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QUATERNIONIC BERTRAND CURVES ACCORDING TO TYPE 2-QUATERNIONIC FRAME IN $\mathbb{R}^4$

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Abstract. In this paper, we give some characterizations of quaternionic Bertrand curves whose torsion is non-zero but bitorsion is zero in $\mathbb{R}^4$ according to Type 2-Quaternionic Frame. One of the most important points in working on quaternionic curves is that given a curve in $\mathbb{R}^4$, the curve in $\mathbb{R}^3$ associated with this curve is determined individually. So, we obtain some relationships between quaternionic Bertrand curve $\alpha^{(4)}$ in $\mathbb{R}^4$and its associated spatial quaternionic curve α in $\mathbb{R}^3$. Also, we support some theorems in the paper by means of an example.

1. Introduction

Bertrand curve was introduced by *Bertrand* in 1850 (see [1]). When a curve is given, if there exists a second curve whose principal normal is the principal normal of that curve, then the first curve is called Bertrand curve and the second curve is called the Bertrand mate of the first curve. The most important properties of Bertrand curves in Euclidean 3-space are that the distance between corresponding points is constant and there is a linear relation between the curvature functions of the first curve, that is, for $\lambda, \mu \in \mathbb{R}$, $\lambda\kappa + \mu\tau = 1$, where κ is curvature and τ is the torsion of the first curve. Also, the absolute value of the real number λ in this linear relation is equal to the distance between corresponding points of Bertrand curves. The Bertrand curves in Euclidean 3-space were extended by L. R. Pears into Riemannian $n-$space and gave general results for Bertrand curves [13]. If these general results were applied to Euclidean $n-$space, then either torsion k_2 or bitorsion k_3 of the curve vanishes. In other words, Bertrand curves in $\mathbb{R}^n$ $(n > 3)$ are degenerate curves. Also, for $n > 3$, some studies about Bertrand curves in Euclidean $n-$space and Lorentzian $n-$space were made in [3], [5], [15].

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Bharathi and Nagaraj introduced spatial quaternionic curve in $\mathbb{R}^3$ and quaternionic curve in $\mathbb{R}^4$. By using the quaternionic multiplication, they obtained the Serret-Frenet equations of the curve in $\mathbb{R}^3$ and then they formed the Serret-Frenet formulae of a quaternionic curve in $\mathbb{R}^4$ by means of the Frenet vectors and curvature functions of the spatial quaternionic curve in $\mathbb{R}^3$ [2]. After then by using these quaternionic frames defined by Bharathi and Nagaraj, a lot of paper about quaternionic curves were made in $\mathbb{R}^3$ and $\mathbb{R}^4$ ([4], [6], [7], [9], [10], [11], [12], [14], [16], [17], [18]).

Kahraman Aksoyak introduced a new quaternionic frame in $\mathbb{R}^4$. This new type of quaternionic frame was called Type 2-Quaternionic Frame [8].

In this paper, we investigate quaternionic Bertrand curves whose torsion is non-zero but bitorsion is zero in $\mathbb{R}^4$ according to Type 2-Quaternionic Frame. One of the most important points in working on quaternionic curves is that given a curve in $\mathbb{R}^4$, the curve in $\mathbb{R}^3$ associated with this curve is determined individually. Hence we obtain some relationships between quaternionic Bertrand curve $\alpha^{(4)}$ in $\mathbb{R}^4$ and spatial quaternionic curve α in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$. For example, we obtain that quaternionic curve $\alpha^{(4)}$ in $\mathbb{R}^4$ is a quaternionic Bertrand curve if and only if the curve α in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$ is a spatial quaternionic Bertrand curve. Also, we show that result: if $\left(\alpha^{(4)}, \beta^{(4)}\right)$ is a quaternionic Bertrand curve couple then (α, β) is a spatial quaternionic Bertrand curve couple, where α and β are curves in $\mathbb{R}^3$ associated with quaternionic curves $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively. And then we give an example about these results.

2. Preliminaries

The quaternion was defined by Hamilton. A real quaternion is as:

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

where $q_i \in \mathbb{R}$ for $0 \leq i \leq 3$ and e_1, e_2, e_3 are unit vectors in usual three dimensional real vector space. Any quaternion q can be divided into two parts such that the scalar part denoted by S_q and the vectorial part denoted by V_q, that is, for $S_q = q_0$ and $V_q = q_1 e_1 + q_2 e_2 + q_3 e_3$ we can express any real quaternion as $q = S_q + V_q$.

If $q = S_q + V_q$ and $q' = S_{q'} + V_{q'}$ are any two quaternions, then equality, addition, the multiplication by a real scalar c and the conjugate of q denoted by γq are as:

$$\begin{aligned} \text{equality} &: \quad q = q' \text{ if and only if } S_q = S_{q'} \text{ and } V_q = V_{q'} \\ \text{addition} &: \quad q + q' = (S_q + S_{q'}) + (V_q + V_{q'}) \\ \text{multiplication by a real scalar} &: \quad cq = cS_q + cV_q \\ \text{conjugate} &: \quad \gamma q = S_q - V_q. \end{aligned}$$

Let us denote the set of quaternions by H. H is a real vector space with above addition and scalar multiplication. A basis of this vector space is $\{1,\ e_1,\ e_2,\ e_3\}$. Hence, we can think of any quaternion q as an element (q_0, q_1, q_2, q_3) of $\mathbb{R}^4$. Even a quaternion whose scalar part is zero (it is called spatial quaternion) can be considered as a ordered triple (q_1, q_2, q_3) of $\mathbb{R}^3$.

The product of two quaternions is defined by means of the multiplication rule between the units e_1, e_2, e_3 are given by:

$$e_1e_1 = e_2e_2 = e_3e_3 = e_1e_2e_3 = -1. \tag{1}$$

So, by using (1), quaternionic multiplication is obtained as:

$$q \times q' = S_qS_{q'} - \langle V_q, V_{q'} \rangle + S_qV_{q'} + S_{q'}V_q + V_q \wedge V_{q'} \text{ for every } q,\ q' \in H, \tag{2}$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and cross products in $\mathbb{R}^3$, respectively. Also, H is a real algebra and it is called quaternion algebra.

Now, by using (2) the symetric, non-degenerate, bilinear form h on H is given by :

$$h : H \times H \to \mathbb{R},$$

$$h(q, q') = \frac{1}{2}(q \times \gamma q' + q' \times \gamma q) \text{ for } q,\ q' \in H \tag{3}$$

and the norm of any q real quaternion is defined by

$$\|q\|^2 = h(q, q) = q \times \gamma q = S_q^2 + \langle V_q, V_q \rangle .$$

So the mapping given by (3) is called the quaternion inner product [2].

We note that a quaternionic curve in $\mathbb{R}^4$ is denoted by $\alpha^{(4)}$ and the spatial quaternionic curve in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$ is denoted by α.

Bharathi and Nagaraj introduced the Serret-Frenet formulas for spatial quaternionic curves in $\mathbb{R}^3$ and quaternionic curves in $\mathbb{R}^4$ follow as:

Theorem 1. *(see [2]) Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and S be the set of spatial quaternionic curve*

$$\alpha : I \subset \mathbb{R} \longrightarrow S,$$

$$s \longrightarrow \alpha(s) = \alpha_1(s)e_1 + \alpha_2(s)e_2 + \alpha_3(s)e_3$$

be an arc-lenghted curve. Then the Frenet equations of α are as follows:

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $t = \alpha'$ is unit tangent, n is unit principal normal, $b = t \times n$ is binormal, where $\times$ denotes the quaternion product. $k = \|t'\|$ is the principal curvature and r is the torsion of the curve γ.

Theorem 2. *(see [2]) Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and*

$$\alpha^{(4)} : I \subset \mathbb{R} \longrightarrow Q,$$

$$s \longrightarrow \alpha^{(4)}(s) = \alpha_0^{(4)}(s) + \alpha_1^{(4)}(s)e_1 + \alpha_2^{(4)}(s)e_2 + \alpha_3^{(4)}(s)e_3$$

be an arc-length curve in $\mathbb{R}^4$. *Then Frenet equations of* $\alpha^{(4)}$ *are given by*

$$\begin{bmatrix} T^{'} \\ N_1^{'} \\ N_2^{\prime} \\ N_3^{\prime} \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (K-r) \\ 0 & 0 & -(K-r) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix},$$

where $T = \frac{d\alpha^{(4)}}{ds}$, N_1, N_2, N_3 *are the Frenet vectors of the curve* $\alpha^{(4)}$ *and* $K = \left\| T^{'} \right\|$ *is the principal curvature,* k *is the torsion and* $(K-r)$ *is the bitorsion of the curve* $\alpha^{(4)}$. *There exists following relations between the Frenet vectors of* $\alpha^{(4)}$ *and the Frenet vectors of* α

$$N_1(s) = t(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = b(s) \times T(s).$$

Type 2-Quaternionic Frame which is introduced by Kahraman Aksoyak in [8] is given as:

Theorem 3. *(see [8]) Let* $I = [0, 1]$ *denote the unit interval in the real line* $\mathbb{R}$ *and*

$$\alpha^{(4)} : I \subset \mathbb{R} \longrightarrow Q,$$
$$s \longrightarrow \alpha^{(4)}(s) = \alpha_0^{(4)}(s) + \alpha_1^{(4)}(s)e_1 + \alpha_2^{(4)}(s)e_2 + \alpha_3^{(4)}(s)e_3$$

be an arc-length curve in $\mathbb{R}^4$. *Then Frenet equations of* $\alpha^{(4)}$ *are given by*

$$\begin{bmatrix} T^{'} \\ N_1^{'} \\ N_2^{\prime} \\ N_3^{\prime} \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & -r & 0 \\ 0 & r & 0 & (K-k) \\ 0 & 0 & -(K-k) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{4}$$

where $T = \frac{d\alpha^{(4)}}{ds}$, N_1, N_2, N_3 *are the Frenet vectors of the curve* $\alpha^{(4)}$ *and* $K = \left\| T^{'} \right\|$ *is the principal curvature,* $-r$ *is the torsion and* $(K-k)$ *is the bitorsion of the curve* $\alpha^{(4)}$. *There exists following relations between the Frenet vectors of* $\alpha^{(4)}$ *and the Frenet vectors of* α

$$N_1(s) = b(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = t(s) \times T(s).$$

3. Characterizations of Quaternionic Bertrand Curve

In this section, we consider the quaternionic curve whose the torsion $(-r)$ is non-zero and bitorsion $(K-k)$ is zero according to Type 2-Quaternionic Frame in $\mathbb{R}^4$ given by (4) and obtain various characterizations for cases where such curves are quaternionic Bertrand curves. Also, we give some relationships between quaternionic Bertrand curves in $\mathbb{R}^4$ and spatial quaternionic curves in $\mathbb{R}^3$ which are related to these curves and discuss some theorems in this section on an example.

Definition 1. *Let* $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ *and* $\beta^{(4)} : \bar{I} \subset \mathbb{R} \to \mathbb{E}^4$ *be quaternionic curves given by the arc-length parameter* s *and* $\bar{s}$, *respectively.*
$\{T(s),\ N_1(s),\ N_2(s),\ N_3(s)\}$ *and* $\{\bar{T}(\bar{s}),\ \bar{N}_1(\bar{s}),\ \bar{N}_2(\bar{s}),\ \bar{N}_3(\bar{s})\}$ *are Frenet vectors*

of these curves. If the principal normal vectors $N_1(s)$ *and* $\bar{N}_1(\bar{s})$ *of the curves* $\alpha^{(4)}$ *and* $\beta^{(4)}$ *are linearly dependent, then these curves are called quaternionic Bertrand curves. Let* $(\alpha^{(4)}, \beta^{(4)})$ *be quaternionic Bertrand curve couple, where* $\alpha^{(4)}$ *is a quaternionic Bertrand curve and* $\beta^{(4)}$ *is quaternionic Bertrand mate of* $\alpha^{(4)}$.

Theorem 4. *Let* $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ *and* $\beta^{(4)} : \bar{I} \subset \mathbb{R} \to \mathbb{E}^4$ *be quaternionic curves with arc-length parameter* s *and* $\bar{s}$*, respectively. If* $(\alpha^{(4)}, \beta^{(4)})$ *is a quaternionic Bertrand curve couple, then the distance at corresponding points is constant, that is*

$$d(\alpha^{(4)}(s), \beta^{(4)}(\bar{s})) = const., \text{ for all } s \in I.$$

Proof. We assume that $\alpha^{(4)}$ is a quaternionic Bertrand curve and $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$. From Definition (1), we can write

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda(s) N(s),$$

where $\lambda : I \to \mathbb{R}$ is a differentiable function. If we take the derivative of the above equation with respect to s and use the equations of Type 2-Quaternionic Frame given by (4), we get

$$\bar{T}(\bar{s}) = \frac{ds}{d\bar{s}} \left[(1 - \lambda(s) K(s)) T(s) + \lambda'(s) N_1(s) - \lambda(s) r(s) N_2(s) \right]. \tag{5}$$

Since $h(\bar{T}(\bar{s}), \bar{N}_1(\bar{s})) = 0$ and $h(N_1(s), \bar{N}_1(\bar{s})) = \pm 1$,

$$\lambda'(s) = 0$$

and we have that λ is a constant function on I. □

Theorem 5. *The measure of the angle between the tangent vector fields of quaternionic Bertrand curve couple* $\left(\alpha^{(4)}, \beta^{(4)}\right)$ *is constant, that is*

$$h\left(T(s), \bar{T}(\bar{s})\right) = \cos \phi_0 = const. \tag{6}$$

Proof. If we derivative $h\left(T(s), \bar{T}(\bar{s})\right)$ and use the equations of Type 2-Quaternionic Frame, we obtain following equality:

$$\begin{aligned} \frac{dh\left(T(s), \bar{T}(\bar{s})\right)}{ds} &= h\left(\frac{dT(s)}{ds}, \bar{T}(\bar{s})\right) + h\left(T(s), \frac{\bar{T}(\bar{s})}{d\bar{s}} \frac{d\bar{s}}{ds}\right) \\ &= h\left(K(s) N_1(s), \bar{T}(\bar{s})\right) + h\left(T(s), \bar{K}(\bar{s}) \bar{N}_1(\bar{s}) \frac{d\bar{s}}{ds}\right). \end{aligned}$$

Since $\bar{N}_1(\bar{s}) = \pm N_1(s)$, we find

$$\frac{dh\left(T(s), \bar{T}(\bar{s})\right)}{ds} = 0$$

which implies that $h\left(T(s), \bar{T}(\bar{s})\right)$ is constant. □

Theorem 6. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic curve with arc-length parameter s whose torsion is non-zero and bitorsion is zero. Then $\alpha^{(4)}$ is a quaternionic Bertrand curve if and only if*

$$\lambda K + \mu r = 1,$$

where λ and μ are real numbers, K is the principal curvature, $-r$ is the torsion of the curve $\alpha^{(4)}$.

Proof. We suppose that $\alpha^{(4)}$ is a quaternionic Bertrand curve such that $r \neq 0$ and $K - k = 0$. Then there exists a quaternionic Bertrand mate of $\alpha^{(4)}$ denoted by $\beta^{(4)}$. $\beta^{(4)}$ can be expressed as:

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N_1(s), \tag{7}$$

where λ is non-zero real number. Since the angle between the tangent vector fields of $\alpha^{(4)}$ and $\beta^{(4)}$ is constant, from (5) and (6), the tangent vector of $\beta^{(4)}$ can be written as:

$$\bar{T}(\bar{s}) = \cos\phi_0 T(s) + \sin\phi_0 N_2(s)$$

in here

$$\cos\phi_0 = (1 - \lambda K(s)) \frac{ds}{d\bar{s}}, \tag{8}$$

$$\sin\phi_0 = -\lambda r(s) \frac{ds}{d\bar{s}}. \tag{9}$$

Since λ and $r(s)$ are non-zero, $\sin\phi_0$ is non-zero. If we take as $-\lambda \frac{\cos\phi_0}{\sin\phi_0} = \mu$ and ratio the equations given by (8) and (9) side by side, we find

$$\lambda K + \mu r = 1.$$

Conversely, let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic curve whose the curvatures K and $-r$ hold the relation $\lambda K + \mu r = 1$ for λ and μ real numbers. Let define a quaternionic curve by using λ real number as:

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N_1(s).$$

It is clearly shown that the principal normal lines of $\alpha^{(4)}$ and $\beta^{(4)}$ are linearly dependent. $\square$

Theorem 7. *Let $\left(\alpha^{(4)}, \beta^{(4)}\right)$ be a quaternionic Bertrand curve couple, then the product of torsions $r(s)$ and $\bar{r}(\bar{s})$ at the corresponding points of the curves α and β is a constant, where α and β are spatial quaternionic curves in $\mathbb{R}^3$ related to quaternionic curves $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively.*

Proof. Let consider that $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$. Then we have

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N_1(s).$$

If we displace the position vectors $\alpha^{(4)}(s)$ and $\beta^{(4)}(s)$, we get

$$\alpha^{(4)}(s) = \beta^{(4)}(s) - \lambda \bar{N}_1(\bar{s}). \tag{10}$$

By differentiating (10) with respect to s and using the equations of Type 2-Quaternionic Frame, we obtain

$$T(s) = \left[(1 + \lambda \bar{K}(\bar{s}))\bar{T}(\bar{s}) + \lambda \bar{r}(\bar{s})\, \bar{N}_2(\bar{s})\right] \frac{d\bar{s}}{ds}.$$

So, we can rewrite

$$T(s) = \cos\phi_0 \bar{T}(\bar{s}) - \sin\phi_0 \bar{N}_2(\bar{s}),$$

where

$$\cos\phi_0 = (1 + \lambda \bar{K}(\bar{s}))\frac{d\bar{s}}{ds}, \tag{11}$$

$$\sin\phi_0 = -\lambda \bar{r}(\bar{s})\, \frac{d\bar{s}}{ds}. \tag{12}$$

Multiplying the equations (9) and (12) side by side, we find

$$r\bar{r} = \frac{\sin^2\phi_0}{\lambda^2} = const.$$

□

Theorem 8. *Let $\left(\alpha^{(4)}, \beta^{(4)}\right)$ be a quaternionic Bertrand curve couple. Then the curvatures $K(s)$, $-r(s)$ and $\bar{K}(\bar{s})$, $-\bar{r}(\bar{s})$ of the curves $\alpha^{(4)}$ and $\beta^{(4)}$, respectively, satisfy the following equation*

$$\lambda\left(K + \bar{K}\right) + \mu\left(r - \bar{r}\right) = 0. \tag{13}$$

Proof. We assume that $\left(\alpha^{(4)}, \beta^{(4)}\right)$ is a quaternionic Bertrand curve couple. Then if we ratio the equations given by (8) and (9) side by side, we find

$$\frac{\cos\phi_0}{\sin\phi_0} = \frac{1 - \lambda K}{-\lambda r}$$

and similarly if we proportion the equation (11) to equation (12),

$$\frac{\cos\phi_0}{\sin\phi_0} = \frac{1 + \lambda \bar{K}}{-\lambda \bar{r}}.$$

If we take as $-\frac{\cos\phi_0}{\sin\phi_0}\lambda = \mu$, we have

$$\lambda K + \mu r = 1 \tag{14}$$

and

$$\lambda \bar{K} - \mu \bar{r} = -1. \tag{15}$$

From (14) and (15), we obtain

$$\lambda\left(K + \bar{K}\right) + \mu\left(r - \bar{r}\right) = 0.$$

□

Theorem 9. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic curve whose the torsion is non-zero and bitorsion is zero and $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a spatial quaternionic curve associated with $\alpha^{(4)}$ quaternionic curve. Then α is a spatial quaternionic Bertrand curve if and only if $\alpha^{(4)}$ is a quaternionic Bertrand curve.*

Proof. We assume that $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is a spatial quaternionic Bertrand curve. Then there are λ, μ are real constants such that the curvatures $k(s)$ and $r(s)$ of α satisfy

$$\lambda k + \mu r = 1. \tag{16}$$

Since the bitorsion of the quaternionic curve $\alpha^{(4)}$ vanishes, we have

$$K = k. \tag{17}$$

From (16) and (17), we get

$$\lambda K + \mu r = 1.$$

From Theorem (6), the above equality says that $\alpha^{(4)}(s)$ is a quaternionic Bertrand curve.

Conversely it is clearly shown that if $\alpha^{(4)}$ is a quaternionic Bertrand curve whose the bitorsion vanishes, then α is a spatial quaternionic Bertrand curve. □

Theorem 10. *If $\left(\alpha^{(4)}, \beta^{(4)}\right)$ is a quaternionic Bertrand curve couple then (α, β) is a spatial quaternionic Bertrand curve couple, where α and β are curves in $\mathbb{R}^3$ associated with quaternionic curves $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively.*

Proof. We consider that $\alpha^{(4)}$ is a quaternionic Bertrand curve and $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$. Then from Definition (1), $N_1(s)$ and $\bar{N}_1(\bar{s})$ are linearly dependent. On the other hand, from Theorem (9), we know that if $\alpha^{(4)}$ and $\beta^{(4)}$ are Bertrand curve then the curves α and β in $\mathbb{R}^3$ which are associated with $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively are Bertrand curves, too. Now, we show that β is quaternionic Bertrand mate of α.

From Type 2- Quaternionic Frame, the binormal $\bar{b}$ of β is written as:

$$\bar{b} = \bar{N}_1 \times \gamma \bar{T} \tag{18}$$

Since $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$, we have $\bar{N}_1 = N_1$ and $\bar{T} = \cos\phi_0 T + \sin\phi_0 N_2$. So we can rewrite (18) following as:

$$\begin{aligned} \bar{b} &= N_1 \times \gamma\left(\cos\phi_0 T + \sin\phi_0 N_2\right) \\ &= \cos\phi_0\left(N_1 \times \gamma T\right) + \sin\phi_0\left(N_1 \times \gamma N_2\right). \end{aligned}$$

In last equality, if we use $N_1 \times \gamma T = b$ and $N_2 = n \times T$, we obtain

$$\bar{b} = \cos\phi_0 b + \sin\phi_0 t. \tag{19}$$

Differentiating (19), we find

$$-\bar{r}\bar{n}\frac{d\bar{s}}{ds} = \left(-\cos\phi_0 r + \sin\phi_0 k\right) n$$

and it implies that $\bar{n} = \pm n$. Hence β is a Bertrand mate of α. $\square$

Now, we will see an application of some theorems in the paper by means of following example.

Example 1. *Let* $\alpha^{(4)}(s) = \left(\cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$ *be a quaternionic curve in* $\mathbb{R}^4$ *which is given by arc-length parameter* s*. The Frenet vectors and the curvatures of the curve* $\alpha^{(4)}$ *in* $\mathbb{R}^4$ *are as:*

$$\begin{aligned}
T(s) &= \frac{1}{\sqrt{3}}\left(-\sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, 1, 1\right), \\
N_1(s) &= \left(-\cos\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}}, 0, 0\right), \\
N_2(s) &= \frac{1}{\sqrt{6}}\left(-2\sin\frac{s}{\sqrt{3}}, 2\cos\frac{s}{\sqrt{3}}, -1, -1\right), \\
N_3(s) &= \frac{1}{\sqrt{2}}(0, 0, -1, 1)
\end{aligned}$$

and

$$k_1 = K = \frac{1}{3},\ k_2 = -r = -\frac{\sqrt{2}}{3},\ k_3 = K - k = 0.$$

By using the definition of Type-2 Quaternionic Frame, the curve α *in* $\mathbb{R}^3$ *which is associated with* $\alpha^{(4)}$ *is obtained as:*

$$\alpha(s) = \frac{1}{\sqrt{2}}\left(2\frac{s}{\sqrt{3}}, -\cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right).$$

The Frenet vectors and the curvatures of α *are computed as:*

$$\begin{aligned}
t(s) &= \frac{1}{\sqrt{6}}\left(2, -\cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, -\cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}\right), \\
n(s) &= \frac{1}{\sqrt{2}}\left(0, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}, -\cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}\right), \\
b(s) &= \frac{1}{\sqrt{3}}\left(1, \cos\frac{s}{\sqrt{3}} - \sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}} + \sin\frac{s}{\sqrt{3}}\right)
\end{aligned}$$

and

$$k = \frac{1}{3},\ \ r = \frac{\sqrt{2}}{3}.$$

From the definition of Type-2 Quaternionic Frame, there exists following relations between Frenet vectors of $\alpha^{(4)}$ *in* $\mathbb{R}^4$ *and* α *in* $\mathbb{R}^3$ *:*

$$N_1(s) = b(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = t(s) \times T(s).$$

$\alpha^{(4)}$ *is a quaternionic curve whose torsion is non zero and bitorsion is zero and we can see that the curvatures of* $\alpha^{(4)}$ *hold* $\lambda K + \mu r = 1$*, for* $\lambda = -2$ *and* $\mu = \frac{5}{\sqrt{2}}$.

So it is a quaternionic Bertrand curve. From Theorem (9), we know that if $\alpha^{(4)}$ is a quaternionic Bertrand curve then α is a spatial quaternionic Bertrand curve. We can easily see that $\lambda k+\mu r=1$, for $\lambda=-2$ and $\mu=\frac{5}{\sqrt{2}}$. Since $\alpha^{(4)}$ is a quaternionic Bertrand curve, we can determine the quaternionic Bertrand mate of it as:

$$\begin{aligned} \beta^{(4)}(s) &= \alpha^{(4)}(s)-2N_1(s) \\ &= \left(3\cos\frac{s}{\sqrt{3}}, 3\sin\frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right), \end{aligned}$$

where $\bar{s}=\varphi(s)=\int\left\|\frac{d\beta^{(4)}(s)}{ds}\right\|ds=\frac{\sqrt{11}}{\sqrt{3}}s$ and $\bar{s}$ is arc-length parameter of $\beta^{(4)}$. Now by using Type-2 Quaternionic Frame, we can determine the Frenet vectors and the curvatures of the curve $\beta^{(4)}$ as follows:

$$\beta^{(4)}(\bar{s})=\left(3\cos\frac{\bar{s}}{\sqrt{11}}, 3\sin\frac{\bar{s}}{\sqrt{11}}, \frac{\bar{s}}{\sqrt{11}}, \frac{\bar{s}}{\sqrt{11}}\right),$$

$$\begin{aligned} \bar{T}(\bar{s}) &= \frac{1}{\sqrt{11}}\left(-3\sin\frac{\bar{s}}{\sqrt{11}}, 3\cos\frac{\bar{s}}{\sqrt{11}}, 1, 1\right), \\ \bar{N}_1(\bar{s}) &= \left(-\cos\frac{\bar{s}}{\sqrt{11}}, -\sin\frac{\bar{s}}{\sqrt{11}}, 0, 0\right), \\ \bar{N}_2(\bar{s}) &= \frac{1}{\sqrt{22}}\left(-2\sin\frac{\bar{s}}{\sqrt{11}}, 2\cos\frac{\bar{s}}{\sqrt{11}}, -3, -3\right), \\ \bar{N}_3(\bar{s}) &= \frac{1}{\sqrt{22}}(0, 0, -11, 11) \end{aligned}$$

and

$$\bar{k}_1=\bar{K}=\frac{3}{11},\ \bar{k}_2=-\bar{r}=-\frac{\sqrt{2}}{11},\ \bar{k}_3=\bar{K}-\bar{k}=0.$$

The curve β which is associated with $\beta^{(4)}$ is found as:

$$\beta(\bar{s})=\frac{1}{\sqrt{2}}\left(2\frac{\bar{s}}{\sqrt{11}}, 3\left(-\sin\frac{\bar{s}}{\sqrt{11}}-\cos\frac{\bar{s}}{\sqrt{11}}\right), 3\left(-\sin\frac{\bar{s}}{\sqrt{11}}+\cos\frac{\bar{s}}{\sqrt{11}}\right)\right).$$

The Frenet vectors and the curvatures of β are found as:

$$\begin{aligned} \bar{t}(\bar{s}) &= \frac{1}{\sqrt{22}}\left(2, 3\left(-\cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}\right), 3\left(-\cos\frac{\bar{s}}{\sqrt{11}}-\sin\frac{\bar{s}}{\sqrt{11}}\right)\right), \\ \bar{n}(\bar{s}) &= \frac{1}{\sqrt{2}}\left(0, \cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}, -\cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}\right), \\ \bar{b}(\bar{s}) &= \frac{1}{\sqrt{11}}\left(3, \cos\frac{\bar{s}}{\sqrt{11}}-\sin\frac{\bar{s}}{\sqrt{11}}, \cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}\right) \end{aligned}$$

and

$$\bar{k}=\frac{3}{11},\ \bar{r}=\frac{\sqrt{2}}{11}.$$

From the definition of Type-2 Quaternionic Frame, there exists following relations between Frenet vectors of $\beta^{(4)}$ *in* $\mathbb{R}^4$ *and* β *in* $\mathbb{R}^3$

$$\bar{N}_1(\bar{s}) = \bar{b}(\bar{s}) \times \bar{T}(\bar{s}),\ \bar{N}_2(\bar{s}) = \bar{n}(\bar{s}) \times \bar{T}(\bar{s}),\ \bar{N}_3(\bar{s}) = \bar{t}(\bar{s}) \times \bar{T}(\bar{s}).$$

Since $\beta^{(4)}$ *is a quaternionic Bertrand curve,* β *is a spatial quaternionic Bertrand curve and the curvatures of* β *satisfy* $\bar{\lambda}\bar{k} + \bar{\mu}\bar{r} = 1$, *for* $\bar{\lambda} = 2$ *and* $\bar{\mu} = \frac{5}{\sqrt{2}}$ *real numbers. From Theorem (10), we know that* β *is Bertrand mate of* α. *In fact* $\bar{n} = n$ *and* $\beta(s) = \alpha(s) - 2n(s)$.

Also, in this example, we can see that the equation (13) in Theorem (8) holds for $\lambda = -2$, $\mu = \frac{5}{\sqrt{2}}$, $K = \frac{1}{3}$, $r = \frac{\sqrt{2}}{3}$, $\bar{K} = \frac{3}{11}$, $\bar{r} = \frac{\sqrt{2}}{11}$, *that is* $\lambda\left(K + \bar{K}\right) + \mu(r - \bar{r}) = 0$.

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