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Stability Properties for the Delay Integro-Differential Equation

Ilhame AMIRALI*

Duzce University, Department of Mathematics, 81620, Duzce, Turkiye

Highlights

• This paper focuses on stability inequalities for the linear VDIDE.
• The solution continuously depends on the right side and initial data.
• A highly precise theoretical result is obtained. Examples illustrate the theoretical result.

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Abstract

In this paper stability inequalities for the linear nonhomogeneous Volterra delay integro-differential equation (VDIDE) is being established. The particular problems are encountered to show the applicability of the method and to confirm the predicted theoretical analysis.

1. INTRODUCTION

Volterra delay integro-differential equations (VDIDEs) have an important impact on the field of science. VDIDEs arise widely in physics and engineering applications. For a detailed results we recommend the reader the books [1-3].

Many research has been studied on the stability of differential equations. In most works on stability of delay differential equations, coefficients and delays are assumed to be continuous. But in some problems, for example, in biological, ecological or in economical models, parameters of differential equations are not continuous. In [4], the authors consider the stability relation between ODEs and DIDEs and shown that under some suitable conditions a DIDE will remain exponential stability of the given ODE. In [5] the author gave conditions for the solution to a first order differential equations to be bounded by an exponential function, uniformly stable, and uniformly asymptotically stable. The author applied the obtained results to the nonlinear equations and systems of equations. Construction of highly stable method based on the energy estimates method for the initial-boundary value problem for linear pseudo-parabolic equation is described in [6] and example of method have good stability properties with respect to the basic equation.

Over the last fifty years, in case of Volterra/Fredholm integral equations substantial efforts on their numerical treatment have been recorded [7-10, 11]. [12] is conserned with the analytical and numerical stability of neutral DIDEs. In [13] the authors consider a new direct numerical method for high-order linear VIDEs. An algorithm based on the use of Taylor polynomials is developed for the numerical analysis of high-order linear VIDEs. The singularly perturbed initial value problem for a linear first order VDIDE

* e-mail: ilhameamirali@duzce.edu.tr

examined in [14]. An extensive overview of assorted techniques for singularly perturbed differential or integro-differential equations can be seen in [15]. The theory and numerical solution of delay differential equations have been extensively analyzed in the [16, 17] and in the references therein.

In this paper, we establish the stability inequalities for the problem (1)-(2), indicating the continuous dependence of the solution on the right side and initial data expressed by the inequality (6).

2. STABILITY BOUNDS

Consider the linear integro-differential equation with constant delay:

$$u'(z) + a(z)u(z) + b(z)u(z-r) + \int_{z-r}^{z} K(z,s)u(s)ds = f(z), \quad z \in I, \tag{1}$$

$$u(z) = \varphi(z), \ -r \le z \le 0, \tag{2}$$

where $I = (0,T] = \cup_{p=1}^{m} I_p,\ I_p = \{z: r_{p-1} < z \le r_p\},\ 1 \le p \le m$ and $r_s = sr,$ for $0 \le s \le m,$
$\bar{I} = [O,T]$ and $I_0 = [-r, 0]$. $a(z) \ge 0,\ f(z)\ \ (z \in \bar{I}),\ b(z)\ \ (z \in \bar{I}),\ \varphi(z)\ \ (z \in I_0)$ and $K(z,s)\ \ ((z,s) \in \bar{I} \times \bar{I})$ are assumed to be sufficiently smooth functions, r is a constant delay.

Lemma 1. Assume that $a(z) \ge 0, |F(z)| \le \mathcal{F}(z)$ and $\mathcal{F}(z)$ is a nondecreasing function. Then the solution of the initial value problem

$$u' + a(z)u = F(z), \quad x_0 < z < X, \tag{3}$$

$$u(x_0) = \mu \tag{4}$$

satisfies

$$|u(z)| \le |\mu| + (X - x_0)\mathcal{F}(z), \quad x_0 < z < X. \tag{5}$$

Proof. From (3)-(4) we have

$$u(z) = \mu e^{-\int_{x_0}^{z} a(\tau)d\tau} + \int_{x_0}^{z} F(\tau) e^{-\int_{\tau}^{t} a(\varsigma)d\varsigma} d\tau.$$

From here it is easy to get

$$|u(z)| \le |\mu| + \int_{x_0}^{z} |F(\tau)| d\tau \le \int_{x_0}^{t} \mathcal{F}(\tau) d\tau,$$

which immediately leads to (5).

Notation. $||g||_\infty$ is a maximum norm for any continuous function g on corresponding closed interval.

Theorem 1. If $a(z), b(z), f(z) \in C(\bar{I}),\ K(z,s) \in C(\bar{I} \times \bar{I})$ and $\varphi(z) \in C(I_0)$, then for the solution $u(z)$ of (1)-(2) holds the following stability inequality

$$||u||_{\infty,p} \le e^{r^2 p \bar{K}} (1 + r||b||_{\infty,\bar{I}} + \bar{K}r^2)^p ||\varphi||_{\infty,0} +$$

$$r e^{r^2 \bar{K}} \frac{e^{r^2 p \bar{K}} (1 + r||b||_{\infty,\bar{I}} + \bar{K}r^2)^p - 1}{e^{r^2 p \bar{K}} (1 + r||b||_{\infty,\bar{I}} + \bar{K}r^2) - 1} ||f||_{\infty,\bar{I}}, \quad 1 \le p \le m, \tag{6}$$

where

$$\overline{K} = \max_{I\times I}|K(z,s)|.$$

Proof. For

$$F(z) = f(z) - b(z)u(z-r) - \int_{z-r}^{z} K(z,\varsigma)u(\varsigma)d\varsigma$$

we can write

$$|F(z)| \leq |f(z)| + |b(z)||u(z-r)| + \int_{z-r}^{z} |K(z,\varsigma)||u(\varsigma)|d\varsigma.$$

Consider this on I_p, we have

$$|F(z)| \leq ||f||_{\infty,p} + ||b||_{\infty,p}\,||u||_{\infty,p-1} + \overline{K}\int_{r_{p-1}}^{t} |u(\varsigma)|d\varsigma + \overline{K}\int_{t-r}^{r_p-1} |u(\varsigma)|d\varsigma$$

$$\leq\ ||f||_{\infty,p} + (||b||_{\infty,p} + r\overline{K})||u||_{\infty,p-1} +\ \overline{K}\int_{r_{p-1}}^{t} |u(\varsigma)|d\varsigma.$$

Therefore using Lemma 1 and Gronwall's inequality we have

$$||u||_{\infty,p} \leq \beta||u||_{\infty,p-1} + \rho$$

with

$$\beta = e^{r^2\overline{K}}\left(1 + r\left|\left|b\right|\right|_{\infty,\bar{I}} + r^2\overline{K}\right), \qquad \rho = re^{r^2\overline{K}}||f||_{\infty,\bar{I}}.$$

After using first-order difference inequality we get

$$||u||_{\infty,p} \leq ||\varphi||_{\infty,0}\beta^p + \frac{\beta^p - 1}{\beta - 1}\rho,$$

which implies the validity of (6).

In order to illustrate the performance of the method proposed above, we give two particular problems.

3. ILLUSTRATIVE EXAMPLES

Example 1. Consider the following problem:

$$u'(z) + z^2u(z) + (1 + e^{-z})u(z-0.5) + \int_{z-0.5}^{z} \sqrt{z+s}\,u(s)ds = sin\pi z, \quad 0 < z \leq 2$$

$$u(z) = 1 + z, \qquad -0.5 \leq z \leq 0.$$

Since

$$r = 0.5, \quad T = 2, \quad 1 \leq p \leq 4, \quad ||b||_{\infty,\bar{I}} = \max_{[0,2]}(1 + e^{-z}) = 2,$$

$||f||_{\infty,\bar{I}} = 1, \qquad \bar{K} = 2, \qquad ||\varphi||_{\infty,0} = 1,$

then the inequality (6) leads to following bounds for the solution $u(z)$

$$||u||_{\infty,1} \leq e^{0,25\times1\times2}(1 + 0,5 \times 2 + 2 \times 0,25)$$

$$+0,5e^{0,25\times2}\frac{e^{0,25\times1\times2}(1+0,5\times2+2\times0,25)-1}{e^{0,25\times2}(1+0,5\times2+2\times0,25)-1} = 4,94,$$

$$||u||_{\infty,2} \leq e^{0,25\times2\times2}(1 + 0,5 \times 2 + 2 \times 0,25)^2$$

$$+0,5e^{0,25\times2}\frac{e^{0,25\times2\times2}(1+0,5\times2+2\times0,5)^2-1}{e^{0,25\times2}(1+0,5\times2+2\times0,25)-1} = 21,21,$$

$$||u||_{\infty,3} \leq e^{0,25\times3\times2}(1 + 0,5 \times 2 + 2 \times 0,25)^3$$

$$+0,5e^{0,25\times2}\frac{e^{0,25\times3\times2}(1+0,5\times2+2\times0,5)^3-1}{e^{0,25\times2}(1+0,5\times2+2\times0,25)-1} = 88,25,$$

$$||u||_{\infty,4} \leq e^{0,25\times4\times2}(1 + 0,5 \times 2 + 2 \times 0,25)^4$$

$$+0,5e^{0,25\times2}\frac{e^{0,25\times4\times2}(1+0,5\times2+2\times0,5)^4-1}{e^{0,25\times2}(1+0,5\times2+2\times0,25)-1} = 364,59.$$

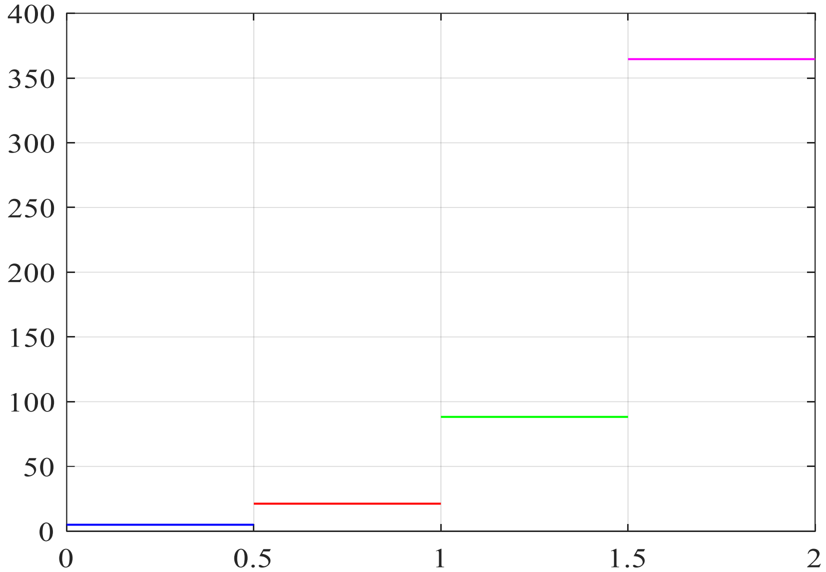

Figure 1. *Bound for the exact solution*

Example 2. Our second example is

$$u'(z) + 4u(z-1) - \int_{z-1}^{z} u(z)dz = z, \quad 0 < z \leq 2$$

$$u(z) = 1, \quad -1 \leq z \leq 0\,.$$

The solution is given by

$$u(z)=\begin{cases}1, & -1\le z\le 0\\ -e^{z}+2e^{-z}, & 0<z\le 1\\ \left(\frac{5}{2}e^{-1}z-\frac{7}{4}e^{-z}-1\right)e^{z}+\left(-3ez+\frac{13}{4}e+2\right)e^{-z}-1, & 1<z\le 2.\end{cases}$$

Since

$r=1,\quad T=2,\quad 1\le p\le 2,\quad ||b||_{\infty,\bar{I}}=4,$

$||\mathrm{f}||_{\infty,\bar{I}}=2,\quad \overline{K}=1,\quad ||\varphi||_{\infty,0}=1,$

the bounds given by Theorem 1 will be

$||u||_{\infty,1}=4{,}71$

$||u||_{\infty,2}=36{,}12.$

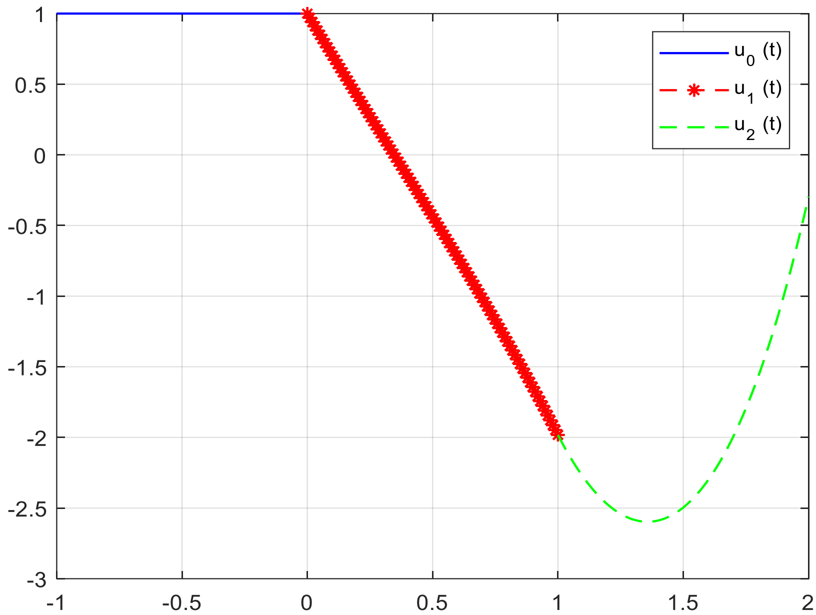

Figure 2. *Bound for the exact solution*

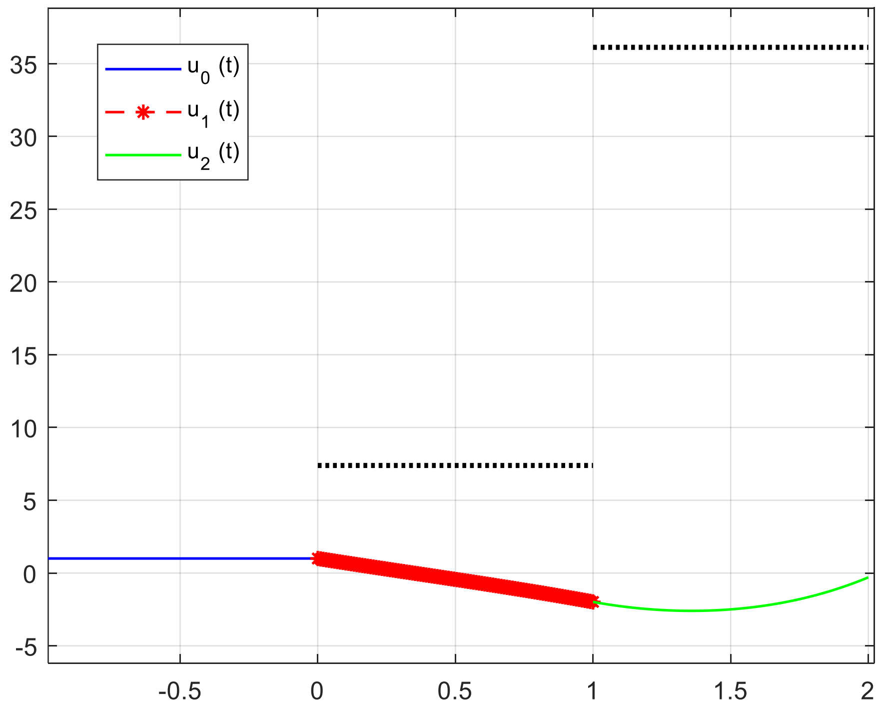

Figure 3. *Bound for the exact solution*

4. CONCLUSION

In this work, we establish the stability inequalities for the linear nonhomogeneous VDIDE. We shown that the solution continuously depends on the right side and initial data expressed by the inequality (6). Finally, examples are performed and illustrate the theoretical result (See: Figure [1-3]).

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

REFERENCES

[1] Arino, O., Hbid, M.L., Dads, E. A., "Delay Differential Equations and Applications", Springer, (2002).

[2] Bellen, A., Zennaro, M., "Numerical Methods for Delay Differential Equations", Oxford: Oxford University Press, (2003).

[3] Driver, R. D., "Ordinary and Delay Differential Equation", Springer, (1977).

[4] Yapman, Ö., Amiraliyev, G. M., Amirali, I., "Convergence Analysis of Fitted Numerical Method for a Singularly Perturbed Nonlinear Volterra Integro-Differential Equation with Delay", Journal of Computational and Applied Mathematics, 355: 301-309, (2019).

[5] Dix, L. G., "Asymptotic Behavior of Solutions to a First-Order Differential Equation with Variable Delays", Computers and Mathematics with Applications, 50: 1791-1800, (2005).

[6] Amirali, I., Cati, S., Amiraliyev, G. M., "Stability Inequalities for the Delay Pseudo-Parabolic Equations'', International Journal of Applied Mathematics, 32(2): 289-294, (2019).

[7] Bellour, A., Bousselsal, M., "Numerical Solution of Delay Integro-Differential Equations by Using Taylor Collocation Method", Mathematical Methods in Applied Science, 37: 1491-1506, (2013).

[8] Zhang, C., Niu, Y., "The Stability Relation Between Ordinary and Delay-Integro-Differential Equations", Mathematical and Computer Modelling, Issues, 12, 49: 13-19, (2009).

[9] Kudu, M., Amirali, I., Amiraliyev, G. M., "A Finite Difference Method for a Singularly Perturbed Delay Integro-Differential Equation", Journal of Computational and Applied Mathematics, 308: 379-390, (2016).

[10] Xiao-yong, Z., "A New Strategy for The Numerical Solution of Nonlinear Volterra Integral Equations with Vanishing Delays", Applied Mathematics and Computation, 365: 124-608, (2020).

[11] Durmaz, M. E., Amiraliyev, G. M., "A Robust Numerical Method for a Singularly Perturbed Fredholm Integro-Differential Equation'', Mediterranean Journal of Mathematics, 18(24): 1-17, (2021).

[12] Wu, S., Gan, S., "Analitical and Numerical Stability of Neutral Delay Integro-Differential Equations and Neutral Delay Partial Differential Equations", Computers and Mathematics with Applications, 55: 2426-2443, (2008).

[13] Laib, H., Bellour, A., Bousselsal, M., "Numerical Solution of High-Order Linear Volterra Integro-Differential Equations by using Taylor Collocation Method'', International Journal of Computer Mathematics, 96 (5): 1066-1085, (2019).

[14] Darania, P., Pishbin, S., "High-Order Collocation Methods for Nonlinear Delay Integral Equations", Computational and Applied Mathematics, 326: 284-295, (2017).

[15] Panda, A, Mohapatra, J., Amirali, I., "A Second Order Post-Processing Technique for Singularly Perturbed Volterra Integro-Differential Equation", Mediterranean Journal of Mathematics, 18(231): 1-25, (2021).

[16] Mohapatra, J., Natesan, S., "Uniform Convergence Analysis of finite Difference Scheme for Singularly Perturbed Delay Differential Equation on an Adaptively Generated Grid", Numerical Mathematics: Theory, Methods and Applications, 3(1): 1-22, (2010).

[17] Amiraliyeva, I. G., Amiraliyev, G. M., "Uniform Difference Method For Parameterized Singularly Perturbed Delay Differential Equations", Numerical Algorithms, 52(4): 509-521, (2009).