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# Generalisations of SDM Methods in *fpfs*-Matrices Space to Render Them Operable in *ifpifs*-Matrices Space and Their Application to Performance Ranking of the Noise-Removal Filters

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**Abstract** − Recently, the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) has allowed to mathematically model some problems in which the parameters and alternatives exhibit intuitionistic fuzzy uncertainties. To this end, the present study aims to generalise 24 soft decision-making (SDM) methods operating in the fuzzy parameterized fuzzy soft matrices space with a single *fpfs*-matrix to the *ifpifs*-matrices spaces. Afterwards, we propose five test scenarios to analyse whether the SDM methods constructed by *ifpifs*-matrices consistently work. Moreover, we make performance comparisons of the generalised SDM methods successful in the five test scenarios by applying them to the performance-based value assignment (PVA) problem of the well-known noise-removal filters. Finally, we discuss the need for further research.

***Keywords*** − *Fuzzy sets, intuitionistic fuzzy sets, soft sets, ifpifs-matrices, soft decision making*

***Mathematics Subject Classification (2020)*** − 03E72, 15B15

## 1. Introduction

The concept of intuitionistic fuzzy sets, characterized by membership and non-membership degree of an element's belonging to a set, has been propounded by Atanassov [1] as a generalization of fuzzy sets [2]. Furthermore, many hybrid versions of this concept, together with soft sets [3], have been introduced, such as intuitionistic fuzzy soft sets [4], intuitionistic fuzzy parameterized soft sets [5], intuitionistic fuzzy parameterized fuzzy soft sets [6], fuzzy parameterized intuitionistic fuzzy soft sets [7], and intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets (*ifpifs*-sets) [8]. Afterwards, the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) [9] modelling all the problems that these concepts can and allowing the data in such problems to be processed in a computer environment has been put forward. Thus, especially for the problems containing a large number of data, it has become possible to process these data on the computer. Therefore, a significant advantage has been gained in decision-making process.

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The recently proposed fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) [10] are capable of modelling situations where parameters and alternatives (objects) are fuzzy. Therefore, 136 soft decision-making (SDM) methods [11-22] constructed with the subspaces [3,23-29] of the *fpfs*-sets/matrices space between 1999 and 2019 have been rendered operable in the *fpfs*-matrices space. Moreover, more successful methods with shorter running time have been produced by mathematically simplifying some configured SDM methods [21,22,30-33]. However, these SDM methods are incapable of modelling the problems in which parameters and alternatives have intuitionistic fuzzy uncertainties. To this end, to model such problems and to achieve the same or similar modelling success of the SDM methods configured in the *fpfs*-matrices space within the *ifpifs*-matrices space, the generalisations of these methods have great importance. Hence, the main motivation of the present study is to generalise the SDM methods [10,11,15,16,20] operating with a single *fpfs*-matrix to the *ifpifs*-matrices space.

Section 2 presents several of the basic notions to be required in the next sections. Section 3 generalises the aforesaid SDM methods to the *ifpifs*-matrices space. Section 4 proposes five test cases to test the generalised SDM methods' performance of ranking the alternatives in the presence of decision-making problems and to determine the successful SDM methods. Section 5 applies the methods passing all the tests to a performance-based value assignment (PVA) problem and compares their ranking performances. The final section discusses the need for further research.

## 2. Preliminaries

This section presents the concepts of fuzzy sets [2], intuitionistic fuzzy sets [1], *ifpifs*-sets [8], and *ifpifs*-matrices [9] to be employed in the next sections. Throughout this study, let $U$ be a universal set and $E$ be a parameter set.

**Definition 2.1.** [2] Let $\mu$ be a function from $E$ to $[0,1]$. Then, the set $\left\{{}^{\mu(x)}x : x \in E\right\}$, being the graphic of $\mu$, is called a fuzzy set over $E$.

**Definition 2.2.** [1] Let $f$ be a function from $E$ to $[0,1] \times [0,1]$. Then, the set $\left\{\left(x, f(x)\right) | \, x \in E\right\}$, being the graphic of $f$, is called an intuitionistic fuzzy set over $E$.

Here, for all $x \in E$, $f(x) \coloneqq \left(\mu(x), \nu(x)\right)$ such that $0 \leq \mu(x) + \nu(x) \leq 1$. Moreover, $\mu$ and $\nu$ are called membership function and non-membership function in an intuitionistic fuzzy set, respectively. Thus, for brevity, we represent an intuitionistic fuzzy set over $E$ with $f \coloneqq \left\{{}^{\mu(x)}_{\nu(x)}x : x \in E\right\}$ instead of $f = \left\{\left(x, \mu(x), \nu(x)\right) : x \in E\right\}$. Besides, $IF(E)$ denotes the set of all the intuitionistic fuzzy sets over $E$. For convenience, we do not display the elements ${}^{0}_{1}x$ in an intuitionistic fuzzy set.

**Definition 2.3.** [8] Let $f \in IF(E)$ and $\alpha$ be a function from $f$ to $IF(U)$. Then, the set $\left\{\left({}^{\mu(x)}_{\nu(x)}x, \alpha\left({}^{\mu(x)}_{\nu(x)}x\right)\right) x \in E\right\}$, being the graphic of $\alpha$, is called an intuitionistic fuzzy parameterized intuitionistic fuzzy soft set (*ifpifs*-set) parameterized via $E$ over $U$ (or briefly over $U$).

In the present study, the set of all the *ifpifs*-sets over $U$ is denoted by $IFPIFS_E(U)$. In $IFPIFS_E(U)$, since the graph$(\alpha)$ and $\alpha$ generate each other uniquely, the notations are interchangeable. Therefore, as long as it causes no confusion, we denote an *ifpifs*-set graph$(\alpha)$ by $\alpha$. Moreover, for convenience, we do not display the elements $({}^{0}_{1}x, 0_U)$ in an *ifpifs*-set. Here, $0_U$ is the empty intuitionistic fuzzy set over $U$.

**Example 2.1.** Let $E = \{x_1, x_2, x_3, x_4\}$ and $U = \{u_1, u_2, u_3, u_4\}$. Then,

$$\alpha = \left\{\left({}^{0.3}_{0.6}x_1, \left\{{}^{0.6}_{0.2}u_1, {}^{0.4}_{0.3}u_3, {}^{0.1}_{0.5}u_4\right\}\right), \left({}^{0}_{1}x_2, \left\{{}^{0.6}_{0.3}u_1, {}^{0}_{0.8}u_2, {}^{0.1}_{0.4}u_4\right\}\right), \left({}^{0.4}_{0.4}x_4, \left\{{}^{0.2}_{0.7}u_1, {}^{1}_{0}u_3, {}^{0.5}_{0.5}u_4\right\}\right)\right\}$$

is an *ifpifs*-set over $U$.

**Definition 2.4.** [9] Let $\alpha \in IFPIFS_E(U)$. Then, $[a_{ij}]$ is called *ifpifs*-matrix of $\alpha$ and is defined by

$$[a_{ij}] := \begin{bmatrix} a_{01} & a_{02} & a_{03} & \cdots & a_{0n} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{0,1,2,\cdots\}$ and $j \in \{1,2,\cdots\}$,

$$a_{ij} := \begin{cases} {}^{\mu(x_j)}_{\nu(x_j)}, & i = 0 \\ \alpha\left({}^{\mu(x_j)}_{\nu(x_j)}x_j\right)(u_i), & i \neq 0 \end{cases}$$

or briefly $a_{ij} := {}^{\mu_{ij}}_{\nu_{ij}}$. Here, if $|U| = m-1$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

From now on, as long as it causes no confusion, the membership and non-membership functions of $[a_{ij}]$, i.e. $\mu_{ij}$ and $\nu_{ij}$, will be represented by $\mu_{ij}^{a}$ and $\nu_{ij}^{a}$, respectively. Moreover, the set of all the *ifpifs*-matrices parameterized via $E$ over $U$ is denoted by $IFPIFS_E[U]$.

**Example 2.2.** The *ifpifs*-matrix of $\alpha$ provided in Example 2.1 is as follows:

$$[a_{ij}] = \begin{bmatrix} {}^{0.3}_{0.6} & {}^{0}_{1} & {}^{0}_{1} & {}^{0.4}_{0.4} \\ {}^{0.6}_{0.2} & {}^{0.6}_{0.3} & {}^{0}_{1} & {}^{0.2}_{0.7} \\ {}^{0}_{1} & {}^{0}_{0.8} & {}^{0}_{1} & {}^{0}_{1} \\ {}^{0.4}_{0.3} & {}^{0}_{1} & {}^{0}_{1} & {}^{1}_{0} \\ {}^{0.1}_{0.5} & {}^{0.1}_{0.4} & {}^{0}_{1} & {}^{0.5}_{0.5} \end{bmatrix}$$

**Proposition 2.1.** [34] Let $IFV([0,1])$ be the set of all the intuitionistic fuzzy values and ${}^{\mu_1}_{\nu_1}, {}^{\mu_2}_{\nu_2} \in IFV([0,1])$. Then, the relation $\widetilde{\leqq}$ defined by

$${}^{\mu_1}_{\nu_1} \widetilde{\leqq} {}^{\mu_2}_{\nu_2} \Leftrightarrow \left[s_1\left({}^{\mu_1}_{\nu_1}\right) < s_1\left({}^{\mu_2}_{\nu_2}\right) \vee \left(s_1\left({}^{\mu_1}_{\nu_1}\right) = s_1\left({}^{\mu_2}_{\nu_2}\right) \wedge s_2\left({}^{\mu_1}_{\nu_1}\right) \leq s_2\left({}^{\mu_2}_{\nu_2}\right)\right)\right]$$

is a linear ordering relation over $IFV([0,1])$. Here, $s_1\left({}^{\mu_1}_{\nu_1}\right) := \mu_1 - \nu_1$ and $s_2\left({}^{\mu_1}_{\nu_1}\right) := \mu_1 + \nu_1$. Moreover, $s_1\left({}^{\mu_1}_{\nu_1}\right)$ and $s_2\left({}^{\mu_1}_{\nu_1}\right)$ are called score value and accuracy value of intuitionistic fuzzy value ${}^{\mu_1}_{\nu_1}$, respectively.

## 3. Generalisations of SDM Methods

This section generalises the SDM methods [10,11,15,16,20] employed a single *fpfs*-matrix and have been constructed with *fpfs*-matrices [10]. Hereinafter, $I_n$ indicates the set of all unsigned integer numbers from 1 to $n$, i.e., $I_n = \{1,2,\cdots,n\}$. Similarly, $I_n^*$ denotes the set of all nonnegative numbers from 0 to $n$, i.e., $I_n^* = \{0,1,2,\cdots,n\}$. Moreover, the variables (inputs) $R$, $w$, $\lambda$, $\lambda_1$, and $\lambda_2$ are used in algorithms. Here, $R$ is a set of indices, $w$ is an intuitionistic fuzzy valued row matrix, $\lambda \in (0,1]$, and $\lambda_1, \lambda_2 \in [0,1]$. Furthermore, the notation of each algorithm is created by inserting the first letter of the word “intuitionistic” at the beginning of the algorithm notation proposed in [16].

**Algorithm 3.1. iMBR01**

**Step 1.** Construct *ifpifs*-matrix $\left[a_{ij}\right]_{m\times n}$

**Step 2.** Obtain $\left[b_{ik}^1\right]_{(m-1)\times(m-1)}$ and $\left[b_{ik}^2\right]_{(m-1)\times(m-1)}$ defined by

$$b_{ik}^1 \coloneqq \sum_{j=1}^{n} \mu_{0j}^a \chi\left(\mu_{ij}^a, \mu_{kj}^a\right) \quad \text{and} \quad b_{ik}^2 \coloneqq \sum_{j=1}^{n} \nu_{0j}^a \psi\left(\nu_{ij}^a, \nu_{kj}^a\right)$$

such that $i, k \in I_{m-1}$,

$$\chi\left(\mu_{ij}^a, \mu_{kj}^a\right) \coloneqq \begin{cases} 1, & \mu_{ij}^a \geq \mu_{kj}^a \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \psi\left(\nu_{ij}^a, \nu_{kj}^a\right) \coloneqq \begin{cases} 0, & \nu_{ij}^a \leq \nu_{kj}^a \\ 1, & \text{otherwise} \end{cases}$$

**Step 3.** Obtain $[c_{i1}]_{(m-1)\times 1}$

$$c_{i1} \coloneqq \sum_{k=1}^{m-1} \left(b_{ik}^1 - b_{ki}^1\right), \quad i \in I_{m-1}$$

**Step 4.** Obtain $[d_{i1}]_{(m-1)\times 1}$

$$d_{i1} \coloneqq \sum_{k=1}^{m-1} \left(b_{ik}^2 - b_{ki}^2\right), \quad i \in I_{m-1}$$

**Step 5.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^s = \begin{cases} \dfrac{c_{i1} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^s = \begin{cases} 1 - \dfrac{c_{i1} + |d_{i1}| + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

**Step 6.** Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s} u_k \middle| u_k \in U\right\}$

**Algorithm 3.2. isMBR01**

**Step 1.** Construct *ifpifs*-matrix $\left[a_{ij}\right]_{m\times n}$

**Step 2.** Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} \coloneqq \sum_{k=1}^{m-1} \sum_{j=1}^{n} \mu_{0j}^a \operatorname{sgn}\left(\mu_{ij}^a - \mu_{kj}^a\right) \quad \text{and} \quad c_{i1} \coloneqq \sum_{k=1}^{m-1} \sum_{j=1}^{n} \nu_{0j}^a \operatorname{sgn}\left(\nu_{ij}^a - \nu_{kj}^a\right)$$

such that $i \in I_{m-1}$

**Step 3.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^s}{\nu_{i1}^s}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^s = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k \in I_{m-1}} \{b_{k1}\}\right|}{\max\limits_{k \in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{b_{k1}\}\right|}, & \max\limits_{k \in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k \in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{b_{k1}\}\right| = 0 \end{cases}$$

**Step 4.** Obtain the decision set $\left\{ {}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}} u_k \middle| u_k \in U \right\}$

**Algorithm 3.3. iMBR01/2**

**Step 1.** Construct *ifpifs*-matrix $\left[a_{ij}\right]_{m \times n}$

**Step 2.** Obtain $\left[b_{ik}^{1}\right]_{(m-1)\times(m-1)}$ and $\left[b_{ik}^{2}\right]_{(m-1)\times(m-1)}$ defined by

$$b_{ik}^{1} \coloneqq \sum_{j=1}^{n} \mu_{0j}^{a} \chi\left(\mu_{ij}^{a}, \mu_{kj}^{a}\right) \quad \text{and} \quad b_{ik}^{2} \coloneqq \sum_{j=1}^{n} \nu_{0j}^{a} \psi\left(\nu_{ij}^{a}, \nu_{kj}^{a}\right)$$

such that $i, k \in I_{m-1}$ and

$$\begin{matrix} \chi\left(\mu_{ij}^{a}, \mu_{kj}^{a}\right) \\ \psi\left(\nu_{ij}^{a}, \nu_{kj}^{a}\right) \end{matrix} \coloneqq \begin{cases} {}^{1}_{0}, & \mu_{ij}^{a} \geq \mu_{kj}^{a} \text{ ve } \nu_{ij}^{a} \leq \nu_{kj}^{a} \\ {}^{0}_{1}, & \text{otherwise} \end{cases}$$

**Step 3.** Obtain $[c_{i1}]_{(m-1)\times 1}$

$$c_{i1} \coloneqq \sum_{k=1}^{m-1} \left(b_{ik}^{1} - b_{ki}^{1}\right), \quad i \in I_{m-1}$$

**Step 4.** Obtain $[d_{i1}]_{(m-1)\times 1}$

$$d_{i1} \coloneqq \sum_{k=1}^{m-1} \left(b_{ik}^{2} - b_{ki}^{2}\right), \quad i \in I_{m-1}$$

**Step 5.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq {}^{\mu_{i1}^{s}}_{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{c_{i1} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k \in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k \in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k \in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{c_{i1} + |d_{i1}| + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k \in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k \in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k \in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k \in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

**Step 6.** Obtain the decision set $\left\{ {}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}} u_k \middle| u_k \in U \right\}$

**Algorithm 3.4. iMRB02($R$)**

**Step 1.** Construct an *ifpifs*-matrix $\left[a_{ij}\right]_{m \times n}$

**Step 2.** Determine a set $R$ of indices such that $R \subseteq I_n$

**Step 3.** Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} \coloneqq \sum_{j \in R} \mu_{0j}^{a} \mu_{ij}^{a} \quad \text{and} \quad c_{i1} \coloneqq \sum_{j \in R} \nu_{0j}^{a} \nu_{ij}^{a}$$

such that $i \in I_{m-1}$

**Step 4.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| = 0 \end{cases}$$

**Step 5.** Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}} u_k | u_k \in U\right\}$

**Algorithm 3.5. iKM11($\boldsymbol{R}$)**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Determine a set $R$ of indices such that $R \subseteq I_n$

**Step 3.** Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} \coloneqq \prod_{j\in R} \mu_{0j}^{a}\mu_{ij}^{a} \quad \text{and} \quad c_{i1} \coloneqq \prod_{j\in R} \nu_{0j}^{a}\nu_{ij}^{a}$$

such that $i \in I_{m-1}$

**Step 4.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}} \{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{b_{k1}\}\right| = 0 \end{cases}$$

**Step 5.** Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}} u_k | u_k \in U\right\}$

**Algorithm 3.6. iCCE11**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$, $K = \{j : \mu_{0j}^{a} \neq 0 \vee \nu_{0j}^{a} \neq 1\}$,

$$\mu_{i1}^{s} \coloneqq \begin{cases} \dfrac{1}{|K|}\sum_{j=1}^{n} \mu_{0j}^{a}\mu_{ij}^{a}, & |K| \neq 0 \\ 0, & |K| = 0 \end{cases} \quad \text{and} \quad \nu_{i1}^{s} \coloneqq \begin{cases} \dfrac{1}{|K|}\sum_{j=1}^{n} \nu_{0j}^{a}\nu_{ij}^{a}, & |K| \neq 0 \\ 0, & |K| = 0 \end{cases}$$

Here, $|K|$ denotes the cardinality of $K$.

**Step 3.** Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

**Algorithm 3.7. iYE12**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} := \begin{cases} \dfrac{1}{\sum_{j=1}^{n}\mu^a_{0j}}\displaystyle\sum_{j=1}^{n}\mu^a_{0j}\mu^a_{ij}, & \displaystyle\sum_{j=1}^{n}\mu^a_{0j} \neq 0 \\ 0, & \displaystyle\sum_{j=1}^{n}\mu^a_{0j} = 0 \end{cases} \quad \text{and} \quad \nu^s_{i1} := \begin{cases} \dfrac{1}{\sum_{j=1}^{n}\nu^a_{0j}}\displaystyle\sum_{j=1}^{n}\nu^a_{0j}\nu^a_{ij}, & \displaystyle\sum_{j=1}^{n}\nu^a_{0j} \neq 0 \\ 0, & \displaystyle\sum_{j=1}^{n}\nu^a_{0j} = 0 \end{cases}$$

**Step 3.** Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

**Algorithm 3.8. iCCE10**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} := \frac{1}{n}\sum_{j=1}^{n}\mu^a_{0j}\mu^a_{ij} \quad \text{and} \quad \nu^s_{i1} := \frac{1}{n}\sum_{j=1}^{n}\nu^a_{0j}\nu^a_{ij}$$

**Step 3.** Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

**Algorithm 3.9. iCEC11**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{1j}]_{1\times n}$ defined by $b_{1j} := {}^{\mu^b_{1j}}_{\nu^b_{1j}}$ such that $j \in I_n$,

$$\mu^b_{1j} := \frac{\mu^a_{0j}}{m-1}\sum_{i=1}^{m-1}\mu^a_{ij} \quad \text{and} \quad \nu^b_{1j} := \frac{\nu^a_{0j}}{m-1}\sum_{i=1}^{m-1}\nu^a_{ij}$$

**Step 3.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := {}^{\mu^s_{i1}}_{\nu^s_{i1}}$ such that $i \in I_{m-1}$,

$$\mu^s_{i1} := \frac{1}{n}\sum_{j=1}^{n}\mu^a_{ij}\mu^b_{1j} \quad \text{and} \quad \nu^s_{i1} := \frac{1}{n}\sum_{j=1}^{n}\nu^a_{ij}\nu^b_{1j}$$

**Step 4.** Obtain the decision set $\left\{{}^{\mu^s_{k1}}_{\nu^s_{k1}}u_k | u_k \in U\right\}$

**Algorithm 3.10. iM11**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b^1_{ik}]_{(m-1)\times(m-1)}$ and $[b^2_{ik}]_{(m-1)\times(m-1)}$ defined by

$$b^1_{ik} := \sum_{j=1}^{n}\mu^a_{0j}\left(\mu^a_{ij} - \mu^a_{kj}\right) \quad \text{and} \quad b^2_{ik} := \sum_{j=1}^{n}\nu^a_{0j}\left(\nu^a_{ij} - \nu^a_{kj}\right)$$

such that $i, k \in I_{m-1}$

**Step 3.** Obtain $[c_{i1}]_{(m-1)\times 1}$ ve $[d_{i1}]_{(m-1)\times 1}$ defined by

$$c_{i1} = \sum_{k=1}^{m-1} b_{ik}^1 \quad \text{and} \quad d_{i1} \coloneqq \sum_{k=1}^{m-1} b_{ik}^2$$

such that $i \in I_{m-1}$

**Step 4.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq {}^{\mu_{i1}^s}_{\nu_{i1}^s}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^s = \begin{cases} \dfrac{c_{i1} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^s = \begin{cases} 1 - \dfrac{c_{i1} + |d_{i1}| + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}{\max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}} \{c_{k1} + |d_{k1}|\} + \left|\min\limits_{k\in I_{m-1}} \{c_{k1}\}\right| = 0 \end{cases}$$

**Step 5.** Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s} u_k | u_k \in U\right\}$

**Algorithm 3.11. iKKT13**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{(n+1)\times n}$

**Step 2.** Obtain $[b_{i1}]_{n\times 1}$ defined by $b_{i1} \coloneqq {}^{\mu_{i1}^b}_{\nu_{i1}^b}$ such that $i \in I_n$,

$$\mu_{i1}^b \coloneqq \frac{1}{n}\sum_{j=1}^{n} \mu_{ij}^a \quad \text{and} \quad \nu_{i1}^b \coloneqq \frac{1}{n}\sum_{j=1}^{n} \nu_{ij}^b$$

**Step 3.** Obtain the score matrix $[s_{i1}]_{n\times 1}$ defined by $s_{i1} \coloneqq {}^{\mu_{i1}^s}_{\nu_{i1}^s}$ such that $i \in I_n$,

$$\mu_{i1}^s \coloneqq \mu_{0i}^a \mu_{i1}^b \quad \text{and} \quad \nu_{i1}^s \coloneqq \nu_{0i}^a + \nu_{i1}^b - \nu_{0i}^a \nu_{i1}^b$$

**Step 4.** Obtain the decision set $\left\{{}^{\mu_{k1}^s}_{\nu_{k1}^s} u_k | u_k \in U\right\}$

iKKT13 is used in decision-making problems containing the same number of alternatives and parameters.

**Algorithm 3.12. iFJLL10$(\boldsymbol{R}, \boldsymbol{w})$**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} \coloneqq {}^{\mu_{1j}}_{\nu_{1j}}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \leq \mu_{1j} + \nu_{1j} \leq 1$, for all $j \in I_n$

**Step 3.** Obtain $[b_{ij}]_{m\times n}$ defined by

$${}^{\mu_{ij}^b}_{\nu_{ij}^b} \coloneqq \begin{cases} {}^{1}_{0}, & \mu_{ij}^a \geq \mu_{1j} \text{ and } \nu_{ij}^a \leq \nu_{1j} \\ {}^{0}_{1}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

**Step 4.** Apply iMRB02$(R)$ to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.13. iFJLL10/2($\boldsymbol{R}, \boldsymbol{w}$)**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} := \begin{smallmatrix}\mu_{1j}\\ \nu_{1j}\end{smallmatrix}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \le \mu_{1j} + \nu_{1j} \le 1$, for all $j \in I_n$

**Step 3.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^b\\ \nu_{ij}^b\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^a\\ \nu_{0j}^a\end{matrix}, & i = 0 \\ \begin{matrix}1\\ 0\end{matrix}, & i \ne 0, \mu_{ij}^a \ge \mu_{1j}, \text{and } \nu_{ij}^a \le \nu_{1j} \\ \begin{matrix}0\\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

**Step 4.** Apply iMRB02($R$) to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.14. iFJLL10/3($\boldsymbol{R}, \boldsymbol{w}$)**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} := \begin{smallmatrix}\mu_{1j}\\ \nu_{1j}\end{smallmatrix}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \le \mu_{1j} + \nu_{1j} \le 1$, for all $j \in I_n$

**Step 3.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^b\\ \nu_{ij}^b\end{matrix} := \begin{cases} \begin{matrix}1\\ 0\end{matrix}, & \mu_{ij}^a \ge \max_{k\in I_n} \mu_{1k} \text{ and } \nu_{ik}^a \le \min_{k\in I_n} \nu_{1k} \\ \begin{matrix}0\\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

**Step 4.** Apply iMRB02($R$) to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.15. iFJLL10/4($\boldsymbol{R}, \boldsymbol{w}$)**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Construct an intuitionistic fuzzy valued row matrix $[w_{1j}]_{1\times n}$ defined by $w_{1j} := \begin{smallmatrix}\mu_{1j}\\ \nu_{1j}\end{smallmatrix}$ such that $\mu_{1j}, \nu_{1j} \in [0,1]$ and $0 \le \mu_{1j} + \nu_{1j} \le 1$, for all $j \in I_n$

**Step 3.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^b\\ \nu_{ij}^b\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^a\\ \nu_{0j}^a\end{matrix}, & i = 0 \\ \begin{matrix}1\\ 0\end{matrix}, & i \ne 0, \mu_{ij}^a \ge \max_{k\in I_n} \mu_{1k}, \text{and } \nu_{ij}^a \le \min_{k\in I_n} \nu_{1k} \\ \begin{matrix}0\\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^*$ and $j \in I_n$

**Step 4.** Apply iMRB02($R$) to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.16. iFJLL10m($\boldsymbol{R}$)**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}1 \\ 0\end{matrix}, & \mu_{ij}^{a} \geq \frac{1}{m-1}\sum_{k=1}^{m-1}\mu_{kj}^{a} \text{ and } \nu_{ij}^{a} \leq \frac{1}{m-1}\sum_{k=1}^{m-1}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

**Step 3.** Apply iMRB02$(R)$ to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.17. iFJLL10/2m$(\boldsymbol{R})$**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^{a} \\ \nu_{0j}^{a}\end{matrix}, & i = 0 \\ \begin{matrix}1 \\ 0\end{matrix}, & i \neq 0, \mu_{ij}^{a} \geq \frac{1}{m-1}\sum_{k=1}^{m-1}\mu_{kj}^{a}, \text{and } \nu_{ij}^{a} \leq \frac{1}{m-1}\sum_{k=1}^{m-1}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

**Step 3.** Apply iMRB02$(R)$ to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.18. iFJLL10max$(\boldsymbol{R})$**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}1 \\ 0\end{matrix}, & \mu_{ij}^{a} \geq \max_{k\in I_{m-1}}\mu_{kj}^{a} \text{ and } \nu_{ij}^{a} \leq \min_{k\in I_{m-1}}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

**Step 3.** Apply iMRB02$(R)$ to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.19. iFJLL10/2max$(\boldsymbol{R})$**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{ij}]_{m\times n}$ defined by

$$\begin{matrix}\mu_{ij}^{b} \\ \nu_{ij}^{b}\end{matrix} := \begin{cases} \begin{matrix}\mu_{0j}^{a} \\ \nu_{0j}^{a}\end{matrix}, & i = 0 \\ \begin{matrix}1 \\ 0\end{matrix}, & i \neq 0, \mu_{ij}^{a} \geq \max_{k\in I_{m-1}}\mu_{kj}^{a}, \text{and } \nu_{ij}^{a} \leq \min_{k\in I_{m-1}}\nu_{kj}^{a} \\ \begin{matrix}0 \\ 1\end{matrix}, & \text{otherwise} \end{cases}$$

such that $i \in I_{m-1}^{*}$ and $j \in I_n$

**Step 3.** Apply iMRB02$(R)$ to $[b_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.20. iF10$(\boldsymbol{\lambda})$**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{1j}]_{1\times n}$ defined by $b_{1j} := \begin{matrix}\mu_{1j}^{b} \\ \nu_{1j}^{b}\end{matrix}$ such that $i \in I_{m-1}$, $j \in I_n$, $\delta_i := f\left(\frac{i}{m-1}\right) - f\left(\frac{i-1}{m-1}\right)$,

$$\mu_{1j}^{b} \coloneqq \sum_{i=1}^{m-1} \mu_j^{a_i}\delta_i \quad \text{and} \quad \nu_{1j}^{b} \coloneqq \sum_{i=1}^{m-1} \nu_j^{a_i}\delta_i$$

Here, for $\lambda \in (0,1]$, $f$ is a function defined by $f(x) = x^{\frac{1-\lambda}{\lambda}}$. Moreover, $\mu_j^{a_i}$ denotes $i^{th}$ largest membership degree of the elements with index nonzero in $j^{th}$ column of $[a_{ij}]$. Similarly, $\nu_j^{a_i}$ indicates $i^{th}$ smallest non-membership degree of the elements with index nonzero in $j^{th}$ column of $[a_{ij}]$.

**Step 3.** Obtain $[c_{ij}]_{m\times n}$ defined by $c_{ij} \coloneqq \frac{\mu_{ij}^{c}}{\nu_{ij}^{c}}$ such that $i \in I_{m-1}^{*}$, $j \in I_n$, and

$$\frac{\mu_{ij}^{c}}{\nu_{ij}^{c}} \coloneqq \begin{cases} \frac{\mu_{0j}^{a}}{\nu_{0j}^{a}}, & i = 0 \\ \frac{1}{0}, & i \neq 0, \mu_{ij}^{a} \geq \mu_{1j}^{b}, \text{and } \nu_{ij}^{a} \leq \nu_{1j}^{b} \\ \frac{0}{1}, & \text{otherwise} \end{cases}$$

**Step 4.** Apply iMRB02$(R)$ to $[c_{ij}]$ such that $R \subseteq I_n$

**Algorithm 3.21. iKSM10**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{ij}]_{m\times n}$ and $[c_{ij}]_{m\times n}$ defined by

$$b_{ij} \coloneqq \begin{cases} \frac{1}{n-1}\left(1 - \frac{b_{1j}^{*}}{\sum_{k=1}^{n} b_{1k}^{*}}\right), & i = 0 \text{ and } \sum_{k=1}^{n} b_{1k}^{*} \neq 0 \\ \frac{1}{n}, & i = 0 \text{ and } \sum_{k=1}^{n} b_{1k}^{*} = 0 \\ \mu_{ij}^{a}, & i \neq 0 \end{cases}$$

and

$$c_{ij} \coloneqq \begin{cases} \frac{1}{n-1}\left(1 - \frac{c_{1j}^{*}}{\sum_{k=1}^{n} c_{1k}^{*}}\right), & i = 0 \text{ and } \sum_{k=1}^{n} c_{1k}^{*} \neq 0 \\ \frac{1}{n}, & i = 0 \text{ and } \sum_{k=1}^{n} c_{1k}^{*} = 0 \\ \nu_{ij}^{a}, & i \neq 0 \end{cases}$$

$i \in I_{m-1}^{*}$ and $j \in I_n$

Here,

$$b_{1j}^{*} \coloneqq \begin{cases} \frac{1}{m-1}\sum_{i=1}^{m-1}\left(\mu_{02}^{a}\mu_{i2}^{a} - \mu_{01}^{a}\mu_{i1}^{a}\right), & j = 1 \\ \frac{1}{2(m-1)}\sum_{i=1}^{m-1}\left(\mu_{0(j+1)}^{a}\mu_{i(j+1)}^{a} - \mu_{0(j-1)}^{a}\mu_{i(j-1)}^{a}\right), & j \in \{2,3,\dots,n-1\} \\ \frac{1}{m-1}\sum_{i=1}^{m-1}\left(\mu_{0n}^{a}\mu_{in}^{a} - \mu_{0(n-1)}^{a}\mu_{i(n-1)}^{a}\right), & j = n \end{cases}$$

and

$$c_{1j}^{*} := \begin{cases} \dfrac{1}{m-1}\sum_{i=1}^{m-1}\left(\nu_{02}^{a}\nu_{i2}^{a}-\nu_{01}^{a}\nu_{i1}^{a}\right), & j=1 \\ \dfrac{1}{2(m-1)}\sum_{i=1}^{m-1}\left(\nu_{0(j+1)}^{a}\nu_{i(j+1)}^{a}-\nu_{0(j-1)}^{a}\nu_{i(j-1)}^{a}\right), & j\in\{2,3,\dots,n-1\} \\ \dfrac{1}{m-1}\sum_{i=1}^{m-1}\left(\nu_{0n}^{a}\nu_{in}^{a}-\nu_{0(n-1)}^{a}\nu_{i(n-1)}^{a}\right), & j=n \end{cases}$$

such that $j \in I_n$

**Step 3.** Obtain $[d_{ij}]_{m\times n}$ defined by $d_{ij} := \begin{matrix}\mu_{ij}^{d}\\ \nu_{ij}^{d}\end{matrix}$ such that $i \in I_{m-1}^{*}$, $j \in I_n$,

$$\mu_{ij}^{d} = \begin{cases} \dfrac{b_{ij}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{kj}+|c_{kj}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{kj}+|c_{kj}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{kj}+|c_{kj}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| = 0 \end{cases}$$

and

$$\nu_{ij}^{d} = \begin{cases} 1-\dfrac{b_{ij}+|c_{ij}|+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{kj}+|c_{kj}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{kj}+|c_{kj}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{b_{kj}+|c_{kj}|\}+\left|\min\limits_{k\in I_{m-1}}\{b_{kj}\}\right| = 0 \end{cases}$$

**Step 4.** Apply iM11 to $[d_{ij}]$

**Algorithm 3.22. iKWW11($\lambda_1, \lambda_2$)**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Apply iMBR01 and iMRB02($R$) to $[a_{ij}]$ such that $R \subseteq I_n$ and obtain the score matrices $[s_{i1}]_{(m-1)\times 1}$ and $[\tilde{s}_{i1}]_{(m-1)\times 1}$, respectively

**Step 3.** Obtain $[b_{i1}^{1}]_{(m-1)\times 1}$, $[b_{i1}^{2}]_{(m-1)\times 1}$, $[c_{i1}^{1}]_{(m-1)\times 1}$, and $[c_{i1}^{2}]_{(m-1)\times 1}$ defined by

$$b_{i1}^{1} := \max_{k\in I_{m-1}} \mu_{k1}^{s} - \mu_{i1}^{s} \quad \text{and} \quad b_{i1}^{2} := \left|\min_{k\in I_{m-1}} \nu_{k1}^{s} - \nu_{i1}^{s}\right|$$

and

$$c_{i1}^{1} := \max_{k\in I_{m-1}} \mu_{k1}^{\tilde{s}} - \mu_{i1}^{\tilde{s}} \quad \text{and} \quad c_{i1}^{2} := \left|\min_{k\in I_{m-1}} \nu_{k1}^{\tilde{s}} - \nu_{i1}^{\tilde{s}}\right|$$

such that $i \in I_{m-1}$

**Step 4.** For $\lambda_1 \in [0,1]$, obtain $[d_{i1}^{1}]_{(m-1)\times 1}$, $[d_{i1}^{2}]_{(m-1)\times 1}$, $[e_{i1}^{1}]_{(m-1)\times 1}$, and $[e_{i1}^{2}]_{(m-1)\times 1}$defined by

$$d_{i1}^{1} := \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}}\{\min\{b_{k1}^{1},c_{k1}^{1}\}\}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{1},c_{k1}^{1}\}\}}{b_{i1}^{1}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{1},c_{k1}^{1}\}\}}, & b_{i1}^{1}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{1},c_{k1}^{1}\}\} \neq 0 \\ 1, & b_{i1}^{1}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{1},c_{k1}^{1}\}\} = 0 \end{cases}$$

$$d_{i1}^{2} := \begin{cases} 1-\dfrac{\min\limits_{k\in I_{m-1}}\{\min\{b_{k1}^{2},c_{k1}^{2}\}\}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{2},c_{k1}^{2}\}\}}{c_{i1}^{2}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{2},c_{k1}^{2}\}\}}, & c_{i1}^{2}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{2},c_{k1}^{2}\}\} \neq 0 \\ 0, & c_{i1}^{2}+\lambda_1\max\limits_{k\in I_{m-1}}\{\max\{b_{k1}^{2},c_{k1}^{2}\}\} = 0 \end{cases}$$

$$e_{i1}^{1} \coloneqq \begin{cases} \dfrac{\min\limits_{k\in I_{m-1}}\left\{\min\left\{b_{k1}^{1}, c_{k1}^{1}\right\}\right\} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{1}, c_{k1}^{1}\right\}\right\}}{b_{i1}^{1} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{1}, c_{k1}^{1}\right\}\right\}}, & b_{i1}^{1} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{1}, c_{k1}^{1}\right\}\right\} \neq 0 \\ 1, & b_{i1}^{1} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{1}, c_{k1}^{1}\right\}\right\} = 0 \end{cases}$$

and

$$e_{i1}^{2} \coloneqq \begin{cases} 1 - \dfrac{\min\limits_{k\in I_{m-1}}\left\{\min\left\{b_{k1}^{2}, c_{k1}^{2}\right\}\right\} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{2}, c_{k1}^{2}\right\}\right\}}{c_{i1}^{2} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{2}, c_{k1}^{2}\right\}\right\}}, & c_{i1}^{2} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{2}, c_{k1}^{2}\right\}\right\} \neq 0 \\ 0, & c_{i1}^{2} + \lambda_1 \max\limits_{k\in I_{m-1}}\left\{\max\left\{b_{k1}^{2}, c_{k1}^{2}\right\}\right\} = 0 \end{cases}$$

such that $i \in I_{m-1}$

**Step 5.** For $\lambda_2 \in [0,1]$, obtain $\left[f_{i1}^{1}\right]_{(m-1)\times 1}$ and $\left[f_{i1}^{2}\right]_{(m-1)\times 1}$ defined by

$$f_{i1}^{1} \coloneqq \lambda_2 d_{i1}^{1} + (1-\lambda_2)e_{i1}^{1} \quad \text{and} \quad f_{i1}^{2} \coloneqq \lambda_2 d_{i1}^{2} + (1-\lambda_2)e_{i1}^{2}$$

such that $i \in I_{m-1}$

**Step 6.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{f_{i1}^{1} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right|}{\max\limits_{k\in I_{m-1}}\left\{f_{k1}^{1} + \left|f_{k1}^{2}\right|\right\} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right|}, & \max\limits_{k\in I_{m-1}}\left\{f_{k1}^{1} + \left|f_{k1}^{2}\right|\right\} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\left\{f_{k1}^{1} + \left|f_{k1}^{2}\right|\right\} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{f_{i1}^{1} + \left|f_{i1}^{2}\right| + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right|}{\max\limits_{k\in I_{m-1}}\left\{f_{k1}^{1} + \left|f_{k1}^{2}\right|\right\} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right|}, & \max\limits_{k\in I_{m-1}}\left\{f_{k1}^{1} + \left|f_{k1}^{2}\right|\right\} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\left\{f_{k1}^{1} + \left|f_{k1}^{2}\right|\right\} + \left|\min\limits_{k\in I_{m-1}}\left\{f_{k1}^{1}\right\}\right| = 0 \end{cases}$$

**Step 7.** Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}}u_k \middle| u_k \in U\right\}$

**Algorithm 3.23. iSM11**

**Step 1.** Construct an *ifpifs*-matrix $\left[a_{ij}\right]_{m\times n}$

**Step 2.** Obtain $\left[b_{1j}^{1}\right]_{1\times n}$ and $\left[b_{1j}^{2}\right]_{1\times n}$ defined by

$$b_{1j}^{1} \coloneqq \max_{i\in I_{m-1}}\left\{\mu_{0j}^{a}\mu_{ij}^{a}\right\} \quad \text{and} \quad b_{1j}^{2} \coloneqq \min_{i\in I_{m-1}}\left\{\nu_{0j}^{a}\nu_{ij}^{a}\right\}$$

such that $j \in I_n$

**Step 3.** Obtain $\left[c_{i1}^{1}\right]_{(m-1)\times 1}$ and $\left[c_{i1}^{2}\right]_{(m-1)\times 1}$ defined by

$$c_{i1}^{1} \coloneqq \min_{j\in I_n}\left\{\max\left\{1-\mu_{0j}^{a}\mu_{ij}^{a}, b_{1j}^{1}\right\}\right\} \quad \text{and} \quad c_{i1}^{2} \coloneqq \max_{j\in I_n}\left\{\min\left\{1-\nu_{0j}^{a}\nu_{ij}^{a}, b_{1j}^{2}\right\}\right\}$$

such that $i \in I_{m-1}$

**Step 4.** Obtain $\left[d_{i1}^{1}\right]_{(m-1)\times 1}$ and $\left[d_{i1}^{2}\right]_{(m-1)\times 1}$ defined by

$$d_{i1}^{1} \coloneqq \max_{j\in I_n}\left\{\min\left\{\mu_{0j}^{a}\mu_{ij}^{a}, b_{1j}^{1}\right\}\right\} \quad \text{and} \quad d_{i1}^{2} \coloneqq \min_{j\in I_n}\left\{\max\left\{\nu_{0j}^{a}\nu_{ij}^{a}, b_{1j}^{2}\right\}\right\}$$

such that $i \in I_{m-1}$

**Step 5.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$

$$\mu_{i1}^{s} \coloneqq c_{i1}^{1} + d_{i1}^{1} - c_{i1}^{1}d_{i1}^{1} \quad \text{and} \quad \nu_{i1}^{s} \coloneqq c_{i1}^{2}d_{i1}^{2}$$

**Step 6.** Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}}u_k \middle| u_k \in U\right\}$

**Algorithm 3.24. iPEM/iEC20**

**Step 1.** Construct an *ifpifs*-matrix $[a_{ij}]_{m\times n}$

**Step 2.** Obtain $[b_{i1}]_{(m-1)\times 1}$ and $[c_{i1}]_{(m-1)\times 1}$ defined by

$$b_{i1} := \sum_{j=1}^{n}\left[\left(\frac{1}{m-1}\sum_{k=1}^{m-1}\mu_{kj}^{a}\right)\left(\frac{1}{n}\sum_{t=1}^{n}\mu_{it}^{a}\right)\mu_{0j}^{a}\mu_{ij}^{a}\right] \quad \text{and} \quad c_{i1} := \sum_{j=1}^{n}\left[\left(\frac{1}{m-1}\sum_{k=1}^{m-1}\nu_{kj}^{a}\right)\left(\frac{1}{n}\sum_{t=1}^{n}\nu_{it}^{a}\right)\nu_{0j}^{a}\nu_{ij}^{a}\right]$$

such that $i \in I_{m-1}$

**Step 3.** Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \frac{\mu_{i1}^{s}}{\nu_{i1}^{s}}$ such that $i \in I_{m-1}$,

$$\mu_{i1}^{s} = \begin{cases} \dfrac{b_{i1} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 1, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

and

$$\nu_{i1}^{s} = \begin{cases} 1 - \dfrac{b_{i1} + |c_{i1}| + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}{\max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right|}, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| \neq 0 \\ 0, & \max\limits_{k\in I_{m-1}}\{b_{k1} + |c_{k1}|\} + \left|\min\limits_{k\in I_{m-1}}\{b_{k1}\}\right| = 0 \end{cases}$$

**Step 4.** Obtain the decision set $\left\{{}^{\mu_{k1}^{s}}_{\nu_{k1}^{s}}u_k | u_k \in U\right\}$

## 4. Proposed Test Cases for Generalised SDM Methods

This section proposes five new test cases by availing of the test cases provided in [13] to compare the decision-making performances of the generalised SDM methods. Each test case generating the same ranking order of the alternatives without using an SDM method consists of $t$ *ifpifs*-matrices $[a_{ij}^{1}]$, $[a_{ij}^{2}]$, …, $[a_{ij}^{t}]$, with the order of $m \times n$. If an SDM method produces the same ranking order of the alternatives presented in a given test case, it means that the method is successful therein. Moreover, Proposition 2.1 of this study is utilised to rank the alternatives in the proposed test cases. Besides, because the numbers of the alternatives and the parameters are required to be equal in Test Case 3 and Test Case 4, we utilise equal numbers of alternatives and parameters in the remaining test cases as well. Therefore, for all the test cases, let $U = \{u_1, u_2, \dots, u_n\}$ be the set of alternatives and $E = \{x_1, x_2, \dots, x_n\}$ be the set of parameters.

### 4.1. Test Case 1

Test case 1 constructs *ifpifs*-matrices $[a_{ij}^{1}]_{(n+1)\times n}$, $[a_{ij}^{2}]_{(n+1)\times n}$, …, $[a_{ij}^{t}]_{(n+1)\times n}$ such that, $\mu_{01}^{a^k} = \mu_{02}^{a^k} = \cdots = \mu_{0n}^{a^k}$, $\nu_{01}^{a^k} = \nu_{02}^{a^k} = \cdots = \nu_{0n}^{a^k}$, $\mu_{1j}^{a^k} < \mu_{2j}^{a^k} < \cdots < \mu_{nj}^{a^k}$, and $\nu_{nj}^{a^k} < \cdots < \nu_{2j}^{a^k} < \nu_{1j}^{a^k}$, for all $k \in I_t$ and $j \in I_n$. Therefore,

$$\mu_{0j}^{a^k}\mu_{1j}^{a^k} - \nu_{0j}^{a^k}\nu_{1j}^{a^k} < \mu_{0j}^{a^k}\mu_{2j}^{a^k} - \nu_{0j}^{a^k}\nu_{2j}^{a^k} < \cdots < \mu_{0j}^{a^k}\mu_{nj}^{a^k} - \nu_{0j}^{a^k}\nu_{nj}^{a^k}$$

for all $k \in I_t$ and $j \in I_n$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_1 \prec u_2 \prec \cdots \prec u_n$.

## 4.2. Test Case 2

Test case 2 constructs *ifpifs*-matrices $[b_{ij}^1]_{(n+1)\times n}, [b_{ij}^2]_{(n+1)\times n}, \ldots, [b_{ij}^t]_{(n+1)\times n}$ such that, $\mu_{01}^{b^k} = \mu_{02}^{b^k} = \cdots = \mu_{0n}^{b^k}$, $\nu_{01}^{b^k} = \nu_{02}^{b^k} = \cdots = \nu_{0n}^{b^k}$, $\mu_{nj}^{b^k} < \cdots < \mu_{2j}^{b^k} < \mu_{1j}^{b^k}$, and $\nu_{1j}^{b^k} < \nu_{2j}^{b^k} < \cdots < \nu_{nj}^{b^k}$, for all $k \in I_t$ and $j \in I_n$. Therefore,

$$\mu_{0j}^{b^k}\mu_{nj}^{b^k} - \nu_{0j}^{b^k}\nu_{nj}^{b^k} < \cdots < \mu_{0j}^{b^k}\mu_{2j}^{b^k} - \nu_{0j}^{b^k}\nu_{2j}^{b^k} < \mu_{0j}^{b^k}\mu_{1j}^{b^k} - \nu_{0j}^{b^k}\nu_{1j}^{b^k}$$

for all $k \in I_t$ and $j \in I_n$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_n \prec \cdots \prec u_2 \prec u_1$.

## 4.3. Test Case 3

Test case 3 constructs *ifpifs*-matrices $[c_{ij}^1]_{(n+1)\times n}, [c_{ij}^2]_{(n+1)\times n}, \ldots, [c_{ij}^t]_{(n+1)\times n}$ such that for all $i,j \in I_n$ and $k \in I_t$, $\mu_{01}^{c^k} < \mu_{02}^{c^k} < \cdots < \mu_{0n}^{c^k}$ and $\nu_{0n}^{c^k} < \cdots < \nu_{02}^{c^k} < \nu_{01}^{c^k}$, $\frac{\mu_{ii}^{c^k}}{\nu_{ii}^{c^k}} = \frac{\lambda}{\varepsilon}$ such that $\lambda, \varepsilon \in [0,1]$ and $\lambda + \varepsilon \leq 1$, and if $i \neq j$, then $\frac{\mu_{ij}^{c^k}}{\nu_{ij}^{c^k}} = \frac{0}{1}$. Therefore,

$$\mu_{01}^{c^k}\mu_{11}^{c^k} - \nu_{01}^{c^k}\nu_{11}^{c^k} < \mu_{02}^{c^k}\mu_{22}^{c^k} - \nu_{02}^{c^k}\nu_{22}^{c^k} < \cdots < \mu_{0n}^{c^k}\mu_{nn}^{c^k} - \nu_{0n}^{c^k}\nu_{nn}^{c^k}$$

and if $i \neq j$, then $\mu_{0j}^{c^k}\mu_{ij}^{c^k} - \nu_{0j}^{c^k}\nu_{ij}^{c^k} = 0 - \nu_{0j}^{c^k} = -\nu_{0j}^{c^k}$, for all $i,j \in I_n$ and $k \in I_t$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_1 \prec u_2 \prec \cdots \prec u_n$.

## 4.4. Test Case 4

Test case 4 constructs *ifpifs*-matrices $[d_{ij}^1]_{(n+1)\times n}, [d_{ij}^2]_{(n+1)\times n}, \ldots, [d_{ij}^t]_{(n+1)\times n}$ such that for all $i,j \in I_n$ and $k \in I_t$, $\mu_{0n}^{d^k} < \cdots < \mu_{02}^{d^k} < \mu_{01}^{d^k}$ and $\nu_{01}^{d^k} < \nu_{02}^{d^k} < \cdots < \nu_{0n}^{d^k}$, $\frac{\mu_{ii}^{d^k}}{\nu_{ii}^{d^k}} = \frac{\lambda}{\varepsilon}$ such that $\lambda, \varepsilon \in [0,1]$ and $\lambda + \varepsilon \leq 1$, and if $i \neq j$, then $\frac{\mu_{ij}^{d^k}}{\nu_{ij}^{d^k}} = \frac{0}{1}$. Therefore,

$$\mu_{0n}^{d^k}\mu_{nn}^{d^k} - \nu_{0n}^{d^k}\nu_{nn}^{d^k} < \cdots < \mu_{02}^{d^k}\mu_{22}^{d^k} - \nu_{02}^{d^k}\nu_{22}^{d^k} < \mu_{01}^{d^k}\mu_{11}^{d^k} - \nu_{01}^{d^k}\nu_{11}^{d^k}$$

and if $i \neq j$, then $\mu_{0j}^{d^k}\mu_{ij}^{d^k} - \nu_{0j}^{d^k}\nu_{ij}^{d^k} = 0 - \nu_{0j}^{d^k} = -\nu_{0j}^{d^k}$, for all $i,j \in I_n$ and $k \in I_t$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_n \prec \cdots \prec u_2 \prec u_1$.

## 4.5. Test Case 5

Test case 5 constructs *ifpifs*-matrices $[e_{ij}^1]_{(n+1)\times n}, [e_{ij}^2]_{(n+1)\times n}, \ldots, [e_{ij}^t]_{(n+1)\times n}$ such that for all $i,j \in I_n$ and $k \in I_t$, $\frac{\mu_{ii}^{e^k}}{\nu_{ii}^{e^k}} = \frac{\lambda}{\varepsilon}$, $\lambda, \varepsilon \in [0,1]$, and $\lambda + \varepsilon \leq 1$. Therefore, $\mu_{ij}^{e^k} - \nu_{ij}^{e^k} = \mu_{lj}^{e^k} - \nu_{lj}^{e^k}$ and $\mu_{ij}^{e^k} + \nu_{ij}^{e^k} = \mu_{lj}^{e^k} + \nu_{lj}^{e^k}$, for all $i,l,j \in I_n$ and $k \in I_t$. For each *ifpifs*-matrix herein, the ranking order of alternatives is $u_1 \approx u_2 \approx \cdots \approx u_n$. Here, $\approx$ denotes the same ranking order.

## 4.6. Results of Test Cases

This subsection tests the generalised SDM methods using aforesaid test cases. Thus, it determines SDM methods being successful in all the test cases. Since the generalised SDM methods herein employ only a single matrix, we consider $t = 1$, $n = 4$, $U = \{u_1, u_2, u_3, u_4\}$, and $E = \{x_1, x_2, x_3, x_4\}$, for all the test cases. Therefore, we use *ifpifs*-matrices provided in Table 1.

**Table 1.** *ifpifs*-matrices employed in the test cases

| Test Case 1 | Test Case 2 | Test Case 3 | Test Case 4 | Test Case 5 |
|---|---|---|---|---|
| $[a_{ij}^1] \coloneqq \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.3 & 0.3 \\[1ex] 0.7 & 0.6 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.25 & 0.3 \\[1ex] 0.8 & 0.7 & 0.6 & 0.5 \\ 0.1 & 0.15 & 0.2 & 0.25 \\[1ex] 0.9 & 0.8 & 0.7 & 0.6 \\ 0.05 & 0.1 & 0.15 & 0.2 \\[1ex] 1 & 0.9 & 0.8 & 0.7 \\ 0 & 0.05 & 0.1 & 0.15 \end{bmatrix}$ | $[b_{ij}^1] \coloneqq \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.8 \\ 0.1 & 0.1 & 0.1 & 0.1 \\[1ex] 1 & 0.9 & 0.8 & 0.7 \\ 0 & 0.05 & 0.1 & 0.15 \\[1ex] 0.9 & 0.8 & 0.7 & 0.6 \\ 0.05 & 0.1 & 0.15 & 0.2 \\[1ex] 0.8 & 0.7 & 0.6 & 0.5 \\ 0.1 & 0.15 & 0.2 & 0.25 \\[1ex] 0.7 & 0.6 & 0.5 & 0.4 \\ 0.15 & 0.2 & 0.25 & 0.3 \end{bmatrix}$ | $[c_{ij}^1] \coloneqq \begin{bmatrix} 0.6 & 0.7 & 0.8 & 0.9 \\ 0.2 & 0.15 & 0.1 & 0.05 \\[1ex] 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\[1ex] 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\[1ex] 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\[1ex] 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ | $[d_{ij}^1] \coloneqq \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 \\ 0.05 & 0.1 & 0.15 & 0.2 \\[1ex] 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\[1ex] 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\[1ex] 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\[1ex] 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ | $[e_{ij}^1] \coloneqq \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\[1ex] 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\[1ex] 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\[1ex] 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \\[1ex] 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$ |

Moreover, since all the parameters should be considered in all the test cases, we determine the variables (inputs) $R$, $R_1$, $R_2$, and $R_3$ as $I_4$. Additionally, the other variables are intently selected so that the methods can pass the largest number of test cases. Table 2 provides the test results of the SDM methods created using MATLAB R2021a. Furthermore, the numbers of the passed tests are presented in the last column of Table 2. 12 of 24 methods are observed to be successful in all the test cases. These methods are iMBR01, isMBR01, iMBR01/2, iMRB02($I_4$), iCCE11, iCCE10, iCEC11, iKKT13, iFJLL10/2$\left(I_4, \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}\right)$ (Test 1, 2), iFJLL10/2$\left(I_4, \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}\right)$ (Test 3, 4, 5), iFJLL10/4$\left(I_4, \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}\right)$ (Test 1, 2), iFJLL10/4$\left(I_4, \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}\right)$ (Test 3, 4, 5), iKWW11($I_4$, 0.5,0.5), and iPEM.

**Table 2.** Performance of the generalised SDM methods in the test cases

| | **Algorithms\Test Cases** | **Test Case 1** | **Test Case 2** | **Test Case 3** | **Test Case 4** | **Test Case 5** | **Numbers of Tests Passed** |
|---|---|---|---|---|---|---|---|
| 1. | iMBR01 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 2. | isMBR01 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 3. | iMBR01/2 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 4. | iMRB02($I_4$) | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 5. | iKM11($I_4$) | ✓ | ✓ | – | – | ✓ | 3 |
| 6. | iCCE11 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 7. | iYE12 | ✓ | ✓ | – | – | ✓ | 3 |
| 8. | iCCE10 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 9. | iCEC11 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 10. | iM11 | – | – | – | – | ✓ | 1 |
| 11. | iKKT13 | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12. | iFJLL10$\left(I_4,\begin{bmatrix}0.7 & 0.7 & 0.7 & 0.7\\ 0.1 & 0.1 & 0.1 & 0.1\end{bmatrix}\right)$ (Test 1, 2)<br>iFJLL10$\left(I_4,\begin{bmatrix}0.4 & 0.4 & 0.4 & 0.4\\ 0.4 & 0.4 & 0.4 & 0.4\end{bmatrix}\right)$ (Test 3, 4, 5) | – | ✓ | – | – | ✓ | 2 |
| 13. | iFJLL10/2$\left(I_4,\begin{bmatrix}0.7 & 0.7 & 0.7 & 0.7\\ 0.1 & 0.1 & 0.1 & 0.1\end{bmatrix}\right)$ (Test 1, 2)<br>iFJLL10/2$\left(I_4,\begin{bmatrix}0.4 & 0.4 & 0.4 & 0.4\\ 0.4 & 0.4 & 0.4 & 0.4\end{bmatrix}\right)$ (Test 3, 4, 5) | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 14. | iFJLL10/3$\left(I_4,\begin{bmatrix}0.7 & 0.7 & 0.7 & 0.7\\ 0.1 & 0.1 & 0.1 & 0.1\end{bmatrix}\right)$ (Test 1, 2)<br>iFJLL10/3$\left(I_4,\begin{bmatrix}0.4 & 0.4 & 0.4 & 0.4\\ 0.4 & 0.4 & 0.4 & 0.4\end{bmatrix}\right)$ (Test 3, 4, 5) | – | ✓ | – | – | ✓ | 2 |
| 15. | iFJLL10/4$\left(I_4,\begin{bmatrix}0.7 & 0.7 & 0.7 & 0.7\\ 0.1 & 0.1 & 0.1 & 0.1\end{bmatrix}\right)$ (Test 1, 2)<br>iFJLL10/4$\left(I_4,\begin{bmatrix}0.4 & 0.4 & 0.4 & 0.4\\ 0.4 & 0.4 & 0.4 & 0.4\end{bmatrix}\right)$ (Test 3, 4, 5) | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 16. | iFJLL10m$(I_4)$ | – | – | – | – | ✓ | 1 |
| 17. | iFJLL10/2m$(I_4)$ | – | – | ✓ | ✓ | ✓ | 3 |
| 18. | iFJLL10max$(I_4)$ | – | – | – | – | ✓ | 1 |
| 19. | iFJLL10/2max$(I_4)$ | – | – | ✓ | ✓ | ✓ | 3 |
| 20. | iF10$(I_4, 0.5)$ | – | – | ✓ | ✓ | ✓ | 3 |
| 21. | iKSM10 | – | – | – | – | ✓ | 1 |
| 22. | iKWW11$(I_4, 0.5, 0.5)$ | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| 23. | iSM11 | – | ✓ | ✓ | ✓ | ✓ | 4 |
| 24. | iPEM | ✓ | ✓ | ✓ | ✓ | ✓ | **5** |
| | Total | 14 | 17 | 16 | 16 | 24 | 12 |

Bold values in the last column indicate the SDM methods passing all the test cases (✓: Successful, –: Unsuccessful)

## 5. An Application of the Generalised SDM Methods Being Successful in All the Test Cases to a PVA Problem

This section applies the SDM methods generalised in Section 3, which is successful in all test cases to a real problem related to performance-based value assignment (PVA) to seven noise-removal filters, namely "Based on Pixel Density Filter (BPDF)" [35], "Decision-Based Algorithm (DBAIN)" [36], "Modified Decision Based Unsymmetrical Trimmed Median Filter (MDBUTMF)" [37], "Noise Adaptive Fuzzy Switching Median Filter (NAFSMF)" [38], "Different Applied Median Filter (DAMF)" [39], "Adaptive Weighted Mean Filter (AWMF)" [40], and "Adaptive Riesz Mean Filter (ARmF)" [41]. In this PVA problem, we indicate the set of alternatives $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ such that $u_1 =$ "BPDF", $u_2 =$ "DBAIN", $u_3 =$ "MDBUTMF", $u_4 =$ "NAFSMF", $u_5 =$ "DAMF", $u_6 =$ "AWMF", and $u_7 =$ "ARmF". Moreover, we denote the parameters set $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ such that $x_1 =$ "noise density 10%", $x_2 =$ "noise density 20%", $x_3 =$ "noise density 30%", $x_4 =$ "noise density 40%", $x_5 =$ "noise density 50%", $x_6 =$ "noise density 60%", $x_7 =$ "noise density 70%", $x_8 =$ "noise density 80%", and $x_9 =$ "noise density 90%". Therefore, we present the Structural Similarity (SSIM) [42] results of aforesaid filters for 20 traditional images, i.e., "Lena", "Cameraman", "Barbara", "Baboon", "Peppers", "Living Room", "Lake", "Plane", "Hill", "Pirate", "Boat", "House", "Bridge", "Elaine", "Flintstones", "Flower", "Parrot", "Dark-Haired Woman", "Blonde Woman", and "Einstein", at noise density ranging from 10% to 90% in Table 3-8. We obtain these results using MATLAB R2021a.

**Table 3.** SSIM results of BPDF for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| **Lena** | 0.9848 | 0.9657 | 0.9411 | 0.9087 | 0.8689 | 0.8120 | 0.7247 | 0.5683 | 0.3063 |
| **Cameraman** | 0.9911 | 0.9782 | 0.9608 | 0.9344 | 0.8966 | 0.8453 | 0.7726 | 0.6722 | 0.5105 |
| **Barbara** | 0.9743 | 0.9427 | 0.9046 | 0.8606 | 0.8024 | 0.7289 | 0.6258 | 0.4597 | 0.2316 |
| **Baboon** | 0.9795 | 0.9516 | 0.9112 | 0.8556 | 0.7812 | 0.6841 | 0.5622 | 0.4080 | 0.1377 |
| **Peppers** | 0.9735 | 0.9460 | 0.9158 | 0.8798 | 0.8363 | 0.7780 | 0.7001 | 0.5584 | 0.2194 |
| **Living Room** | 0.9747 | 0.9432 | 0.9056 | 0.8569 | 0.7962 | 0.7153 | 0.6012 | 0.4372 | 0.2337 |
| **Lake** | 0.9795 | 0.9526 | 0.9218 | 0.8796 | 0.8253 | 0.7468 | 0.6464 | 0.4839 | 0.2226 |
| **Plane** | 0.9885 | 0.9733 | 0.9533 | 0.9220 | 0.8797 | 0.8194 | 0.7309 | 0.5631 | 0.1894 |
| **Hill** | 0.9761 | 0.9480 | 0.9129 | 0.8676 | 0.8062 | 0.7275 | 0.6232 | 0.4954 | 0.3573 |
| **Pirate** | 0.9801 | 0.9549 | 0.9232 | 0.8817 | 0.8266 | 0.7506 | 0.6494 | 0.4797 | 0.2741 |
| **Boat** | 0.9753 | 0.9456 | 0.9085 | 0.8608 | 0.8010 | 0.7245 | 0.6155 | 0.4697 | 0.2851 |
| **House** | 0.9938 | 0.9858 | 0.9730 | 0.9550 | 0.9241 | 0.8835 | 0.8113 | 0.7002 | 0.4932 |
| **Bridge** | 0.9705 | 0.9335 | 0.8856 | 0.8269 | 0.7503 | 0.6452 | 0.5159 | 0.3648 | 0.1815 |
| **Elaine** | 0.9707 | 0.9405 | 0.9052 | 0.8649 | 0.8149 | 0.7517 | 0.6628 | 0.4927 | 0.2911 |
| **Flintstones** | 0.9726 | 0.9417 | 0.9021 | 0.8550 | 0.7912 | 0.7099 | 0.5908 | 0.4125 | 0.1259 |
| **Flower** | 0.9808 | 0.9618 | 0.9346 | 0.8998 | 0.8446 | 0.7718 | 0.6634 | 0.4970 | 0.2249 |
| **Parrot** | 0.9791 | 0.9663 | 0.9490 | 0.9270 | 0.8992 | 0.8580 | 0.7955 | 0.6816 | 0.3541 |
| **Dark-Haired Woman** | 0.9909 | 0.9802 | 0.9665 | 0.9471 | 0.9200 | 0.8789 | 0.8100 | 0.6828 | 0.4483 |
| **Blonde Woman** | 0.9657 | 0.9385 | 0.9055 | 0.8664 | 0.8191 | 0.7561 | 0.6624 | 0.5003 | 0.2184 |
| **Einstein** | 0.9830 | 0.9614 | 0.9361 | 0.9051 | 0.8640 | 0.8085 | 0.7315 | 0.5892 | 0.3465 |

**Table 4.** SSIM results of DBA for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| **Lena** | 0.9885 | 0.9741 | 0.9555 | 0.9291 | 0.8989 | 0.8560 | 0.7942 | 0.7139 | 0.5979 |
| **Cameraman** | 0.9948 | 0.9867 | 0.9758 | 0.9586 | 0.9332 | 0.8977 | 0.8452 | 0.7805 | 0.6917 |
| **Barbara** | 0.9769 | 0.9502 | 0.9174 | 0.8762 | 0.8279 | 0.7662 | 0.6880 | 0.5882 | 0.4589 |
| **Baboon** | 0.9844 | 0.9644 | 0.9352 | 0.8933 | 0.8373 | 0.7605 | 0.6587 | 0.5422 | 0.4161 |
| **Peppers** | 0.9742 | 0.9508 | 0.9239 | 0.8909 | 0.8535 | 0.8034 | 0.7387 | 0.6565 | 0.5402 |
| **Living Room** | 0.9802 | 0.9557 | 0.9251 | 0.8857 | 0.8368 | 0.7693 | 0.6888 | 0.5838 | 0.4565 |
| **Lake** | 0.9768 | 0.9565 | 0.9315 | 0.8988 | 0.8561 | 0.7984 | 0.7228 | 0.6267 | 0.5053 |
| **Plane** | 0.9885 | 0.9781 | 0.9642 | 0.9423 | 0.9124 | 0.8706 | 0.8139 | 0.7343 | 0.6268 |
| **Hill** | 0.9801 | 0.9578 | 0.9287 | 0.8912 | 0.8410 | 0.7784 | 0.6997 | 0.6036 | 0.4833 |
| **Pirate** | 0.9832 | 0.9637 | 0.9387 | 0.9062 | 0.8605 | 0.8017 | 0.7286 | 0.6247 | 0.5002 |
| **Boat** | 0.9767 | 0.9532 | 0.9239 | 0.8844 | 0.8396 | 0.7785 | 0.6968 | 0.5992 | 0.4825 |
| **House** | 0.9969 | 0.9920 | 0.9832 | 0.9703 | 0.9522 | 0.9238 | 0.8777 | 0.8142 | 0.7234 |
| **Bridge** | 0.9728 | 0.9424 | 0.9047 | 0.8552 | 0.7917 | 0.7104 | 0.6060 | 0.4880 | 0.3518 |
| **Elaine** | 0.9746 | 0.9483 | 0.9173 | 0.8800 | 0.8358 | 0.7832 | 0.7157 | 0.6292 | 0.5121 |
| **Flintstones** | 0.9769 | 0.9533 | 0.9210 | 0.8793 | 0.8239 | 0.7487 | 0.6490 | 0.5308 | 0.3807 |
| **Flower** | 0.9854 | 0.9722 | 0.9517 | 0.9259 | 0.8841 | 0.8330 | 0.7579 | 0.6588 | 0.5230 |
| **Parrot** | 0.9840 | 0.9741 | 0.9607 | 0.9440 | 0.9209 | 0.8900 | 0.8467 | 0.7871 | 0.6951 |
| **Dark-Haired Woman** | 0.9925 | 0.9850 | 0.9754 | 0.9614 | 0.9414 | 0.9133 | 0.8715 | 0.8065 | 0.7056 |
| **Blonde Woman** | 0.9666 | 0.9449 | 0.9184 | 0.8856 | 0.8441 | 0.7938 | 0.7259 | 0.6470 | 0.5432 |
| **Einstein** | 0.9867 | 0.9706 | 0.9500 | 0.9236 | 0.8881 | 0.8449 | 0.7839 | 0.7102 | 0.6142 |

**Table 5.** SSIM results of MDBUTMF for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| **Lena** | 0.9865 | 0.9479 | 0.8498 | 0.8155 | 0.8655 | 0.8898 | 0.8668 | 0.7830 | 0.4010 |
| **Cameraman** | 0.9911 | 0.9412 | 0.8160 | 0.7818 | 0.8653 | 0.9174 | 0.9052 | 0.8265 | 0.4667 |
| **Barbara** | 0.9741 | 0.9228 | 0.8235 | 0.7757 | 0.7962 | 0.7914 | 0.7477 | 0.6573 | 0.3884 |
| **Baboon** | 0.9727 | 0.9321 | 0.8655 | 0.8228 | 0.8126 | 0.7869 | 0.7317 | 0.6333 | 0.3625 |
| **Peppers** | 0.9794 | 0.9331 | 0.8263 | 0.7884 | 0.8321 | 0.8484 | 0.8206 | 0.7382 | 0.4131 |
| **Living Room** | 0.9764 | 0.9338 | 0.8567 | 0.8137 | 0.8251 | 0.8066 | 0.7621 | 0.6682 | 0.3744 |
| **Lake** | 0.9802 | 0.9275 | 0.8097 | 0.7749 | 0.8177 | 0.8374 | 0.8066 | 0.7192 | 0.4084 |
| **Plane** | 0.9884 | 0.9317 | 0.7907 | 0.7539 | 0.8392 | 0.8978 | 0.8833 | 0.7857 | 0.3518 |
| **Hill** | 0.9781 | 0.9340 | 0.8335 | 0.7938 | 0.8193 | 0.8220 | 0.7827 | 0.6976 | 0.3921 |
| **Pirate** | 0.9813 | 0.9381 | 0.8418 | 0.8072 | 0.8363 | 0.8430 | 0.8096 | 0.7178 | 0.4185 |
| **Boat** | 0.9783 | 0.9353 | 0.8450 | 0.8064 | 0.8268 | 0.8243 | 0.7833 | 0.6906 | 0.3796 |
| **House** | 0.9950 | 0.9491 | 0.8178 | 0.7831 | 0.8833 | 0.9449 | 0.9425 | 0.8641 | 0.4270 |
| **Bridge** | 0.9699 | 0.9236 | 0.8433 | 0.7994 | 0.7855 | 0.7572 | 0.6950 | 0.6000 | 0.3651 |
| **Elaine** | 0.9774 | 0.9324 | 0.8347 | 0.7965 | 0.8224 | 0.8295 | 0.7925 | 0.6973 | 0.3492 |
| **Flintstones** | 0.9764 | 0.9304 | 0.8315 | 0.7932 | 0.8169 | 0.8128 | 0.7671 | 0.6735 | 0.3965 |
| **Flower** | 0.9820 | 0.9486 | 0.8681 | 0.8407 | 0.8732 | 0.8832 | 0.8523 | 0.7679 | 0.4292 |
| **Parrot** | 0.9771 | 0.9334 | 0.8242 | 0.7958 | 0.8655 | 0.9042 | 0.8911 | 0.8123 | 0.4008 |
| **Dark-Haired Woman** | 0.9923 | 0.9395 | 0.7833 | 0.7576 | 0.8620 | 0.9294 | 0.9272 | 0.8566 | 0.4772 |
| **Blonde Woman** | 0.9642 | 0.9236 | 0.8294 | 0.7952 | 0.8214 | 0.8258 | 0.7936 | 0.7017 | 0.3539 |
| **Einstein** | 0.9833 | 0.9418 | 0.8476 | 0.8127 | 0.8528 | 0.8677 | 0.8393 | 0.7561 | 0.4127 |

**Table 6.** SSIM results of NAFSMF for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| **Lena** | 0.9839 | 0.9669 | 0.9485 | 0.9279 | 0.9080 | 0.8821 | 0.8511 | 0.8040 | 0.6862 |
| **Cameraman** | 0.9798 | 0.9643 | 0.9500 | 0.9340 | 0.9177 | 0.8988 | 0.8727 | 0.8325 | 0.7207 |
| **Barbara** | 0.9749 | 0.9472 | 0.9174 | 0.8843 | 0.8483 | 0.8039 | 0.7533 | 0.6896 | 0.5729 |
| **Baboon** | 0.9612 | 0.9216 | 0.8767 | 0.8305 | 0.7800 | 0.7211 | 0.6540 | 0.5777 | 0.4671 |
| **Peppers** | 0.9772 | 0.9551 | 0.9328 | 0.9068 | 0.8810 | 0.8512 | 0.8154 | 0.7665 | 0.6470 |
| **Living Room** | 0.9704 | 0.9382 | 0.9047 | 0.8687 | 0,8301 | 0.7839 | 0.7329 | 0.6678 | 0.5472 |
| **Lake** | 0.9754 | 0.9489 | 0.9210 | 0.8925 | 0.8588 | 0.8229 | 0.7805 | 0.7221 | 0.6021 |
| **Plane** | 0.9845 | 0.9685 | 0.9524 | 0.9334 | 0.9136 | 0.8892 | 0.8596 | 0.8175 | 0.7019 |
| **Hill** | 0.9733 | 0.9451 | 0.9148 | 0.8824 | 0.8463 | 0.8064 | 0.7585 | 0.7010 | 0.5843 |
| **Pirate** | 0.9766 | 0.9511 | 0.9248 | 0.8970 | 0.8635 | 0.8251 | 0.7844 | 0.7227 | 0.6093 |
| **Boat** | 0.9723 | 0.9422 | 0.9115 | 0.8766 | 0.8414 | 0.8005 | 0.7528 | 0.6898 | 0.5778 |
| **House** | 0.9914 | 0.9831 | 0.9733 | 0.9643 | 0.9535 | 0.9405 | 0.9210 | 0.8918 | 0.7827 |
| **Bridge** | 0.9631 | 0.9222 | 0.8788 | 0.8337 | 0.7818 | 0.7237 | 0.6544 | 0.5766 | 0.4578 |
| **Elaine** | 0.9774 | 0.9542 | 0.9295 | 0.9025 | 0.8730 | 0.8404 | 0.8010 | 0.7470 | 0.6310 |
| **Flintstones** | 0.9659 | 0.9333 | 0.8983 | 0.8631 | 0.8220 | 0.7743 | 0.7165 | 0.6464 | 0.5215 |
| **Flower** | 0.9763 | 0.9568 | 0.9363 | 0.9143 | 0.8883 | 0.8600 | 0.8218 | 0.7682 | 0.6492 |
| **Parrot** | 0.9785 | 0.9653 | 0.9519 | 0.9380 | 0.9209 | 0.9030 | 0.8774 | 0.8418 | 0.7331 |
| **Dark-Haired Woman** | 0.9906 | 0.9815 | 0.9723 | 0.9622 | 0.9513 | 0.9361 | 0.9192 | 0.8891 | 0.7756 |
| **Blonde Woman** | 0.9606 | 0.9366 | 0.9104 | 0.8833 | 0.8526 | 0.8184 | 0.7805 | 0.7259 | 0.6113 |
| **Einstein** | 0.9801 | 0.9591 | 0.9364 | 0.9132 | 0.8878 | 0.8591 | 0.8231 | 0.7732 | 0.6698 |

**Table 7.** SSIM results of DAMF for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| Lena | 0.9902 | 0.9789 | 0.9653 | 0.9488 | 0.9310 | 0.9085 | 0.8796 | 0.8396 | 0.7657 |
| Cameraman | 0.9961 | 0.9908 | 0.9844 | 0.9759 | 0.9652 | 0.9512 | 0.9321 | 0.9012 | 0.8347 |
| Barbara | 0.9815 | 0.9588 | 0.9327 | 0.9013 | 0.8675 | 0.8261 | 0.7786 | 0.7176 | 0.6308 |
| Baboon | 0.9884 | 0.9748 | 0.9572 | 0.9356 | 0.9086 | 0.8738 | 0.8237 | 0.7466 | 0.6037 |
| Peppers | 0.9804 | 0.9594 | 0.9372 | 0.9110 | 0.8835 | 0.8515 | 0.8152 | 0.7707 | 0.7018 |
| Living Room | 0.9846 | 0.9654 | 0.9422 | 0.9152 | 0.8824 | 0.8443 | 0.7976 | 0.7325 | 0.6295 |
| Lake | 0.9856 | 0.9690 | 0.9499 | 0.9285 | 0.9020 | 0.8689 | 0.8293 | 0.7737 | 0.6842 |
| Plane | 0.9938 | 0.9861 | 0.9769 | 0.9648 | 0.9505 | 0.9331 | 0.9086 | 0.8714 | 0.7987 |
| Hill | 0.9841 | 0.9656 | 0.9438 | 0.9181 | 0.8875 | 0.8515 | 0.8075 | 0.7495 | 0.6571 |
| Pirate | 0.9875 | 0.9722 | 0.9542 | 0.9332 | 0.9063 | 0.8744 | 0.8362 | 0.7784 | 0.6853 |
| Boat | 0.9833 | 0.9634 | 0.9407 | 0.9123 | 0.8829 | 0.8463 | 0.8011 | 0.7419 | 0.6514 |
| House | 0.9982 | 0.9955 | 0.9912 | 0.9861 | 0.9796 | 0.9709 | 0.9577 | 0.9376 | 0.8852 |
| Bridge | 0.9798 | 0.9560 | 0.9276 | 0.8953 | 0.8563 | 0.8072 | 0.7465 | 0.6667 | 0.5415 |
| Elaine | 0.9774 | 0.9534 | 0.9270 | 0.8961 | 0.8620 | 0.8230 | 0.7784 | 0.7248 | 0.6584 |
| Flintstones | 0.9840 | 0.9658 | 0.9430 | 0.9173 | 0.8865 | 0.8464 | 0.7980 | 0.7268 | 0.6061 |
| Flower | 0.9878 | 0.9786 | 0.9662 | 0.9513 | 0.9321 | 0.9089 | 0.8772 | 0.8290 | 0.7404 |
| Parrot | 0.9839 | 0.9763 | 0.9666 | 0.9563 | 0.9423 | 0.9270 | 0.9064 | 0.8775 | 0.8226 |
| Dark-Haired Woman | 0.9950 | 0.9891 | 0.9826 | 0.9743 | 0.9647 | 0.9525 | 0.9362 | 0.9134 | 0.8664 |
| Blonde Woman | 0.9700 | 0.9518 | 0.9301 | 0.9053 | 0.8764 | 0.8424 | 0.8015 | 0.7505 | 0.6753 |
| Einstein | 0.9894 | 0.9765 | 0.9619 | 0.9445 | 0.9244 | 0.8989 | 0.8666 | 0.8208 | 0.7472 |

**Table 8.** SSIM results of AWMF for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| Lena | 0.9822 | 0.9740 | 0.9636 | 0.9497 | 0.9349 | 0.9134 | 0.8852 | 0.8447 | 0.7737 |
| Cameraman | 0.9883 | 0.9849 | 0.9813 | 0.9759 | 0.9681 | 0.9563 | 0.9371 | 0.9059 | 0.8401 |
| Barbara | 0.9718 | 0.9540 | 0.9331 | 0.9065 | 0.8762 | 0.8366 | 0.7879 | 0.7250 | 0.6382 |
| Baboon | 0.9720 | 0.9616 | 0.9487 | 0.9343 | 0.9135 | 0.8824 | 0.8331 | 0.7550 | 0.6108 |
| Peppers | 0.9609 | 0.9560 | 0.9410 | 0.9204 | 0.8952 | 0.8633 | 0.8256 | 0.7789 | 0.7096 |
| Living Room | 0.9693 | 0.9539 | 0.9358 | 0.9144 | 0.8879 | 0.8523 | 0.8062 | 0.7394 | 0.6356 |
| Lake | 0.9742 | 0.9620 | 0.9474 | 0.9297 | 0.9067 | 0.8758 | 0.8361 | 0.7799 | 0.6904 |
| Plane | 0.9850 | 0.9796 | 0.9733 | 0.9645 | 0.9532 | 0.9376 | 09133 | 0.8760 | 0.8055 |
| Hill | 0.9724 | 0.9576 | 0.9409 | 0.9195 | 0.8929 | 0.8593 | 0.8152 | 0.7562 | 0.6632 |
| Pirate | 0.9753 | 0.9624 | 0.9489 | 0.9322 | 0.9088 | 0.8790 | 0.8417 | 0.7834 | 0.6913 |
| Boat | 0.9706 | 0.9555 | 0.9375 | 0.9146 | 0.8887 | 0.8543 | 0.8091 | 0.7483 | 0.6571 |
| House | 0.9933 | 0.9924 | 0.9905 | 0.9878 | 0.9834 | 0.9760 | 0.9630 | 0.9426 | 0.8948 |
| Bridge | 0.9638 | 0.9440 | 0.9209 | 0.8948 | 0.8611 | 0.8148 | 0.7551 | 0.6736 | 0.5469 |
| Elaine | 0.9684 | 0.9514 | 0.9296 | 0.9021 | 0.8696 | 0.8313 | 0.7857 | 0.7310 | 0.6640 |
| Flintstones | 0.9551 | 0.9502 | 0.9364 | 0.9167 | 0.8908 | 0.8541 | 0.8058 | 0.7334 | 0.6118 |
| Flower | 0.9752 | 0.9684 | 0.9594 | 0.9488 | 0.9333 | 0.9126 | 0.8820 | 0.8340 | 0.7459 |
| Parrot | 0.9779 | 0.9727 | 0.9655 | 0.9572 | 0.9457 | 0.9316 | 0.9112 | 0.8828 | 0.8309 |
| Dark-Haired Woman | 0.9910 | 0.9870 | 0.9823 | 0.9761 | 0.9678 | 0.9565 | 0.9404 | 0.9177 | 0.8744 |
| Blonde Woman | 0.9579 | 0.9450 | 0.9273 | 0.9061 | 0.8802 | 0.8476 | 0.8069 | 0.7554 | 0.6814 |
| Einstein | 0.9798 | 0.9701 | 0.9588 | 0.9450 | 0.9280 | 0.9043 | 0.8724 | 0.8259 | 0.7531 |

**Table 9.** SSIM results of ARmF for 20 traditional images

| Images/Noise Densities | 10% | 20% | 30% | 40% | 50% | 60% | 70% | 80% | 90% |
|---|---|---|---|---|---|---|---|---|---|
| **Lena** | 0.9910 | 0.9810 | 0.9697 | 0.9554 | 0.9398 | 0.9176 | 0.8885 | 0.8471 | 0.7752 |
| **Cameraman** | 0.9970 | 0.9933 | 0.9890 | 0.9828 | 0.9743 | 0.9614 | 0.9416 | 0.9093 | 0.8426 |
| **Barbara** | 0.9841 | 0.9654 | 0.9438 | 0.9172 | 0.8861 | 0.8450 | 0.7949 | 0.7291 | 0.6394 |
| **Baboon** | 0.9915 | 0.9818 | 0.9689 | 0.9523 | 0.9294 | 0.8960 | 0.8442 | 0.7630 | 0.6150 |
| **Peppers** | 0.9826 | 0.9640 | 0.9439 | 0.9205 | 0.8939 | 0.8618 | 0.8241 | 0.7779 | 0.7096 |
| **Living Room** | 0.9856 | 0.9699 | 0.9514 | 0.9294 | 0.9018 | 0.8653 | 0.8171 | 0.7483 | 0.6418 |
| **Lake** | 0.9867 | 0.9716 | 0.9553 | 0.9361 | 0.9113 | 0.8793 | 0.8391 | 0.7828 | 0.6926 |
| **Plane** | 0.9947 | 0.9887 | 0.9816 | 0.9719 | 0.9599 | 0.9433 | 0.9182 | 0.8795 | 0.8080 |
| **Hill** | 0.9860 | 0.9703 | 0.9526 | 0.9310 | 0.9038 | 0.8690 | 0.8240 | 0.7626 | 0.6672 |
| **Pirate** | 0.9884 | 0.9750 | 0.9600 | 0.9424 | 0.9181 | 0.8875 | 0.8487 | 0.7886 | 0.6939 |
| **Boat** | 0.9842 | 0.9664 | 0.9467 | 0.9223 | 0.8957 | 0.8606 | 0.8142 | 0.7529 | 0.6610 |
| **House** | 0.9987 | 0.9970 | 0.9946 | 0.9913 | 0.9863 | 0.9786 | 0.9652 | 0.9446 | 0.8962 |
| **Bridge** | 0.9823 | 0.9621 | 0.9385 | 0.9113 | 0.8762 | 0.8285 | 0.7663 | 0.6819 | 0.5515 |
| **Elaine** | 0.9773 | 0.9532 | 0.9272 | 0.8971 | 0.8630 | 0.8239 | 0.7791 | 0.7270 | 0.6631 |
| **Flintstones** | 0.9847 | 0.9688 | 0.9491 | 0.9267 | 0.8987 | 0.8608 | 0.8112 | 0.7381 | 0.6154 |
| **Flower** | 0.9877 | 0.9796 | 0.9696 | 0.9577 | 0.9411 | 0.9195 | 0.8876 | 0.8384 | 0.7489 |
| **Parrot** | 0.9851 | 0.9786 | 0.9706 | 0.9621 | 0.9499 | 0.9351 | 0.9141 | 0.8848 | 0.8320 |
| **Dark-Haired Woman** | 0.9956 | 0.9909 | 0.9854 | 0.9787 | 0.9701 | 0.9585 | 0.9420 | 0.9189 | 0.8753 |
| **Blonde Woman** | 0.9718 | 0.9551 | 0.9355 | 0.9132 | 0.8864 | 0.8531 | 0.8114 | 0.7582 | 0.6825 |
| **Einstein** | 0.9911 | 0.9805 | 0.9687 | 0.9543 | 0.9367 | 0.9121 | 0.8788 | 0.8305 | 0.7551 |

In this PVA problem, we construct an *ifpifs*-matrix $[a_{ij}]_{8\times 9}$ by using multiple fuzzy values provided in Table 3-9. We calculate the other rows of this *ifpifs*-matrix except for its zero-index row by employing the membership function and non-membership function defined by

$$\mu_{ij}^{a} := \min_{t} S_{ij}^{t} \quad \text{and} \quad \nu_{ij}^{a} := 1 - \max_{t} S_{ij}^{t}$$

such that $i \in I_7$ , $j \in I_9$, and $t \in I_{20}$. Here, $(S_{ij}^{t})$ denotes ordered $s$-tuples such that $S_{ij}^{t}$ corresponds to SSIM results originating from $t^{\text{th}}$ image for $i^{\text{th}}$ filter and $j^{\text{th}}$ noise density. Moreover, $s$ is the number of images. That is, $s = 20$. For instance,

$$(S_{11}^{t}) = (0.9848, 0.9911, 0.9743, 0.9795, 0.9735, 0.9747, 0.9795, 0.9885, 0.9761, 0.9801, 0.9753, 0.9938, 0.9705, 0.9707, 0.9726, 0.9808, 0.9791, 0.9909, 0.9657, 0.9830)$$

Hence, $\mu_{11}^{a} = 0.9657$ and $\nu_{11}^{a} = 0.0062$. Similarly, the values of the other alternatives can be calculated. Moreover, suppose that the noise removal success of the filters at high noise densities is more significant than at the other densities, it is anticipated that the membership degrees at high noise densities are greater than the non-membership degrees and the former at low noise densities are smaller than the latter. In other words, we consider the first row of $[a_{ij}]_{8\times 9}$ to be

$$\begin{bmatrix} 0.05 & 0.15 & 0.25 & 0.35 & 0.5 & 0.65 & 0.75 & 0.85 & 0.9 \\ 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.3 & 0.2 & 0.1 & 0.05 \end{bmatrix}$$

Thus, *ifpifs*-matrix $[a_{ij}]_{8\times 9}$ is constructed as follows:

$$
[a_{ij}] = \begin{bmatrix}
0.05 & 0.15 & 0.25 & 0.35 & 0.5 & 0.65 & 0.75 & 0.85 & 0.9 \\
0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.3 & 0.2 & 0.1 & 0.05 \\
0.9657 & 0.9335 & 0.8856 & 0.8269 & 0.7503 & 0.6452 & 0.5159 & 0.3648 & 0.1259 \\
0.0062 & 0.0142 & 0.0270 & 0.0450 & 0.0759 & 0.1165 & 0.1887 & 0.2998 & 0.4895 \\
0.9666 & 0.9424 & 0.9047 & 0.8552 & 0.7917 & 0.7104 & 0.6060 & 0.4880 & 0.3518 \\
0.0031 & 0.0080 & 0.0168 & 0.0297 & 0.0478 & 0.0762 & 0.1223 & 0.1858 & 0.2766 \\
0.9642 & 0.9228 & 0.7833 & 0.7539 & 0.7855 & 0.7572 & 0.6950 & 0.6000 & 0.3492 \\
0.0050 & 0.0509 & 0.1319 & 0.1593 & 0.1167 & 0.0551 & 0.0575 & 0.1359 & 0.5228 \\
0.9606 & 0.9216 & 0.8767 & 0.8305 & 0.7800 & 0.7211 & 0.6540 & 0.5766 & 0.4578 \\
0.0086 & 0.0169 & 0.0267 & 0.0357 & 0.0465 & 0.0595 & 0.0790 & 0.1082 & 0.2173 \\
0.9700 & 0.9518 & 0.9270 & 0.8953 & 0.8563 & 0.8072 & 0.7465 & 0.6667 & 0.5415 \\
0.0018 & 0.0045 & 0.0088 & 0.0139 & 0.0204 & 0.0291 & 0.0423 & 0.0624 & 0.1148 \\
0.9551 & 0.9440 & 0.9209 & 0.8948 & 0.8611 & 0.8148 & 0.7551 & 0.6736 & 0.5469 \\
0.0067 & 0.0076 & 0.0095 & 0.0122 & 0.0166 & 0.0240 & 0.0370 & 0.0574 & 0.1052 \\
0.9718 & 0.9532 & 0.9272 & 0.8971 & 0.8630 & 0.8239 & 0.7663 & 0.6819 & 0.5515 \\
0.0013 & 0.0030 & 0.0054 & 0.0087 & 0.0137 & 0.0214 & 0.0348 & 0.0554 & 0.1038
\end{bmatrix}
$$

In Table 10, $w = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$. Moreover, since the number of alternatives and parameters is not equal in this PVA problem, iKKT13 is not applied to $[a_{ij}]$.

**Table 10.** Decision sets produced by SDM methods*

| **Algorithms** | **Matrix** | **Decision Sets** |
|---|---|---|
| iMBR01 | $[a_{ij}]$ | $\{{}^{0}_{0.8098}\text{BPDF}, {}^{0.1770}_{0.8005}\text{DBAIN}, {}^{0.2008}_{0.5865}\text{MDBUTMF}, {}^{0.1955}_{0.6433}\text{NAFSMF}, {}^{0.4571}_{0.3699}\text{DAMF}, {}^{0.5271}_{0.3884}\text{AWMF}, {}^{0.6711}_{0}\text{ARmF}\}$ |
| isMBR01 | $[a_{ij}]$ | $\{{}^{0}_{0.8098}\text{BPDF}, {}^{0.1770}_{0.8005}\text{DBAIN}, {}^{0.2008}_{0.5865}\text{MDBUTMF}, {}^{0.1955}_{0.6433}\text{NAFSMF}, {}^{0.4571}_{0.3699}\text{DAMF}, {}^{0.5271}_{0.3884}\text{AWMF}, {}^{0.6711}_{0}\text{ARmF}\}$ |
| iMBR01/2 | $[a_{ij}]$ | $\{{}^{0}_{0.8210}\text{BPDF}, {}^{0.1582}_{0.8257}\text{DBAIN}, {}^{0.1508}_{0.6475}\text{MDBUTMF}, {}^{0.2131}_{0.6005}\text{NAFSMF}, {}^{0.4444}_{0.3720}\text{DAMF}, {}^{0.5255}_{0.4088}\text{AWMF}, {}^{0.6662}_{0}\text{ARmF}\}$ |
| iMRB02($\boldsymbol{I_9}$) | $[a_{ij}]$ | $\{{}^{0.8080}_{0.1520}\text{BPDF}, {}^{0.8878}_{0.0872}\text{DBAIN}, {}^{0.9085}_{0.0285}\text{MDBUTMF}, {}^{0.9209}_{0.0546}\text{NAFSMF}, {}^{0.9833}_{0.0064}\text{DAMF}, {}^{0.9869}_{0.0025}\text{AWMF}, {}^{0.9924}_{0}\text{ARmF}\}$ |
| iCCE11 | $[a_{ij}]$ | $\{{}^{0.2560}_{0.0253}\text{BPDF}, {}^{0.3065}_{0.0158}\text{DBAIN}, {}^{0.3197}_{0.0399}\text{MDBUTMF}, {}^{0.3275}_{0.0155}\text{NAFSMF}, {}^{0.3670}_{0.0066}\text{DAMF}, {}^{0.3693}_{0.0067}\text{AWMF}, {}^{0.3728}_{0.0048}\text{ARmF}\}$ |
| iCCE10 | $[a_{ij}]$ | $\{{}^{0.2560}_{0.0253}\text{BPDF}, {}^{0.3065}_{0.0158}\text{DBAIN}, {}^{0.3197}_{0.0399}\text{MDBUTMF}, {}^{0.3275}_{0.0155}\text{NAFSMF}, {}^{0.3670}_{0.0066}\text{DAMF}, {}^{0.3693}_{0.0067}\text{AWMF}, {}^{0.3728}_{0.0048}\text{ARmF}\}$ |
| iCEC11 | $[a_{ij}]$ | $\{{}^{0.2129}_{0.0035}\text{BPDF}, {}^{0.2469}_{0.0021}\text{DBAIN}, {}^{0.2556}_{0.0055}\text{MDBUTMF}, {}^{0.2598}_{0.0021}\text{NAFSMF}, {}^{0.2895}_{0.0009}\text{DAMF}, {}^{0.2912}_{0.0009}\text{AWMF}, {}^{0.2938}_{0.0007}\text{ARmF}\}$ |
| iFJLL10/2($\boldsymbol{I_9}, \boldsymbol{w}$) | $[a_{ij}]$ | $\{{}^{0.7552}_{0.2238}\text{BPDF}, {}^{0.8741}_{0.1189}\text{DBAIN}, {}^{0.8741}_{0.1189}\text{MDBUTMF}, {}^{1}_{0}\text{NAFSMF}, {}^{1}_{0}\text{DAMF}, {}^{1}_{0}\text{AWMF}, {}^{1}_{0}\text{ARmF}\}$ |
| iFJLL10/4($\boldsymbol{I_9}, \boldsymbol{w}$) | $[a_{ij}]$ | $\{{}^{0.7552}_{0.2238}\text{BPDF}, {}^{0.8741}_{0.1189}\text{DBAIN}, {}^{0.8741}_{0.1189}\text{MDBUTMF}, {}^{1}_{0}\text{NAFSMF}, {}^{1}_{0}\text{DAMF}, {}^{1}_{0}\text{AWMF}, {}^{1}_{0}\text{ARmF}\}$ |
| iKWW11($\boldsymbol{I_9}, \mathbf{0.5}, \mathbf{0.5}$) | $[a_{ij}]$ | $\{{}^{0.6198}_{0.0827}\text{BPDF}, {}^{0.6793}_{0.0542}\text{DBAIN}, {}^{0.6950}_{0.0968}\text{MDBUTMF}, {}^{0.7018}_{0.0662}\text{NAFSMF}, {}^{0.8114}_{0.0325}\text{DAMF}, {}^{0.8429}_{0}\text{AWMF}, {}^{0.9430}_{0.0569}\text{ARmF}\}$ |
| iPEM | $[a_{ij}]$ | $\{{}^{0.7485}_{0.2507}\text{BPDF}, {}^{0.8465}_{0.1532}\text{DBAIN}, {}^{0.8602}_{0.1388}\text{MDBUTMF}, {}^{0.8803}_{0.1195}\text{NAFSMF}, {}^{0.9850}_{0.0149}\text{DAMF}, {}^{0.9886}_{0.0113}\text{AWMF}, {}^{1}_{0}\text{ARmF}\}$ |

*In the event that noise removal performance at high noise densities is more important.

The intuitionistic fuzzy values in the decision sets provided in Table 10 are generated on MATLAB R2021a. Moreover, using the relation in Proposition 2.1, the ranking orders of the alternatives are presented in Table 11. The number of the algorithms producing the same ranking order is signified in the last column of Table 11. According to the table, iMBR01/2, iCEC11, and iPEM have the same ranking orders just as iCCE11, iCCE10, and iKWW11$(I_9, 0.5, 0.5)$ do. Moreover, these six methods produce the same ranking orders with the exception of DBAIN and MDBUTMF's ranks. Besides, iMBR01, isMBR01, and iMRB02$(I_9)$ generate the same ranking orders. However, iFJLL10/2$(I_9, w)$ and iFJLL10/4$(I_9, w)$ have anomalous ranking orders unlike the other SDM methods. Although the decision-making abilities of all the SDM methods herein differ, all signify that BPDF has the lowest noise removal performance. Similarly, all the SDM methods but iFJLL10/2$(I_9, w)$ and iFJLL10/4$(I_9, w)$ yield that ARmF has the highest noise removal performance.

**Table 11.** Ranking orders of SDM methods*

| **Algorithms** | **Ranking Orders** | **Frequency** |
|---|---|---|
| iMBR01 | BPDF≺DBAIN≺NAFSMF≺MDBUTMF≺DAMF≺AWMF≺ARmF | 3 |
| isMBR01 | BPDF≺DBAIN≺NAFSMF≺MDBUTMF≺DAMF≺AWMF≺ARmF | 3 |
| iMBR01/2 | BPDF≺DBAIN≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF | 3 |
| iMRB02$(\boldsymbol{I_9})$ | BPDF≺DBAIN≺NAFSMF≺MDBUTMF≺DAMF≺AWMF≺ARmF | 3 |
| iCCE11 | BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF | 3 |
| iCCE10 | BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF | 3 |
| iCEC11 | BPDF≺DBAIN≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF | 3 |
| iFJLL10/2$(\boldsymbol{I_9}, \boldsymbol{w})$ | BPDF≺DBAIN≈MDBUTMF≺NAFSMF≈DAMF≈AWMF≈ARmF | 2 |
| iFJLL10/4$(\boldsymbol{I_9}, \boldsymbol{w})$ | BPDF≺DBAIN≈MDBUTMF≺NAFSMF≈DAMF≈AWMF≈ARmF | 2 |
| iKWW11$(\boldsymbol{I_9}, \mathbf{0.5}, \mathbf{0.5})$ | BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF | 3 |
| iPEM | BPDF≺DBAIN≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF | 3 |

*In the event that noise removal performance at high noise densities is more important.

On the other hand, suppose that the noise removal success of the filters at low noise densities are more significant than at the other densities, it is anticipated that the membership degrees at high noise densities are smaller than the non-membership degrees and the former at low noise densities are greater than the latter. In other words, we consider the first row of $[b_{ij}]_{8\times 9}$ to be

$$\begin{bmatrix} 0.9 & 0.85 & 0.75 & 0.65 & 0.5 & 0.35 & 0.25 & 0.15 & 0.05 \\ 0.05 & 0.1 & 0.2 & 0.3 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \end{bmatrix}$$

Thereby, *ifpifs*-matrix $[b_{ij}]_{8\times 9}$ is constructed as follows:

$$[b_{ij}]=\begin{bmatrix}
0.9 & 0.85 & 0.75 & 0.65 & 0.5 & 0.35 & 0.25 & 0.15 & 0.05\\
0.05 & 0.1 & 0.2 & 0.3 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9\\[6pt]
0.9657 & 0.9335 & 0.8856 & 0.8269 & 0.7503 & 0.6452 & 0.5159 & 0.3648 & 0.1259\\
0.0062 & 0.0142 & 0.0270 & 0.0450 & 0.0759 & 0.1165 & 0.1887 & 0.2998 & 0.4895\\[6pt]
0.9666 & 0.9424 & 0.9047 & 0.8552 & 0.7917 & 0.7104 & 0.6060 & 0.4880 & 0.3518\\
0.0031 & 0.0080 & 0.0168 & 0.0297 & 0.0478 & 0.0762 & 0.1223 & 0.1858 & 0.2766\\[6pt]
0.9642 & 0.9228 & 0.7833 & 0.7539 & 0.7855 & 0.7572 & 0.6950 & 0.6000 & 0.3492\\
0.0050 & 0.0509 & 0.1319 & 0.1593 & 0.1167 & 0.0551 & 0.0575 & 0.1359 & 0.5228\\[6pt]
0.9606 & 0.9216 & 0.8767 & 0.8305 & 0.7800 & 0.7211 & 0.6540 & 0.5766 & 0.4578\\
0.0086 & 0.0169 & 0.0267 & 0.0357 & 0.0465 & 0.0595 & 0.0790 & 0.1082 & 0.2173\\[6pt]
0.9700 & 0.9518 & 0.9270 & 0.8953 & 0.8563 & 0.8072 & 0.7465 & 0.6667 & 0.5415\\
0.0018 & 0.0045 & 0.0088 & 0.0139 & 0.0204 & 0.0291 & 0.0423 & 0.0624 & 0.1148\\[6pt]
0.9551 & 0.9440 & 0.9209 & 0.8948 & 0.8611 & 0.8148 & 0.7551 & 0.6736 & 0.5469\\
0.0067 & 0.0076 & 0.0095 & 0.0122 & 0.0166 & 0.0240 & 0.0370 & 0.0574 & 0.1052\\[6pt]
0.9718 & 0.9532 & 0.9272 & 0.8971 & 0.8630 & 0.8239 & 0.7663 & 0.6819 & 0.5515\\
0.0013 & 0.0030 & 0.0054 & 0.0087 & 0.0137 & 0.0214 & 0.0348 & 0.0554 & 0.1038
\end{bmatrix}$$

In Table 12, $w=\begin{bmatrix}0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4\\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4\end{bmatrix}$. Moreover, since the numbers of the alternatives and the parameters are not equal in this PVA problem, iKKT13 is not applied to $[b_{ij}]$.

**Table 12.** Decision sets of SDM methods*

| **Algorithms** | **Matrix** | **Decision Sets** |
|---|---|---|
| iMBR01 | $[b_{ij}]$ | $\{{}^{0.0491}_{0.6250}\text{BPDF}, {}^{0.2217}_{0.6042}\text{DBAIN}, {}^{0.0238}_{0.7589}\text{MDBUTMF}, {}^{0}_{0.9271}\text{NAFSMF}, {}^{0.4449}_{0.4092}\text{DAMF}, {}^{0.2827}_{0.4702}\text{AWMF}, {}^{0.6027}_{0}\text{ARmF}\}$ |
| isMBR01 | $[b_{ij}]$ | $\{{}^{0.0491}_{0.6250}\text{BPDF}, {}^{0.2217}_{0.6042}\text{DBAIN}, {}^{0.0238}_{0.7589}\text{MDBUTMF}, {}^{0}_{0.9271}\text{NAFSMF}, {}^{0.4449}_{0.4092}\text{DAMF}, {}^{0.2827}_{0.4702}\text{AWMF}, {}^{0.6027}_{0}\text{ARmF}\}$ |
| iMBR01/2 | $[b_{ij}]$ | $\{{}^{0.0255}_{0.6294}\text{BPDF}, {}^{0.2078}_{0.6242}\text{DBAIN}, {}^{0}_{0.8237}\text{MDBUTMF}, {}^{0.0173}_{0.8762}\text{NAFSMF}, {}^{0.4314}_{0.4164}\text{DAMF}, {}^{0.2993}_{0.4576}\text{AWMF}, {}^{0.5994}_{0}\text{ARmF}\}$ |
| iMRB02($\boldsymbol{I_9}$) | $[b_{ij}]$ | $\{{}^{0.8760}_{0}\text{BPDF}, {}^{0.8913}_{0.0340}\text{DBAIN}, {}^{0.8755}_{0.0202}\text{MDBUTMF}, {}^{0.8868}_{0.0585}\text{NAFSMF}, {}^{0.9125}_{0.0589}\text{DAMF}, {}^{0.9104}_{0.0638}\text{AWMF}, {}^{0.9148}_{0.0607}\text{ARmF}\}$ |
| iCCE11 | $[b_{ij}]$ | $\{{}^{0.3830}_{0.1084}\text{BPDF}, {}^{0.3963}_{0.0653}\text{DBAIN}, {}^{0.3826}_{0.0911}\text{MDBUTMF}, {}^{0.3924}_{0.0479}\text{NAFSMF}, {}^{0.4149}_{0.0250}\text{DAMF}, {}^{0.4130}_{0.0226}\text{AWMF}, {}^{0.4169}_{0.0214}\text{ARmF}\}$ |
| iCCE10 | $[b_{ij}]$ | $\{{}^{0.3830}_{0.1084}\text{BPDF}, {}^{0.3963}_{0.0653}\text{DBAIN}, {}^{0.3826}_{0.0911}\text{MDBUTMF}, {}^{0.3924}_{0.0479}\text{NAFSMF}, {}^{0.4149}_{0.0250}\text{DAMF}, {}^{0.4130}_{0.0226}\text{AWMF}, {}^{0.4169}_{0.0214}\text{ARmF}\}$ |
| iCEC11 | $[b_{ij}]$ | $\{{}^{0.3775}_{0.0311}\text{BPDF}, {}^{0.3882}_{0.0184}\text{DBAIN}, {}^{0.3735}_{0.0267}\text{MDBUTMF}, {}^{0.3832}_{0.0134}\text{NAFSMF}, {}^{0.4032}_{0.0071}\text{DAMF}, {}^{0.4010}_{0.0065}\text{AWMF}, {}^{0.4049}_{0.0063}\text{ARmF}\}$ |
| iFJLL10/2($\boldsymbol{I_9}, \boldsymbol{w}$) | $[b_{ij}]$ | $\{{}^{0.8205}_{0}\text{BPDF}, {}^{0.8308}_{0.0769}\text{DBAIN}, {}^{0.8308}_{0.0769}\text{MDBUTMF}, {}^{0.8359}_{0.1641}\text{NAFSMF}, {}^{0.8359}_{0.1641}\text{DAMF}, {}^{0.8359}_{0.1641}\text{AWMF}, {}^{0.8359}_{0.1641}\text{ARmF}\}$ |
| iFJLL10/4($\boldsymbol{I_9}, \boldsymbol{w}$) | $[b_{ij}]$ | $\{{}^{0.8205}_{0}\text{BPDF}, {}^{0.8308}_{0.0769}\text{DBAIN}, {}^{0.8308}_{0.0769}\text{MDBUTMF}, {}^{0.8359}_{0.1641}\text{NAFSMF}, {}^{0.8359}_{0.1641}\text{DAMF}, {}^{0.8359}_{0.1641}\text{AWMF}, {}^{0.8359}_{0.1641}\text{ARmF}\}$ |
| iKWW11($\boldsymbol{I_9}, \mathbf{0.5}, \mathbf{0.5}$) | $[b_{ij}]$ | $\{{}^{0.7130}_{0.1208}\text{BPDF}, {}^{0.7508}_{0.0657}\text{DBAIN}, {}^{0.7097}_{0.0985}\text{MDBUTMF}, {}^{0.7158}_{0.0588}\text{NAFSMF}, {}^{0.8317}_{0}\text{DAMF}, {}^{0.7801}_{0.0391}\text{AWMF}, {}^{0.9332}_{0.0331}\text{ARmF}\}$ |
| iPEM | $[b_{ij}]$ | $\{{}^{0.8585}_{0.1366}\text{BPDF}, {}^{0.9154}_{0.0829}\text{DBAIN}, {}^{0.8968}_{0.0988}\text{MDBUTMF}, {}^{0.9211}_{0.0779}\text{NAFSMF}, {}^{0.9918}_{0.0080}\text{DAMF}, {}^{0.9891}_{0.0107}\text{AWMF}, {}^{0.9998}_{0}\text{ARmF}\}$ |

*In the event that noise removal performance at low noise densities is more important.

The intuitionistic fuzzy values in the decision sets provided in Table 12 are obtained with MATLAB R2021a. Moreover, using the relation in Proposition 2.1, the ranking orders of the alternatives are presented in Table 13. The number of the algorithms producing the same ranking orders is signified in the last column of Table 13. According to these ranking orders, iMBR01, isMBR01, and iMBR01/2 produce the same ranking orders just as iCEC11 and iKWW11($I_9$, 0.5,0.5) do. Furthermore, these methods have the same ranking orders with the exception of BPDF and NAFSMF's ranks. Besides, iCCE11 and iCCE10 generate the same ranking orders. However, iFJLL10/2($I_9, w$) and iFJLL10/4($I_9, w$) have anomalous ranking order unlike the other SDM methods. Additionally, all the SDM methods except for iMBR01, isMBR01, iMBR01/2, iMRB02($I_9$), iFJLL10/2($I_9, w$), and iFJLL10/4($I_9, w$) confirm that BPDF exhibits the lowest performance. Further, all the SDM methods but iMRB02($I_9$), iFJLL10/2($I_9, w$), and iFJLL10/4($I_9, w$) validate that ARmF performs better than the other filters.

**Table 13.** Ranking orders of SDM methods*

| **Algorithms** | **Ranking Orders** | **Frequency** |
|---|---|---|
| iMBR01 | NAFSMF≺MDBUTMF≺BPDF≺DBAIN≺AWMF≺DAMF≺ARmF | 3 |
| isMBR01 | NAFSMF≺MDBUTMF≺BPDF≺DBAIN≺AWMF≺DAMF≺ARmF | 3 |
| iMBR01/2 | NAFSMF≺MDBUTMF≺BPDF≺DBAIN≺AWMF≺DAMF≺ARmF | 3 |
| iMRB02($\boldsymbol{I_9}$) | NAFSMF≺AWMF≺DAMF≺ARmF≺MDBUTMF≺DBAIN≺BPDF | 1 |
| iCCE11 | BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF | 2 |
| iCCE10 | BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺DAMF≺AWMF≺ARmF | 2 |
| iCEC11 | BPDF≺MDBUTMF≺NAFSMF≺DBAIN≺AWMF≺DAMF≺ARmF | 2 |
| iFJLL10/2($\boldsymbol{I_9, w}$) | NAFSMF≈DAMF≈AWMF≈ARmF≺DBAIN≈MDBUTMF≺BPDF | 2 |
| iFJLL10/4($\boldsymbol{I_9, w}$) | NAFSMF≈DAMF≈AWMF≈ARmF≺DBAIN≈MDBUTMF≺BPDF | 2 |
| iKWW11($\boldsymbol{I_9, 0.5, 0.5}$) | BPDF≺MDBUTMF≺NAFSMF≺DBAIN≺AWMF≺DAMF≺ARmF | 2 |
| iPEM | BPDF≺MDBUTMF≺DBAIN≺NAFSMF≺AWMF≺DAMF≺ARmF | 1 |

*In the event that noise removal performance at low noise densities is more important.

## 6. Conclusion

In this study, we generalised 24 SDM methods [10,11,15,16,20], constructed by the concept of *fpfs*-matrices, in the *ifpifs*-matrices space. We then suggested five new test scenarios by inspiring from the scenarios in [13] to examine the performance consistency of the SDM methods in decision-making problems. Thus, we determined the SDM methods which successfully passed all the tests. Afterwards, we applied the successful SDM methods to a PVA problem to rank the state-of-the-art noise removal filters according to their noise removal performance.

The present study encourages researchers to generalised other SDM methods to render them operable in the *ifpifs*-matrices space. Researchers can also focus on SDM methods constructed with intuitionistic fuzzy sets, soft sets, or their hybrid versions [4-8]. Moreover, classification algorithms can be developed using a generalised method (for more on classification methods, see [43-47]). This study ignored the SDM methods suggested by using the superstructures of *ifpifs*-sets/matrices. Thereby, future papers can study the generalisations of SDM methods for such spaces as interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets/matrices space [48,49] and the hybrid versions of picture fuzzy sets [50,51] and soft sets.

## Author Contributions

S. Enginoğlu directed the project and supervised the process whereby the findings were obtained. T. Aydın and B. Arslan generalised the SDM methods. S. Memiş and B. Arslan produced the application results of the SDM methods by writing their MATLAB codes. T. Aydın and B. Arslan wrote the manuscript with the support of S. Enginoğlu and S. Memiş. S. Enginoğlu reviewed and edited the manuscript. All the authors discussed the results and contributed to the final manuscript.

## Conflict of Interest

The authors declare no conflict of interest.

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