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A Brief Look at OU, Vasicek, CIR and Hull-White Models Through Their Actuarial Applications

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OU, Vasicek, CIR ve Hull-White Modellerine Aktüeryal Uygulamaları Üzerinden Kısa Bir Bakış

Sinem KOZPINAR
Başkent Üniversitesi, Ticari Bilimler Fakültesi, Ankara, Türkiye
s_kozpinar@hotmail.com
ORCID: 0000-0002-8136-0328

Özet

Amaç: Orntein-Uhlenbeck (OU), Vasicek, Cox-Ingersoll-Ross (CIR) ve Hull-White afin süreçleri, önemli özelliklerine kısaca değinilerek ele alınmıştır. Bu makalenin temel amacı söz konusu afin süreçlerin, farklı stokastik modeller ve matematiksel metotların kullanıldığı altı yakın dönem aktüeryal uygulamasını tartışmaktır.

Sonuç ve Katkılar: Bu uygulamalar, bir yandan söz konusu afin süreçlerin modelleme sürecine nasıl dahil edildiğini göstermekte, diğer yandan ise matematiksel hesaplamalar/veri analizi yoluyla bu afin süreçleri kullanmanın avantajları hakkında fikir vermektedir.

Anahtar Kelimeler: Ornstein-Uhlenbeck modeli, Vasicek modeli, Cox-Ingersoll-Ross modeli, Hull-White modeli.

Jel Kodu: G22, G13, G12

Abstract

Aim: A brief overview of the affine processes, namely the Orntein-Uhlenbeck (OU) process, the Vasicek process, the Cox-Ingersoll-Ross (CIR) process and the Hull-White process, is presented through their important features. The main purpose of this paper is to discuss six very recent actuarial applications of these affine processes that focus on different problems with different stochastic models and different mathematical methods.

Conclusion and Contributions: On one hand, these applications show how to incorporate the corresponding affine processes into the modelling framework. On the one hand they give an insight about the advantages of using these affine processes through mathematical calculations/data analysis.

Keywords: Ornstein-Uhlenbeck model, Vasicek model, Cox-Ingersoll-Ross model, Hull-White model.

Jel Codes: G22, G13, G12

1. Introduction

Practitioners and researchers have shown many interest in adapting the affine processes such as Ornstein-Uhlenbeck (Uhlenbeck and Ornstein, 1930), Vasicek (Vasicek, 1977), Cox-Ingersoll-Ross (Cox, Ingersoll and Ross, 1985) and Hull-White processes (Hull and White, 1990) into financial and actuarial applications. These processes are simple and easy to tackle with as a result of the affine structure they have. In particular, it is known that all these processes provided closed-form solutions for the zero-coupon price as well as for the European vanilla options in case the underlying asset is modelled by these processes. Such features of these processes made them very popular among the researchers and therefore they are ubiquitously used in the field of actuarial sciences. This paper reviews some very recent publications that deal with an actuarial research problem, associated with these affine processes. Specifically, they will be used to model the stochastic interest rates, stochastic volatility, mortality intensity and the mean growth of an asset in an investment strategy.

Section 2-5 gives a brief overview about the affine processes driven by the Ornstein-Uhlenbeck, Vasicek, Cox-Ingersoll-Ross and Hull-White models. Section 6 presents the recent studies incorporating these affine processes to an actuarial research problem. Section 7 concludes the paper.

2. Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck (OU) process is one of the most popular Gaussian-Markov stochastic processes, that is also known for allowing mean-reversion. By mean-reversion, it is meant that the process tends to revert to its mean in the long-run. This mean-reversion property is a result of the following stochastic differential equation (SDE):

$$dS(t) = -\theta S(t)dt + \sigma dW(t), \qquad (1)$$

where $S(t)$ denotes the value of an OU process at time t, $\theta > 0$ is the rate of mean-reversion, $\sigma > 0$ is the volatility and $W(t)$ is a standard Brownian motion (Uhlenbeck and Ornstein, 1930). Here, the model parameters θ and σ are assumed to be constant.

By this stochastic differential equation, one can easily observe that:

1. The unique solution for $S(t)$ is given by

$$S(t) = S(0)\, e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-u)} dW(u).$$

This representation explains the Gaussian property of the OU process. Remark that the OU model allows negative values, since the probability of having negative values is positive for a Gaussian random variable (for a more detailed discussion see, e.g., Lamberton and Lapeyre, 2011).

2. The conditional mean $\mathbb{E}[\, S(t) \mid S(s) = x]$ for $s < t$ reads:

$$\mathbb{E}[\, S(t) \mid S(s) = x] = x e^{-\theta(t-s)}.$$

This conditional mean function addresses a downward (upward) drift in the process when $x > 0$ ($x < 0$) since as $t \to \infty$, the conditional mean function goes to zero. That is, it is likely that the extreme movements are followed by the movements that cause the process to move around the average level, pointing out a tendency to return to zero (see, e.g., Beekman and Shiu, 1988).

It is worth noting that in the seminal paper of Uhlenbeck and Ornstein (1930), this process was indeed used to model the velocity of a Brownian particle. Although the model itself was originally used for a physics problem, it with some modifications has found many applications in the field of Actuarial sciences. In this paper, we aim to give an insight into the actuarial use of a few extensions of OU model. For this purpose, we will first present 3 popular models associated with OU processes, i.e. Vasicek, Cox-Ingersoll-Ross and Hull-White Model, and then review some papers that deal with the actuarial application of the corresponding models. It is clear that it is not possible to discuss every paper in these contexts, and even within the papers we consider, we cannot give the whole mathematical arguments. Instead, we present the models and their actuarial applications through a brief description of the methodology and/or numerical experiments, but intend to give a clear motivation behind them.

We start with the Vasicek model (Vasicek, 1977) that is constructed by adding a drift term into the OU-SDE given in (1).

3. Vasicek Model

Vasicek model is originally proposed to describe the term structure of interest rates by the following SDE (Vasicek, 1977):

$$dr(t) = \theta(\kappa - r(t))dt + \sigma dW(t), \tag{2}$$

where $r(t)$ represents the instantaneous interest rate with $r(0) > 0$, $\theta > 0$ is the rate of mean-reversion, $\sigma > 0$ is the volatility term, $\kappa > 0$ is the long-term mean level and

$W(t)$ is a standard Brownian motion. Here, θ, σ and κ are all constant. Note that, as mentioned earlier, when $\kappa = 0$, the model is reduced to the OU model in (1). In other words, OU model in (1) can be considered a special case of the Vasicek model.

Before continuing with another OU-type model, we find it useful to make a few remarks here: The first remark is that there is a unique solution for the interest rate process $r(t)$, which is one of the reasons that makes the model very popular among the researchers and practitioners. More precisely (Vasicek, 1977),

$$r(t) = \kappa(1 - e^{-\theta t}) + r(0)e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-u)} dW(u). \quad (3)$$

The second remark is about a natural consequence of the closed-form solution (3). Namely, the interest rate process $r(t)$ is normally distributed, and therefore, it can reach negative values, which have been considered a drawback of the Vasicek model before 2008 crisis (Grasselli and Lipton, 2019). However, as shown empirically in the paper of Jackson (2015), negative interest rates have also been observed. Third remark is dedicated to the mean-reversion that can be represented by the conditional mean function $\mathbb{E}[\, r(t) |\, r(s) = x] = xe^{-\theta(t-s)} + \kappa(1 - e^{-\theta(t-s)})$ (Beekman and Shiu, 1988). Note that when $t \to \infty$, the conditional mean $\mathbb{E}[\, r(t) |\, r(s) = x]$ tends to κ. This implies that, when $r > \kappa$ ($r < \kappa$), a negative (positive) drift is observed by means of the parameter θ, pushing the process into the long-term mean level κ. Therefore, the Vasicek model is said to have the mean-reverting feature. As a fourth remark, the attractiveness of the Vasicek model also lies in the fact that it provides an affine representation of the zero-coupon bond prices. By using this affine representation, one can easily calculate the price of an European option written on a pure discount bond (see, Jamshidian, 1985; Zeytun and Gupta, 2007). Finally, we conclude our remarks by pointing out that the Vasicek model fails to fit the term structure in the market, which is viewed as one of the major drawbacks of the model. Since it is not the scope of this paper to overview all the advantages and limitations of the stochastic models of our interest, we refer to Brigo and Mercurio (2007) for a more detailed discussion.

4. Cox-Ingersoll-Ross (CIR) model

As mentioned above, the possibility of taking a negative value for interest rates that follow a Vasicek model is nonzero. To overcome this drawback (after 2008 crisis it is observed that it is not a drawback at all), Cox, Ingersoll and Ross (1985) introduce an SDE for modelling short-rate processes $r(t)$:

$$dr(t) = \theta(\kappa - r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad (4)$$

extending the Vasicek model into a mean-reverting model with a standard deviation term $\sigma\sqrt{r(t)}$. Here, $\theta > 0$ is the rate of mean-reversion, $\sigma > 0$ is the volatility and $\kappa > 0$ is the long-term mean level with all being constant, and $W(t)$ is a standard Brownian motion. Since the drift term $\theta(\kappa - r(t))$ of the CIR process is the same as the one defined in the Vasicek model, one can conclude that the mean-reverting property is preserved. Importantly, notice that $Y(t) = r^2(t)$ is an OU process of the form (1), indicating the relation between these two models.

Note that nonnegativity of this interest rate process is a consequence of the Feller condition. As shown in Feller (1951), if the condition $2\theta\kappa \geq \sigma^2$ is satisfied, the CIR process can never hit zero. Otherwise, the interest rate $r(t)$ can get a zero value. But, in Cox, Ingersoll and Ross (1985), it is noted that the interest rate, which is initially nonnegative, can never subsequently have a negative value. That is, the CIR process always takes nonnegative values, unlike the Vasicek process.

It is also worth mentioning that, as a result of the standard deviation term $\sigma\sqrt{r(t)}$, the CIR process is not normally distributed anymore, but instead has a noncentral chi-square distribution. Although the normality arguments of the Vasicek model are not valid for the CIR process, the zero-coupon bond prices still have an affine representation under the CIR model. This affine representation, importantly, yields a closed-form solution for the price of an European option written on a zero coupon bond price (Cox, Ingersoll and Ross, 1985).

All these properties mentioned above make the CIR model very attractive for the researchers and practitioners. However, it does not remedy one drawback of the Vasicek model: the CIR model may be inadequate to match the term structure of the interest rates. For this issue, Hull and White (1990) propose an extension of the Vasicek and CIR models by adapting the time-dependent parameters into the modeling framework. In this paper, we give our attention into the extended-Vasicek model, which is also referred as the Hull-White model.

5. Hull-White model (Extended Vasicek Model)

In order to lead a model that carries the nice properties of the Vasicek model as well as provides a good fit to the term structure of interest rates, Hull and White (1990) introduce the following SDE with time-dependent parameters:

$$dr(t) = (a(t) + \theta(t)(\kappa - r(t)))dt + \sigma(t)dW(t), \qquad (5)$$

where $a(t)$, $\theta(t)$ and $\sigma(t)$ are all deterministic functions of time, and $W(t)$ is a standard Brownian motion. Here, the deterministic function $a(t)$ is included in the model so as to match the term structure of the interest rates, while the time-dependent volatility $\sigma(t)$ is incorporated into the model to match the current and future volatilities of the interest rates.

Like the Vasicek model, the Hull-White model is mean-reverting and very tractable in the sense that there is a closed-form pricing formula for zero-coupon bonds and European options written on them. This model also shares with the Vasicek model the possibility of negative interest rates, which has been regarded as a disadvantage of the Hull-White model (see, e.g., Hull and White, 1990; Hull and White, 1994; Hull, 1996, as the very first studies on this model and its properties).

There is one more issue we want to remark on: As already mentioned above, in the original paper of Hull and White (1990), all parameters are allowed to be time dependent. Hull and White (1996) show that this time-dependency in the parameters $\theta(t)$ and $\sigma(t)$ can yield a nonstationary volatility term structure, which is undesirable when pricing instruments whose value depends on the term structure of future volatility. They illustrate that a nonstationary term structure of volatility can lead to mispricing of such instruments, and the only case to observe a stationary volatility term structure is when $\theta(t)$ and $\sigma(t)$ are constants. Note that by nonstationarity, it is meant that the volatility structure today may not be preserved in the future, as already addressed in the previous work of authors (Hull and White, 1994). This feature, therefore, motivated to use the model also with constant parameters $\theta(t)$ and $\sigma(t)$.

The next section presents some recent application of the affine processes previously mentioned within the field of the actuarial sciences. It is worth noting that although we did not derive anything new in this paper nor carry out a data analysis, we find it useful to discuss the recent studies, in order to give an insight to how popular these processes are still in this area.

6. Some Recent Actuarial Applications

6.1. Pricing Survival Forward and Survival Swap

Under the assumption that there is no basis and counterparty default risk, Zeddouk and Devolder (2019) examine the valuation of two longevity-linked instruments, namely Survival forward (S-forward) and Survival swap (S-swap), by proposing a Cost of Capital method in the line of Solvency II.

For pricing these two instruments, they model the longevity risk through the mortality intensity, which is interpreted as the force of mortality for an individual aged $x+t$ at time t. In this paper, longevity has stochastic behaviour since the dynamics of the mortality intensity is considered to be driven by two continuous-time stochastic models, one of which is the Hull-White model. Denoting $\mu_x(t)$ as the mortality intensity, the corresponding Hull-White dynamics are defined as:

$$d\mu_x(t) = b(\xi(t) - \mu_x(t))dt + \sigma dW(t),$$

where $\xi(t) = \frac{Ae^{Bt}}{b}$, A, B, b and σ are all positive constants, and $W(t)$ is a standard Brownian motion. One advantage of using the Hull-White model in the mortality intensity is that it gives an explicit representation of the expectation of survival index, resulting from the affine structure of this model. Here, the survival index at time t is defined as $e^{-\int_t^T \mu_x(u)du}$ for an individual initially aged x, alive at time t and surviving $T-t$ years more.

Since the price of an S-swap is equal to the sum of S-forwards, the authors give their attention first to pricing an S-forward contract. To be consistent with Solvency II, the price of an S-forward is expressed as the sum of a best estimate and a risk-margin. The best estimate is determined by discounting the expectation of the S-forward payoff. On the other hand, risk margin is specified by using a Cost of Capital approach that accounts for the future Solvency Capital Requirements. This Solvency Capital Requirement is defined as the capital that an insurer should set aside to be able to cover unexpected losses with a 99.5% probability. Since these future Solvency Capital Requirements are random variables, the computation of the risk margin is performed by replacing Solvency Capital Requirements with their estimates. This estimation is done by the VaR (Value at Risk) method. Then, by following some mathematical technicalities such as the calculation of $VaR_{99.5\%}(e^{-\int_i^{i+1} \mu_x(u)du})$ for time i, a pricing formula is derived for S-forwards. This price expression relies on the expectation of mortality intensity process as well as on the coefficients that appears in the explicit expression of the expectation of survival index.

After presenting a price formula for S-forwards, the paper continues with the valuation of S-swaps. They develop a price expression for S-swaps by summing up the price of S-forwards exchanged at $t_1, t_2, \ldots, t_n$. This pricing problem is easy to tackle with when using S-forward price expression.

Another core of the paper is to discuss the consistency of the Cost of Capital approach with three pricing methods: The risk-neutral approach, The Wang transform and Sharpe ratio. In order to observe the consistency of the Cost of Capital approach with these three methods, the computations are centered around the market price of longevity for the risk-neutral approach, Sharpe parameter for the Sharpe ratio and Wang parameter for the Wang transform. These parameter values are the ones that provide the same price as the Cost of Capital approach. The reason for focusing on these three parameters is that their stability for different ages and maturities is a sign of how much the corresponding approach is consistent with the Cost of Capital method.

To this end, based on a Belgian population data, numerical experiments are first carried out to obtain prices for S-forwards and S-swaps under the Cost of Capital approach. It is assumed that there are 10,000 initial policyholders for each cohort aged 65, 70, or 75 years old in 2015. The models of mortality intensity are calibrated on a projected data from the IABE unisex projected generational mortality table, by using Least Square Estimation method. After the calibration procedure, prices obtained from the Cost of Capital approach are reported for age groups 65, 70 and 75 and maturities $T = 5$ and 10. These results are followed by the calculation of the market price of longevity, Sharpe parameter and Wang parameter. It is concluded that the Cost of Capital method is not so consistent with the three methods mentioned above. The method having the worst performance in terms of consistency is revealed to be the risk-neutral method.

6.2. Optimal Proportional Reinsurance and Investment

Li et al. (2020) focus on an optimal proportional reinsurance and investment strategy for the insurer through the maximization of the expected utility of terminal wealth, when the surplus process follows a compound Poisson model.

The investment strategy in the paper decides how the insurer can invest the surplus in a portfolio of a risky asset and a riskless asset. The dynamics of the risky asset $S(t)$ is driven by a continuous-time stochastic model that accounts for the dividend income and the transaction cost:

$$dS(t) = (a(t) + c - \theta)S(t)dt + \sigma S(t)dW^{(2)}(t),$$

where $c > 0$ is the dividend income, $\theta > 0$ is the transaction cost, $\sigma > 0$ is the volatility term, and $W^{(2)}(t)$ is a standard Brownian motion. Herewith, $a(t)$, the mean growth, is assumed to be a Vasicek process of the form:

$$da(t) = \alpha(\bar{a} - a(t))dt + \beta dW^{(3)}(t), \quad a(0) = a_0,$$

where α, $\bar{a}$ and β are positive constants, and $W^{(3)}(t)$ is a standard Brownian motion correlated with $W^{(2)}(t)$. Here, $\bar{a}$ is the mean growth of the risky asset. The reason behind the choice of such a model is to determine a realistic risky asset process both in bull and bear market. When $a(t)$ is much larger than $\bar{a}$, the risky asset can be viewed as in a bull market. If $a(t)$ is much less than $\bar{a}$, then it is considered to be in a bear market.

The proportional reinsurance, on the other hand, is purchased by the insurer in order to hedge against the insurance risks. In the paper, it is described by the retention level and the premium payment of the reinsurance, which is computed by the variance principle depending on the safety loading of the reinsurer. Along with the reinsurance premium payments, the net profit condition is presented, which is required to determine the optimal proportional reinsurance strategy.

The main aim of the paper is to find an optimal proportional reinsurance and investment strategy by maximizing the expected utility of terminal wealth $U(T)$ with $U(x) = \lambda_1 - \frac{\eta}{n} e^{-nx}$. Here, $\lambda_1 > 0, \eta > 0$ and $n > 0$ are all constants with n denoting the constant absolute risk aversion (CARA) parameter. The corresponding optimization problem is solved by dynamic programming techniques, which in turn, provide analytical expressions for the optimal strategy.

It is noted that an increase in the dividend income yields an investment with a higher number of shares in the risky asset, whereas an increase in the transaction cost and in the value of riskless asset cause a decrease in the risky asset investment. Without an attempt to use financial data (they use the parameters reported in another study), they also illustrate two numerical experiments to study the impact of risk aversion and safety loadings on the reinsurance proportion. They conclude that a higher level of risk-aversion addresses a less optimal proportion reinsurance. On the contrary, greater safety loadings are followed by a greater retention level.

6.3. Performance of Affine Processes for the Force of Mortality

Zeddouk and Devolder (2020) study 5 affine processes in order to model the force of mortality. These are: Ornstein-Uhlenbeck process, Vasicek process, Feller process, Hull-White process and extended CIR process. Denoting $\mu_x(t)$ by the mortality intensity of an individual aged $x + t$ at time t, the corresponding dynamics are given as:

1. (Orntsein-Uhlenbeck):
$$d\mu_x(t) = a\mu_x(t)dt + \sigma dW(t)$$
2. (Feller):
$$d\mu_x(t) = a\mu_x(t)dt + \sigma\sqrt{\mu_x(t)}dW(t)$$
3. (Vasicek):
$$d\mu_x(t) = b\left(\frac{a}{b} - \mu_x(t)\right)dt + \sigma dW(t)$$
4. (Hull-White):
$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma dW(t)$$
5. (Extended-CIR):
$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dW(t)$$

In this paper, $\xi(t)$ is considered as the Gombertz function $\xi(t) = Ae^{Bt}$ with A and B being positive constants. Since our intention in this review is to discuss some actuarial applications of the models in 1-4, we give the dynamics of the Extended-CIR process, but not give the results on it (all the numerical results reported for this model are in the same line with the ones of Hull-White model).

The advantage of choosing these affine processes is that one can obtain the closed-form expression for the survival probability.

The main aim of this paper is to show how appropriate these models in order to describe the dynamics of the force of mortality. For this purpose various statistical tests are carried out based on a historical data for old generations of Belgian population and a projected mortality table for the younger individuals. They start with the calibration of the models on the Belgian historical and projected mortality data. It is assumed that there are four cohorts of Belgian individuals, namely, 1900, 1915, 1965 and 1970 generations. For the 1900 and 1915 generations, a historical data from Mortality database is used whereas for the 1965 and 1970 generations a projected data from the IABE projected generational mortality table is considered. The model parameters are calibrated on those datas by the Least Square Estimation method. The resulting mean square errors address a poor fit for young individuals under the OU and Feller models, but very poor for all four generations under the Vasicek model. The results for Hull-White model are, instead, very promising compared with these three models. This calibration procedure is then followed by a robustness test to examine the quality of calibrations and by a backtesting to predict the mortality. All these results implies that Hull-White model is more successful when modeling mortality, compared with OU, Feller and Vasicek model.

The statistical tests covered in Zeddouk and Devolder (2020) are not limited to the ones mentioned in this review, but since the details are beyond the scope of this review, they are skipped.

6.4. Pricing Vulnerable Options

Vulnerable option is a financial instrument that takes into account the default risk. Here, a default event is interpreted as follows: A default event occurs if the market value of the assets of the option writer falls below a default threshold level. In this case, only a proportion of the nominal claim is paid. All nominal claim is paid when the market value of the assets of the option writer is higher than or equal to this threshold level.

Yoon and Kim (2015) consider the valuation of vulnerable options under framework with constant as well as stochastic interest rates. In this paper, the stochastic interest rates are assumed to follow the Hull-White model. Since Hull-White model is one of the models that we focus on this review, the case of stochastic interest rates will be introduced.

Under the stochastic interest rate framework, the value of the underlying asset, $S(t)$, and the market value of the assets, $V(t)$, are given by:

$$dS(t) = r(t)S(t)dt + \sigma_s S(t)dW(t)^{s^*}$$

$$dV(t) = r(t)V(t)dt + \sigma_v V(t)dW(t)^{v^*}$$

$$dr(t) = (b(t) - ar(t))dt + \check{\sigma} dW(t)^{r^*}$$

where $\sigma_s, \sigma_v, \check{\sigma}$ and a are all positive constants that represents the volatility of the underlying asset, volatility of the market value of asset, volatility of the interest rate and mean reversion rate, respectively. The time-dependent function $b(t)$ matches the term structure of interest rates.

In this paper, Mellin transform is favored in order obtain a closed-form pricing formula for vulnerable options. Although an analytic pricing formula is derived under this stochastic interest rate framework, the paper does not include a numerical study nor a data analysis.

Wang et al. (2017) examines vulnerable options with a stochastic volatility framework under which the stochastic volatility is decomposed into the long-term and short-term volatility. In the paper, the long-term volatility is interpreted with a positive constant and the short-term volatility is described by a CIR process. This description of the stochastic volatility is incorporated into the model dynamics of the underlying asset $S(t)$ as well as the market value of the assets of the option writer $V(t)$. Precisely:

$$dS(t) = \mu_S S(t)dt + \sigma_s S(t)dW^{(1)}(t) + \sqrt{Y^{(1)}(t)}\, S(t)dB^{(1)}(t)$$

$$dY^{(1)}(t) = \left(\gamma_1 - \beta_1 Y^{(1)}(t)\right)dt + \sigma_{1,Y}\sqrt{Y^{(1)}(t)}dL^{(1)}(t)$$

where μ_S is the average appreciation rate, σ_s is the long-term volatility; $\gamma_{1,}$ β_1 and $\sigma_{1,Y}$ are CIR process parameters, $W^{(1)}(t)$, $B^{(1)}(t)$ and $L^{(1)}(t)$ are standard Brownian motions. Here, $B^{(1)}(t)$ and $L^{(1)}(t)$ are correlated with the parameter ρ_1, on the other hand, $W^{(1)}(t)$ is independent of these two $B^{(1)}(t)$ and $L^{(1)}(t)$. The rationale behind the description of short-term and long-term volatility is that short-term volatility considers the trading activities of the investors while the economic states and corporate performances represent long-term volatility.

Similarly, the dynamics of the market value of assets are governed by:

$$dV(t) = \mu_V V(t)dt + \sigma_V V(t)dW^{(2)}(t) + \sqrt{Y^{(2)}(t)}\, V(t)dB^{(2)}(t)$$

$$dY^{(1)}(t) = \left(\gamma_2 - \beta_2 Y^{(2)}(t)\right)dt + \sigma_{2,Y}\sqrt{Y^{(2)}(t)}dL^{(2)}(t)$$

where μ_V is the average appreciation rate, σ_V is the long-term volatility; $\gamma_{2,}$ β_2 and $\sigma_{2,Y}$ are CIR process parameters, $W^{(2)}(t)$, $B^{(2)}(t)$ and $L^{(2)}(t)$ are standard Brownian motions. Furthermore, they assume that $B^{(2)}(t)$ and $L^{(2)}(t)$ are correlated with the parameter ρ_2, and the pairs $(W^{(1)}(t), W^{(2)}(t))$, $(B^{(1)}(t), L^{(1)}(t))$, $(B^{(2)}(t), L^{(2)}(t))$ have mutual independence.

The core of the paper is to investigate the effects of short-term and long-term volatility on the vulnerable option price. Therefore, following the structural approach, a price formula is obtained for the special case $\rho_1 = \rho_2 = 0$ and, based on this price formulation, several numerical experiments are performed to illustrate the corresponding effects. For instance, the sensitivity of price of the at-the-money option to the stochastic volatility of the option writer's asset is not as strong as that of the underlying asset. Additionally, it is observed that long-term mean reverting value and long-term volatility of the underlying asset (asset of the counterparty) affect the option price positively (negatively) while the mean reverting speed affects negatively (positively).

Ma et al. (2020) focus on the pricing of vulnerable options under a stochastic volatility and interest rate framework. They assume that the underlying asset is modelled as:

$$dS(t) = r(t)S(t)dt + \sqrt{Z(t)}\, S(t)dW_S(t)$$

$$dZ(t) = \kappa_z\big(\theta - Z(t)\big)dt + \sigma_z Z(t)\left(\rho_1 dW_S(t) + \sqrt{(1-\rho_1^2)}dW_z(t)\right)$$

$$dr(t) = \big(b - ar(t)\big)dt + \sigma_r dW_r(t)$$

where $S(t)$, $Z(t)$ and $r(t)$ denote the time t-values of the underlying asset, the latent instantaneous variance and the stochastic interest rate, respectively. Regarding the variance process, κ_z is the mean-reverting rate, θ is the long-run mean level, σ_z is the volatility term. The interest rate process evolves according to the Vasicek model, defining the mean-reverting rate by a, the long-run mean by $\frac{b}{a}$ and the volatility by σ_r. All the parameters given above are assumed to be constant. Herewith, $W_S(t)$, $W_z(t)$ and $W_r(t)$ are the standard Brownian motions.

The dynamics of the market value of the counterparty's assets, $V(t)$, is governed by the following SDE:

$$dV(t) = r(t)V(t)dt + \sigma_V V(t)\left(\rho_2 dW_S(t) + \sqrt{(1-\rho_2^2)}dW_V(t)\right)$$

where σ_V is the volatiltiy, ρ_2 is the correlation coefficient between the value of underlying asset and counterparty's assets, and $W_V(t)$ is a standard Brownian motion. All the Brownian motions defined above, namely $W_V(t), W_z(t)\,, W_S(t), \text{and } W_r(t)$, are supposed to be mutually independent.

Following the classical approach in the presence of stochastic interest rates, the valuation of vulnerable options is examined under the T-forward measure, rather than the risk-neutral probability measure. T-forward measure is determined by a Radon-Nikodym derivative in which the zero-coupon bond price is used as the numeriare. Since the interest rate process is driven by a Vasicek model, this zero coupon price can be expressed in an affine form.

The paper continues with the formulation of the option price under T-forward measure by using fast Fourier transform technique. Here, the key point to obtaining an analytical option price is to compute the joint characteristic function of $\ln S(T)$ and $\ln V(T)$, whose dynamics under the T-forward measure are represented by using Girsanov theorem. The calculation of the joint characteristic function is done by solving a non-linear PDE, for which a closed-form solution does not exist, via a perturbation method. After the calculation of this joint characteristic function, an analytical option price formula is derived, relying on the FFT arguments. Along with the option price formula, the Greeks (Delta, Gamma, Rho, Theta) are computed.

The paper ends with various numerical experiments: First, the performance of the FFT method is tested by comparing the FFT-prices with those of Monte Carlo simulations. To this end, vulnerable and non-vulnerable option prices are calculated via these two methods for varying strike prices. Numerical tests shows that FFT is accurate and faster than Monte Carlo simulations. Secondly, the vulnerable option prices are compared with the ones of Klein and those of the non-vulnerable option in order to investigate the impact of stochastic interest rates, stochastic volatility and credit risk on the option price. Interestingly, it is revealed that non-vulnerable option prices are higher than vulnerable option prices. It is due to the possible credit loss of the option writer followed by a default event. The paper includes several figures that show the effect of the model parameters on the option price. For instance, greater values of the long-run mean level of the stochastic interest rate turn out to increase the option price. Indeed, when stochastic interest rates take greater values, price of the underlying asset as well as the market value of the assets of the option writer tend to increase which makes difficult to fall below the debt threshold level. Therefore, the default probability decreases, making the option more valuable.

7. Conclusion

In this paper, we have reviewed some very recent research publications in which OU,Vasicek, CIR and Hull-White models are incorporated into the modeling framewok within the context of actuarial sciences. The affine processes in these publications are used to model the stochastic interest rates, stochastic volatility, mortality intensity and the mean growth. Other than the models, also the mathematical approaches discussed in these papers are different. These methods include the Cost of Capital approach, fast Fourier transform, Mellin transform, a perturbation method, change of measure technique. There are publications, also favoring a data analysis. In one of the papers, a very detailed data analysis, including calibration, robustness test and backtesting, is carried out in order to compare the performance of the affine processes we favor in this review to model the mortality intensity. Precisely, Hull and White model is revealed to overperform compared with OU and Vasicek process when modeling the mortality intensity. By reviewing these publications, we aimed to give an insight about the use of these affine processes in actuarial applications.

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