f^2}R\left(u,Y\right)R\left(u,Z\right)X\right\}^{H_0},$$

$$6)\overline{R}\left(X^{H_2},Y^{H_0}\right)Z^{H_0}$$
$$=\left\{-\frac{1}{2f}\left(\nabla_Y R\right)\left(\omega,X,Z\right)-(\nabla_Y\frac{1}{2f})R\left(\omega,X\right)Z+\frac{1}{2f}R\left(\omega,X\right)A_f\left(Y,Z\right)\right.$$
$$\left.-\frac{1}{2f}A_f\left(Y,R\left(\omega,Z\right)X\right)\right\}^{H_0}+\left\{\frac{1}{4f}R\left(Y,R\left(\omega,X\right)Z\right)u\right\}^{H_1}$$

$$+\left\{\frac{1}{2}R(Z,Y)X+\frac{1}{4f}R(Y,R(\omega,X)Z)\omega\right\}^{H_2},$$

$$\begin{aligned}
&7)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_2}\\
=\ &\left\{\frac{1}{2f}\nabla_X R(\omega,Z)Y-\frac{1}{2f}\nabla_Y R(\omega,Z)X+(\nabla_X\frac{1}{2f})R(\omega,Z)Y\right.\\
&\left.+(\nabla_Y\frac{1}{2f})R(\omega,Z)X+\frac{1}{2f}A_f(X,R(\omega,Z)Y)-\frac{1}{2f}A_f(Y,R(\omega,Z)X)\right\}^{H_0}\\
&+\frac{1}{4f}\left\{R(R(\omega,Z)Y,X)u-R(R(\omega,Z)X,Y)u\right\}^{H_1}\\
&+\left\{R(X,Y)Z-\frac{1}{4f}\left[R(R(\omega,Z)Y,X)\omega+R(R(\omega,Z)X,Y)\omega\right]\right\}^{H_2},
\end{aligned}$$

$$\begin{aligned}
&8)\overline{R}\left(X^{H_2},Y^{H_2}\right)Z^{H_0}\\
=\ &\left\{\frac{1}{f}R(X,Y)Z+\frac{1}{4f^2}\left[R(\omega,X)R(\omega,Y)Z-R(\omega,Y)R(\omega,X)Z\right]\right\}^{H_0},
\end{aligned}$$

$$9)\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_2}=\left\{-\frac{1}{2f}R(Y,Z)X-\frac{1}{4f^2}R(\omega,Y)R(\omega,Z)X\right\}^{H_0},$$

$$10)\overline{R}\left(X^{H_1},Y^{H_0}\right)Z^{H_2}=\left\{\frac{1}{4f^2}R(u,X)R(\omega,Z)Y\right\}^{H_0},$$

$$11)\overline{R}\left(X^{H_1},Y^{H_2}\right)Z^{H_0}=\frac{1}{4f^2}\left\{R(u,X)R(\omega,Y)Z-R(\omega,Y)R(u,X)Z\right\}^{H_0},$$

$$12)\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_1}=\left\{-\frac{1}{4f^2}R(\omega,Y)R(u,Z)X\right\}^{H_0},$$

$$13)\overline{R}\left(X^{H_2},Y^{H_0}\right)Z^{H_1}=\left\{\frac{1}{4f^2}R(\omega,X)R(u,Z)Y\right\}^{H_0},$$

$$14)\overline{R}\left(X^{H_1},Y^{H_1}\right)Z^{H_1}=0,\overline{R}\left(X^{H_1},Y^{H_1}\right)Z^{H_2}=0\ ,\overline{R}\left(X^{H_1},Y^{H_2}\right)Z^{H_2}=0,$$

$$15)\overline{R}\left(X^{H_1},Y^{H_2}\right)Z^{H_1}=0,\overline{R}\left(X^{H_2},Y^{H_2}\right)Z^{H_1}=0,\overline{R}\left(X^{H_2},Y^{H_2}\right)Z^{H_2}=0$$

for all vector fields X,Y on M.

Proposition 4. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle with the deformed Sasaki metric $\overline{g}$. The sectional curvature tensor $\overline{K}$ on T^2M satisfies*

$$\begin{aligned}
1)\overline{K}\left(X^{H_0},Y^{H_0}\right)\ =\ &\frac{1}{f}K(X,Y)+\frac{1}{f}\{g(\nabla_X A_f(Y,Y),X)-g(\nabla_Y A_f(X,Y),X)\\
&+g(A_f(X,A_f(Y,Y),X))-g(A_f(Y,A_f(X,Y),X))\}
\end{aligned}$$

$$-\frac{3}{4f^2}\left[\left|R(X,Y)u\right|^2+\left|R(X,Y)\omega\right|^2\right],$$

$$\begin{aligned}
2)\overline{K}\left(X^{H_1},Y^{H_0}\right) &= \frac{1}{4f}\left|R(u,X)Y\right|^2,\\
3)\overline{K}\left(X^{H_2},Y^{H_0}\right) &= \frac{1}{4f}\left|R(\omega,X)Y\right|^2,\\
4)\overline{K}\left(X^{H_1},Y^{H_1}\right) &= 0,\overline{K}\left(X^{H_1},Y^{H_2}\right)=0,\overline{K}\left(X^{H_2},Y^{H_2}\right)=0
\end{aligned}$$

for all orthonormal vector fields X, Y on M.

Proposition 5. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki Metric $\overline{g}$. The scalar curvature $\overline{S}$ of T^2M with the metric $\overline{g}$ given by*

$$\begin{aligned}
\overline{S} &= \frac{1}{f^2}S+\frac{1}{f^2}\{g\left(\nabla_{X_i}A_f\left(X_j,X_j\right),X_i\right)-g\left(\nabla_{X_j}A_f\left(X_i,X_j\right),X_i\right)\\
&\quad +g\left(A_f\left(X_i,A_f\left(X_j,X_j\right),X_i\right)\right)-g\left(A_f\left(X_j,A_f\left(X_i,X_j\right),X_i\right)\right)\}\\
&\quad -\frac{3-2f\sqrt{f}}{4f^3}\sum_{i,j=1}^{m}\left|R\left(X_i,X_j\right)u\right|^2+\left|R\left(X_i,X_j\right)\omega\right|^2,
\end{aligned}$$

where $\{X_1,\ldots,X_m\}$ is a local orthonormal frame for M and S is the scalar curvature of M.

Proof. The set $\{Y_1,\ldots,Y_{3m}\}$ is an orthonormal basis on T^2M with $\frac{1}{\sqrt{f}}X_i^{H_0}=Y_i$, $X_i^{H_1}=Y_{m+i}$ and $X_i^{H_2}=Y_{2m+i}$ we get

$$\begin{aligned}
\overline{S} &= \sum_{i,j=1}^{3m}\overline{K}\left(Y_i,Y_j\right)=\sum_{i,j=1}^{m}\overline{K}\left({X_i}^{H_0},{X_j}^{H_0}\right)+\overline{K}\left({X_i}^{H_1},{X_j}^{H_1}\right)\\
&\quad +\overline{K}\left({X_i}^{H_2},{X_j}^{H_2}\right)+2\overline{K}\left({X_i}^{H_0},{X_j}^{H_1}\right)+2\overline{K}\left({X_i}^{H_0},{X_j}^{H_2}\right)+2\overline{K}\left({X_i}^{H_1},{X_j}^{H_2}\right)\\
&= \sum_{i,j=1}^{m}\{\frac{1}{f^2}K\left(X_i,X_j\right)+\frac{1}{f^2}\{g\left(\nabla_{X_i}A_f\left(X_j,X_j\right),X_i\right)-g\left(\nabla_{X_j}A_f\left(X_i,X_j\right),X_i\right)\\
&\quad +g\left(A_f\left(X_i,A_f\left(X_j,X_j\right),X_i\right)\right)-g\left(A_f\left(X_j,A_f\left(X_i,X_j\right),X_i\right)\right)\\
&\quad -\frac{3}{4f^3}\left[\left|R\left(X_i,X_j\right)u\right|^2+\left|R\left(X_i,X_j\right)\omega\right|^2\right]\}\\
&\quad +2\frac{1}{4f\sqrt{f}}\left|R\left(u,X_j\right)X_i\right|^2+2\frac{1}{4f\sqrt{f}}\left|R\left(\omega,X_j\right)X_i\right|^2.
\end{aligned}$$

Standard calculations give that

$$\begin{aligned}
\overline{S} &= \sum_{i,j=1}^{m}\frac{1}{f^2}K(X_i,X_j)+\frac{1}{f^2}\{g\left(\nabla_{X_i}A_f\left(X_j,X_j\right),X_i\right)-g\left(\nabla_{X_j}A_f\left(X_i,X_j\right),X_i\right)\\
&\quad +g\left(A_f\left(X_i,A_f\left(X_j,X_j\right),X_i\right)\right)-g\left(A_f\left(X_j,A_f\left(X_i,X_j\right),X_i\right)\right)\}
\end{aligned}$$

$$-\frac{3-2f\sqrt{f}}{4f^3}\sum_{i,j=1}^{m}\left|R\left(X_i,X_j\right)u\right|^2+\left|R\left(X_i,X_j\right)\omega\right|^2.$$

□

Let Ric_g and $Ric_{\overline{g}}$ denote Ricci tensors of (M,g) and $\left(T^2M,\overline{g}\right)$, respectively. We can write

$$\begin{aligned} Ric_{\overline{g}}\left(X^{H_a},Y^{H_b}\right) &= \sum_{i=1}^{m}\overline{g}\left(\overline{R}\left(X^{H_a},\frac{1}{\sqrt{f}}{X_i}^{H_0}\right)\frac{1}{\sqrt{f}}{X_i}^{H_0},Y^{H_b}\right) \\ &\quad+\sum_{i=1}^{m}\overline{g}\left(\overline{R}\left(X^{H_a},{X_i}^{H_1}\right){X_i}^{H_1},Y^{H_b}\right) \\ &\quad+\sum_{i=1}^{m}\overline{g}\left(\overline{R}\left(X^{H_a},{X_i}^{H_2}\right){X_i}^{H_2},Y^{H_b}\right)\end{aligned}$$

for orthonormal basis $\frac{1}{\sqrt{f}}{X_i}^{H_0}=Y_i$, ${X_i}^{H_1}=Y_{m+i}$ and ${X_i}^{H_2}=Y_{2m+i}$, where $a,b=0,1,2$.

After a straightforward computation, the components of the Ricci tensor $Ric_{\overline{g}}$ of the deformed Sasaki metric $\overline{g}$ are characterized by

$$\begin{aligned} & Ric_{\overline{g}}\left(X^{H_0},Y^{H_0}\right) \qquad (4)\\ =\ & Ric_g\left(X,Y\right) \\ & +\sum_{i=1}^{m}g\left(\left(\nabla_X A_f\right)\left(X_i,X_i\right),Y\right)-\sum_{i=1}^{m}g\left(\left(\nabla_{X_i}A_f\right)\left(X,X_i\right),Y\right) \\ & +\sum_{i=1}^{m}g\left(A_f\left(X,A_f\left(X_i,X_i\right)\right),Y\right)-\sum_{i=1}^{m}g\left(A_f\left(X_i,A_f\left(X,X_i\right)\right),Y\right) \\ & +\frac{3}{4f}\sum_{i=1}^{m}g\left(R\left(X_i,X\right)u,R\left(X_i,Y\right)u\right)+\frac{3}{4f}\sum_{i=1}^{m}g\left(R\left(X_i,X\right)\omega,R\left(X_i,Y\right)\omega\right) \\ & -\frac{1}{4f}\sum_{i=1}^{m}g\left(R\left(u,X_i\right)R\left(u,X_i\right)X,Y\right)-\frac{1}{4f}\sum_{i=1}^{m}g\left(R\left(\omega,X_i\right)R\left(\omega,X_i\right)X,Y\right),\end{aligned}$$

$$\begin{aligned} Ric_{\overline{g}}\left(X^{H_0},Y^{H_1}\right) &= \frac{1}{2f}\sum_{i=1}^{m}g\left(\nabla_{X_i}R\left(X,X_i\right)u,Y\right) \qquad (5)\\ &\quad+\frac{1}{2f}\sum_{i=1}^{m}g\left(R\left(X_i,A_f\left(X,X_i\right)\right)u,Y\right) \\ &\quad-\frac{1}{2f}\sum_{i=1}^{m}g\left(R\left(X,A_f\left(X_i,X_i\right)\right)u,Y\right),\end{aligned}$$

$$\begin{aligned} Ric_{\overline{g}}\left(X^{H_0},Y^{H_2}\right) &= \frac{1}{2f}\sum_{i=1}^{m} g\left(\nabla_{X_i}R\left(X,X_i\right)\omega,\ Y\right) \\ &+\frac{1}{2f}\sum_{i=1}^{m} g\left(R\left(X_i,A_f\left(X,X_i\right)\right)\omega,Y\right) \\ &-\frac{1}{2f}\sum_{i=1}^{m} g\left(R\left(X,A_f\left(X_i,X_i\right)\right)\omega,Y\right), \end{aligned} \tag{6}$$

$$Ric_{\overline{g}}\left(X^{H_1},Y^{H_1}\right)=\frac{1}{4f^2}\sum_{i=1}^{m} g\left(R\left(u,X\right)X_i,R\left(u,Y\right)X_i\right),$$

$$Ric_{\overline{g}}\left(X^{H_2},Y^{H_1}\right)=\frac{1}{4f^2}\sum_{i=1}^{m} g\left(R\left(\omega,X\right)X_i,R\left(u,Y\right)X_i\right),$$

$$Ric_{\overline{g}}\left(X^{H_2},Y^{H_2}\right)=\frac{1}{4f^2}\sum_{i=1}^{m} g\left(R\left(\omega,X\right)X_i,R\left(\omega,Y\right)X_i\right),$$

for all vector field X,Y,Z on M and $u,\omega\in T^2M$.

Theorem 1. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki metric $\overline{g}$. $\left(T^2M,\overline{g}\right)$ is an Einstein manifold if and only if (M,g) is flat and*

$$\begin{aligned} &\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j}A_f\right)\left(X_i,X_i\right),X_j\right)-\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i}A_f\right)\left(X_j,X_i\right),X_j\right) \\ &+\sum_{i,j=1}^{m} g\left(A_f\left(X_j,A_f\left(X_i,X_i\right)\right),X_j\right)-\sum_{i,j=1}^{m} g\left(A_f\left(X_i,A_f\left(X_j,X_i\right)\right),X_j\right) \\ = &\ 0. \end{aligned}$$

Proof. The set $\{Y_1,\ldots,Y_{3m}\}$ be an local orthonormal basis on T^2M with $\frac{1}{\sqrt{f}}{X_i}^{H_0}=Y_i$, ${X_i}^{H_1}=Y_{m+i}$ and ${X_i}^{H_2}=Y_{2m+i}$. At first, suppose that $\left(T^2M,\overline{g}\right)$ is an Einstein manifold. Then it must be

$$Ric_{\overline{g}}\left(\overline{X},\overline{Y}\right)=\lambda\overline{g}\left(\overline{X},\overline{Y}\right)$$

for all vector field $\overline{X}$, $\overline{Y}$ on T^2M, where λ is a constant. If $X=Y=X_j$ is put into (4), (5) and (6), it follows that

$$\begin{aligned} \lambda &= Ric_g\left(X_j,X_j\right) \\ &+\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j}A_f\right)\left(X_i,X_i\right),X_j\right) \\ &-\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i}A_f\right)\left(X_j,X_i\right),X_j\right) \end{aligned} \tag{7}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right) \\
&-\sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right) \\
&+\frac{1}{2f}\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right) u\right|^2+\frac{1}{2f}\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right) \omega\right|^2 \\
= & \frac{1}{4f^2}\sum_{i,j=1}^{m}\left|R\left(\omega, X_j\right) X_i\right|^2 \\
= & \frac{1}{4f^2}\sum_{i,j=1}^{m}\left|R\left(u, X_j\right) X_i\right|^2 .
\end{aligned}
$$

Restricting the last identity to the zero section of T^2M, it follows

$$
\begin{aligned}
Ric_g\left(X_j, X_j\right) = & -\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j} A_f\right)\left(X_i, X_i\right), X_j\right)+\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i} A_f\right)\left(X_j, X_i\right), X_j\right) \\
&-\sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right) \\
&+\sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right)
\end{aligned}
$$

and we obtain $\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right) u\right|^2=0$ and $\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right) \omega\right|^2=0$. Replacing u and ω by X_k in the last identiy we see that

$$
\sum_{i,j,k=1}^{m}\left|R\left(X_i, X_j\right) X_k\right|^2=0 .
$$

Thus $R\left(X_i, X_j\right) X_k=0$ for all $i, j, k=1 \ldots m$ and we deduce $R=0$. (M, g) is flat. If we reconsider the equation (7), we obtain

$$
\begin{aligned}
&\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j} A_f\right)\left(X_i, X_i\right), X_j\right)-\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i} A_f\right)\left(X_j, X_i\right), X_j\right) \\
&+\sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right)-\sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right) \\
= & 0 .
\end{aligned}
$$

Conversely, we assume that $R=0$ and the equation above, then Ricci formulas become $Ric_{\overline{g}}=\lambda\overline{g}$ and $\lambda=0$. Thus T^2M is an Einstein manifold. □

5. Weakly Symmetry Properties of the Deformed Sasaki Metric

The Riemannian manifold (M, g) is called weakly symmetric if there exist two 1−forms $\alpha_1,\ \alpha_2$ and a vector field A, all on M, such that:

$$\begin{aligned} & (\nabla_W R)(X,Y,Z) \\ = \; & \alpha_1(W) R(X,Y) Z + \alpha_2(X) R(W,Y) Z \\ & +\alpha_2(Y) R(X,W) Z + \alpha_2(Z) R(X,Y) W + g(R(X,Y)Z,W)(\alpha_2)\#, \end{aligned} \tag{8}$$

where $A = (\alpha_2)^{\#}$ and $\alpha_i g^{ij} = \alpha^j = \alpha\#$, that is, A is the $g-$dual vector field of the 1−form α_2. In [1], Bejan and Crasmareanu proved that the weakly symmetry property of the Sasaki metric on the tangent bundle over base manifold, generalising the result obtained in [2]. The weakly symmetry property of Sasaki metric on the second-order tangent bundle T^2M proved in [8]. In this section, we consider the result for the second-order tangent bundle with the deformed Sasaki metric $(T^2M, \overline{g})$.

Theorem 2. *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki metric $\overline{g}$. $\left(T^2M, \overline{g}\right)$ is weakly symmetric if and only if the base manifold (M, g) is flat and*

$$(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) + A_f(X, A_f(Y,Z)) - A_f(Y, A_f(X,Z)) = 0,$$

where $A_f(X,Y) = \frac{1}{2f}\left(X(f)Y - Y(f)X - g(X,Y)\circ(df)^*\right)$ is a $(1,2)-$tensor field. On account of this, $\left(T^2M, \overline{g}\right)$ is flat.

Proof. In the proof, we apply the method used in [1]. If $R = 0$ then $\overline{R} = 0$ and so we have (8) as null equality. Primarily we take into account the condition (8) for $W^{H_0},\ X^{H_0},\ Y^{H_2}$ and Z^{H_2} and we obtain

$$\begin{aligned} & \alpha_1\left(W^{H_0}\right)\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_2} + \alpha_2\left(X^{H_0}\right)\overline{R}\left(W^{H_0},Y^{H_2}\right)Z^{H_2} \\ & +\alpha_2\left(Y^{H_2}\right)\overline{R}\left(X^{H_0},W^{H_0}\right)Z^{H_0} + \alpha_2\left(Z^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_2}\right)W^{H_0} \\ & +\overline{g}\left(\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_2},W^{H_0}\right)(a_2)^{\#} \\ = \; & -\overline{\nabla}_{W^{H_0}}\left[-\frac{1}{2f}R(Y,Z)X - \frac{1}{4f^2}R(\omega,Y)R(\omega,Z)X\right]^{H_0} \\ & -\overline{R}\begin{pmatrix} (\nabla_W X)^{H_0} + (A_f(W,X))^{H_0} + \left(\frac{1}{2}R(X,W)u\right)^{H_1} \\ +\left(\frac{1}{2}R(X,W)\omega\right)^{H_2}, Y^{H_2} \end{pmatrix} Z^{H_2} \\ & -\overline{R}\left(X^{H_0}, \left(\frac{1}{2f}R(\omega,Y)W\right)^{H_0} + (\nabla_W Y)^{H_2}\right)Z^{H_2} \\ & -\overline{R}\left(X^{H_0},Y^{H_2}\right)\left(\frac{1}{2f}(R(\omega,Z)W)^{H_0} + (\nabla_W Z)^{H_2}\right). \end{aligned} \tag{9}$$

Thereafter we consider the H_2 part of both sides of the above equation and we get

$$\begin{aligned}
& \alpha_2\left(Y^{H_2}\right)\left(R(X,W)Z-\frac{1}{4f}R(R(\omega,Z)W,X)\omega-\frac{1}{4f}R(R(\omega,Z)X,W)\omega\right) \\
& +\alpha_2\left(Z^{H_2}\right)\left(-\frac{1}{2}R(W,X)Y-\frac{1}{4f}R(X,R(\omega,Y)W)\omega\right) \\
& -\frac{1}{f}g\left(\frac{1}{2f}R(Y,Z)X+\frac{1}{4f^2}R(\omega,Y)R(\omega,Z)X,W\right)\alpha_2{}^{\#} \\
= & -\frac{1}{4f}R(R(Y,Z)X,W)\omega-\frac{1}{8f^2}R(R(\omega,Y)R(\omega,Z)X,W)\omega \\
& -\frac{1}{2f}\left(\begin{array}{c} R(X,R(\omega,Y)W)Z+\frac{1}{4f}R(R(\omega,Z)R(\omega,Y)W,X)\omega \\ +\frac{1}{4f}R(R(\omega,Z)X,R(\omega,Y)W)\omega \end{array}\right) \\
& -\frac{1}{2f}\left(-\frac{1}{2}R(R(\omega,Z)W,X)Y+\frac{1}{4f}R(R(\omega,Y)R(\omega,Z)W,X)\omega\right). \qquad (10)
\end{aligned}$$

By setting $Y=\omega$ and $Z=\omega$ respectively we get

$$\begin{aligned}
& \alpha_2\left(\omega^{H_2}\right)\left(R(X,W)Z+\frac{1}{4f}\left(R(R(\omega,Z)W,X)\omega-R(R(\omega,Z)X,W)\omega\right)\right) \\
& +\alpha_2\left(Z^{H_2}\right)\left(-\frac{1}{2}R(W,X)\omega\right)-\frac{1}{f}g\left(\frac{1}{2f}R(\omega,Z)X,W\right)\alpha_2{}^{\#} \\
= & \ \frac{1}{4f}R(R(\omega,Z)X,W)\omega+\frac{1}{4f}R(R(\omega,Z)W,X)\omega \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
& \alpha_2\left(Y^{H_2}\right)(R(X,W)\omega) \\
& +\alpha_2\left(\omega^{H_2}\right)\left(-\frac{1}{2}R(W,X)Y-\frac{1}{4f}R(X,R(\omega,Y)W)\omega\right) \\
& -\frac{1}{f}g\left(\frac{1}{2f}R(Y,\omega)X,W\right)\alpha_2{}^{\#} \\
= & \ \frac{1}{4f}R(R(Y,\omega)X,W)\omega-\frac{1}{2f}R(X,R(\omega,Y)W)\omega. \qquad (12)
\end{aligned}$$

Now we replace Y by Z in (12) equation

$$\begin{aligned}
& \alpha_2\left(Z^{H_2}\right)R(X,W)\omega \\
& +\alpha_2\left(\omega^{H_2}\right)\left(-\frac{1}{2}R(W,X)Z-\frac{1}{4f}R(X,R(\omega,Z)W)\omega\right) \\
& -\frac{1}{f}g\left(\frac{1}{2f}R(Z,\omega)X,W\right)\alpha_2{}^{\#} \\
= & \ \frac{1}{4f}R(R(Z,\omega)X,W)\omega-\frac{1}{2f}R(X,R(\omega,Z)W)\omega. \qquad (13)
\end{aligned}$$

And by adding (11) and (13) we produce

$$\begin{aligned} &\frac{3}{2}\alpha_2\left(Z^{H_2}\right)R\left(X,W\right)\omega \\ &+\alpha_2\left(\omega^{H_2}\right)\left(\frac{3}{2}R\left(X,W\right)Z+\frac{1}{2f}R\left(R\left(\omega,Z\right)W,X\right)\omega-\frac{1}{4f}R\left(R\left(\omega,Z\right)X,W\right)\omega\right) \\ = &\frac{3}{4f}R\left(R\left(\omega,Z\right)W,X\right)\omega. \end{aligned} \tag{14}$$

The equation (14) with $Z=\omega$ we obtain that:

$$\alpha_2\left(\omega^{H_2}\right)R\left(X,W\right)\omega=0. \tag{15}$$

If $\alpha_2\left(\omega^{H_2}\right)\neq 0$, then we have result. Suppose now that $\alpha_2\left(\omega^{H_2}\right)=0$ then $\left((\alpha_2)^{\#}\right)^{H_2}=0$.

Returning to equation (11) it results

$$\alpha_2\left(Z^{H_2}\right)\left(-\frac{1}{2}R\left(W,X\right)\omega\right)=\frac{1}{4f}R\left(R\left(\omega,Z\right)X,W\right)\omega+\frac{1}{4f}R\left(R\left(\omega,Z\right)W,X\right)\omega.$$

By setting now $W=X$ we get

$$R\left(R\left(\omega,Z\right)X,X\right)\omega=0.$$

And we take the inner product with Z, it follows that:

$$g\left(R\left(\omega,Z\right)X,R\left(\omega,Z\right)X\right)=0.$$

Thus

$$R\left(\omega,Z\right)X=0.$$

Now the inner product with an arbitary Y gives

$$g\left(R\left(X,Y\right)\omega,Z\right)=0.$$

For Z being an arbitary vector field we get $R\left(X,Y\right)\omega=0$, for every $X,\ Y$ and ω. Hence, we have $R=0$. In the case the Riemannian curvature tensor reduce to

$$\begin{aligned} \overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0} &= \{(\nabla_X A_f)(Y,Z)-(\nabla_Y A_f)(X,Z) \\ &\quad +A_f\left(X,A_f\left(Y,Z\right)\right)-A_f\left(Y,A_f(X,Z)\right)\}^{H_0} \end{aligned}$$

and also Levi-Civita connection is

$$\overline{\nabla}_{X^{H_0}}Y^{H_0}=\left(\nabla_X Y+A_f\left(X,Y\right)\right)^{H_0}.$$

Next we again consider the equation (8) for $X^{H_0},\ Y^{H_0},\ Z^{H_0},\ W^{H_2}$

$$\begin{aligned} &\alpha_1\left(W^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0}+\alpha_2\left(X^{H_0}\right)\overline{R}\left(W^{H_2},Y^{H_0}\right)Z^{H_0} \\ &+\alpha_2\left(Y^{H_0}\right)\overline{R}\left(X^{H_0},W^{H_2}\right)Z^{H_0}+\alpha_2\left(Z^{H_0}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)W^{H_2} \\ &+\overline{g}\left(R\left(X^{H_0},Y^{H_0}\right)Z^{H_0},W^{H_2}\right)(\alpha_2)\# \end{aligned}$$

$$= \quad -\overline{\nabla}_{W^{H_2}}\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0}-\overline{R}\left(\overline{\nabla}_{W^{H_2}}X^{H_0},Y^{H_0}\right)Z^{H_0}$$
$$-\overline{R}\left(X^{H_0},\overline{\nabla}_{W^{H_2}}Y^{H_0}\right)Z^{H_0}-\overline{R}\left(X^{H_0},Y^{H_0}\right)\overline{\nabla}_{W^{H_2}}Z^{H_0}.$$

Hence, we get

$$\alpha_1\left(W^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0}+g\left(\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0},W^{H_2}\right)(\alpha_2)\#=0,$$
$$\alpha_1\left(W^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0}=0,$$
$$\alpha_1\left(W^{H_2}\right)\left[\left(\nabla_X A_f\right)(Y,Z)-\left(\nabla_Y A_f\right)(X,Z)+A_f\left(X,A_f\left(Y,Z\right)\right)-A_f\left(Y,A_f(X,Z)\right)\right]^{H_0}=0.$$

Since α_1 is arbitrary, we find

$$\left(\nabla_X A_f\right)(Y,Z)-\left(\nabla_Y A_f\right)(X,Z)+A_f\left(X,A_f\left(Y,Z\right)\right)-A_f\left(Y,A_f\left(X,Z\right)\right)=0.$$

The proof is complete. □

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