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ON DIFFERENCE OF BIVARIATE LINEAR POSITIVE OPERATORS

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Abstract. In the present paper we give quantitative type theorems for the differences of different bivariate positive linear operators by using weighted modulus of continuity. Similar estimates are obtained via K-functional and for Chebyshev functionals. Moreover, an example involving Szász and Szász-Kantorovich operators is given.

1. Introduction

Studies in the theory of approximations have been going on for many years. During these times, the most well-known operator Bernstein operators, the best-known theorem for convergence was the Korovkin Theorem. Then, Szasz, Baskakov, Kantorovich operators are defined and their convergence properties are examined. Many researchers have defined various modification forms of these operators and examined their convergence properties and their applications are given. In recent years, some studies have been carried out to obtain general information between the convergence speeds of the operators by taking the difference of any two operators.

In the recent past, there is a growing interest in studying the difference of linear positive operators in approximation theory (see [1], [2], [3] and [6])

In 2006, Gonska et al., using Taylor's expansion with Peano remainder, gave a Theorem showing that the difference of two operators A and B can be limited by the concave majorant $\widetilde{\omega}$, where ω_k is the k-th order modulus of smoothness [11].

In 2016, A. M. Acu and I. Raşa obtained some inequalities using Taylor's formula and obtained some estimations by applying these inequalities on the differences of Linear Positive operators [1].

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In 2019, A. Aral et al. obtained some estimates for the difference of two general linear positive operators on unbounded interval [5].

In 2021, A. M. Acu et al. gave some theorems for the difference of linear positive operators of two variables defined on a simplex [4].

In this study, we will give some theorems given by A. Aral et al. [5] for univariate operators for bivariate operators.

This paper deals with the difference of certain bivariate operators defined on unbounded intervals. The differences are estimated in terms of weighted moduli of smoothness for the operators constructed with the same fundamental functions and different functionals in front of them.

2. Auxiliary Results

If we can calculate that the difference between the A and B operators is very small, we can learn the properties of the other by looking at the properties of one.

It is well-known that classical modulus of continuity is a very useful tool in order to determine the rate of convergence of the corresponding sequence of linear positive operators defined bounded interval, in case of unbounded intervals, It would be more appropriate to use a defined modulus of continuity in weighted function spaces. This allows to enlarge the continuous function space to weighted function space in approximation problems. For this purpose, we consider the modulus of continuity defined in suitable polynomial weighted space, defined for univariate case in [10] by Gadjieva and Doğru and for bivariate case in [12] by İspir and Atakut.

Let $\mathcal{D} := [0,\infty) \times [0,\infty)$ and $\rho(x,y) := 1 + x^2 + y^2$, $(x,y) \in \mathcal{D}$. Throughout the paper; $C(\mathcal{D})$ will denote the space of real-valued continuous functions on $\mathcal{D}$ and $C_B(\mathcal{D})$ will denote the space of all $f \in C(\mathcal{D})$ that are bounded on $\mathcal{D}$. Let $B_\rho(\mathcal{D})$ denote the space of functions f satisfying the inequality

$$|f(x,y)| \leq m_f \rho(x,y), \quad (x,y) \in \mathcal{D},$$

where m_f is a positive constant which depend on the function f. $B_\rho(\mathcal{D})$ is a linear normed space with the norm

$$\|f\|_\rho = \sup_{(x,y)\in\mathcal{D}} \frac{|f(x,y)|}{\rho(x,y)}. \tag{1}$$

Let $C_\rho(\mathcal{D})$ denote the subspace of all continuous functions belonging to $B_\rho(\mathcal{D})$. Also, let $C_\rho^*(\mathcal{D})$ denote the subspace of all functions $f \in C_\rho(\mathcal{D})$ for which there exists a constant k_f such that

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{|f(x,y)|}{\rho(x,y)} = k_f < \infty.$$

In the case of $k_f = 0$, we will write $C_\rho^0(\mathcal{D})$.

We use the weighted modulus of continuity, considered in [10] and [12], denoted by $\Omega_\rho(f,\cdot,\cdot)$ and given by

$$\Omega_\rho(f,\delta_1,\delta_2)=\sup_{(x,y)\in\mathcal{D},\ |h_1|<\delta_1,\ |h_2|<\delta_2}\frac{f(x+h_1,y+h_2)-f(x,y)}{(1+x^2+y^2)(1+h_1^2+h_2^2)};\ f\in C_\rho(\mathcal{D}). \tag{2}$$

The weighted modulus of continuity Ω_ρ satisfies the following properties for $f\in C_\rho^*(\mathcal{D})$:

i: $\Omega_\rho(f,\delta_1,\delta_2)\to 0$ as $\delta_1\to 0$ and $\delta_2\to 0$ for $\delta_1,\delta_2>0$.

ii: For any positive real numbers λ_1, λ_2, δ_1 and δ_2 the following relation

$$\Omega_\rho(f,\lambda_1\delta_1,\lambda_2\delta_2)\le 4(1+\lambda_1)(1+\lambda_2)\,\Omega_\rho(f,\delta_1,\delta_2) \tag{3}$$

holds.

In the sequel, we will use the notation that $e_{i,j}(x,y):=x^iy^j$, $i,j\in\mathbb{N}$, $(x,y)\in\mathcal{D}$, $\mathbf{1}$ denotes the constant function

$$\mathbf{1}:\mathcal{D}\to\mathbb{R},\ \mathbf{1}(x,y)=1,\ (x,y)\in\mathcal{D}, \tag{4}$$

and $\mathbb{D}$ denotes a linear subspace of $C(\mathcal{D})$, which contains $C_\rho(\mathcal{D})$. We also consider the positive linear functional $F:\mathbb{D}\to\mathbb{R}$ such that $F(\mathbf{1})=1$. Denoting

$$\theta_1^F:=F(e_{1,0})\ ,\ \theta_2^F:=F(e_{0,1}) \tag{5}$$

and

$$\mu_{i,j}^F:=F\left(\left(e_{1,0}-\theta_1^F\mathbf{1}\right)^i\left(e_{0,1}-\theta_2^F\mathbf{1}\right)^j\right),\quad i,j\in\mathbb{N}, \tag{6}$$

then one has

$$\begin{aligned}\mu_{1,0}^F&=0,\ \ \mu_{2,0}^F=F(e_{1,0})^2-\left(\theta_1^F\right)^2\ge 0,\\ \mu_{0,1}^F&=0,\ \ \mu_{0,2}^F=F(e_{0,1})^2-\left(\theta_2^F\right)^2\ge 0.\end{aligned} \tag{7}$$

Lemma 1. *For $(x,y)\in\mathcal{D}$, $f\in C_\rho^*(\mathcal{D})$ and $0<\delta_1,\delta_2\le 1$, we have*

$$|f(t,s)-f(x,y)|\le 256\rho(x,y)\left(1+\frac{(t-x)^4}{\delta_1^4}\right)\left(1+\frac{(s-y)^4}{\delta_2^4}\right)\Omega_\rho(f,\delta_1,\delta_2).$$

Proof. Using the inequality [5] with $\lambda_1=\frac{|t-x|}{\delta_1}$ ve $\lambda_2=\frac{|s-y|}{\delta_2}$, from (2) and (3), we have

$$\begin{aligned}|f(t,s)-f(x,y)|&\le 4\rho(x,y)\,\Omega_\rho(f,\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\\&\quad\times\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)\end{aligned}$$

$$\le\begin{cases}16\rho(x,y)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega_\rho(f,\delta_1,\delta_2); & |t-x|\le\delta_1,|s-y|\le\delta_2\\ 16\rho(x,y)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega_\rho(f,\delta_1,\delta_2)\frac{(t-x)^4}{\delta_1^4}\frac{(s-y)^4}{\delta_2^4}; & |t-x|>\delta_1,|s-y|>\delta_2\end{cases}.$$

Therefore

$$|f(t,s)-f(x,y)| \leq 16\rho(x,y)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\left(1+\tfrac{(t-x)^4}{\delta_1^4}\right)\left(1+\tfrac{(t-y)^4}{\delta_2^4}\right)\Omega_\rho(f,\delta_1,\delta_2).$$

Choosing $0<\delta_1\leq 1,\ 0<\delta_2\leq 1$ for $f\in C_\rho^*(\mathcal{D}),\ (x,y)\in\mathcal{D}$, we get

$$|f(t,s)-f(x,y)| \leq 256\rho(x,y)\left(1+\frac{(t-x)^4}{\delta_1^4}\right)\left(1+\frac{(s-y)^4}{\delta_2^4}\right)\Omega_\rho(f,\delta_1,\delta_2).$$

□

Now, we present the following estimate for the difference $\left|F(f)-f\left(\theta_1^F,\theta_2^F\right)\right|$.

Lemma 2. *Let f and all of its partial derivatives of order ≤ 2 belong to the space $C_\rho(\mathcal{D})$ and $0<\delta_1\leq 1,\ 0<\delta_2\leq 1$. Then we have*

$$\left|F(f)-f\left(\theta_1^F,\theta_2^F\right)\right| \leq M_f\rho\left(\theta_1^F,\theta_2^F\right)\left[\mu_{2,0}^F+\mu_{0,2}^F\right],$$

where

$$M_f := \max\left\{\|f_{xx}\|_\rho, \|f_{xy}\|_\rho, \|f_{yy}\|_\rho\right\}.$$

Proof. For $f\in C_\rho(\mathcal{D})$, $(t,s)\in\mathcal{D}$, using the Taylor formula we have

$$\begin{aligned}
& f(t,s)-f\left(\theta_1^F,\theta_2^F\right)\\
= & f_x\left(\theta_1^F,\theta_2^F\right)\left(t-\theta_1^F\right)+f_y\left(\theta_1^F,\theta_2^F\right)\left(s-\theta_2^F\right)+\frac{1}{2}\left\{f_{xx}(c_1,c_2)\left(t-\theta_1^F\right)^2\right.\\
& \left.+2f_{xy}(c_1,c_2)\left(t-\theta_1^F\right)\left(s-\theta_2^F\right)+f_{yy}(c_1,c_2)\left(s-\theta_2^F\right)^2\right\},
\end{aligned}$$

where (c_1,c_2) is a point on the line connecting $\left(\theta_1^F,\theta_2^F\right)$ and (t,s). Taking into account of the fact that $F(\mathbf{1})=1$ and (5), one has

$$\begin{aligned}
& F(f)-f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\\
= & f_x\left(\theta_1^F,\theta_2^F\right)\left(F(e_{1,0})-\theta_1^F F(\mathbf{1})\right)-f_y\left(\theta_1^F,\theta_2^F\right)\left(F(e_{0,1})-\theta_2^F F(\mathbf{1})\right)\\
& +\frac{1}{2}\left\{f_{xx}(c_1,c_2)\mu_{2,0}^F+2f_{xy}(c_1,c_2)\mu_{1,1}^F+f_{yy}(c_1,c_2)\mu_{0,2}^F\right\}. \qquad (8)
\end{aligned}$$

Using the facts

$$|f_{xx}(c_1,c_2)| \leq M_f\left(1+\left(\theta_1^F\right)^2+\left(\theta_2^F\right)^2\right),$$

$$|f_{xy}(c_1,c_2)| \leq M_f\left(1+\left(\theta_1^F\right)^2+\left(\theta_2^F\right)^2\right),$$

and

$$|f_{yy}(c_1,c_2)| \leq M_f\left(1+\left(\theta_1^F\right)^2+\left(\theta_2^F\right)^2\right),$$

and since

$$2\mu_{1,1}^{F} \leq \mu_{2,0}^{F} + \mu_{0,2}^{F},$$

from (8) we get

$$\begin{aligned} \left|F(f) - f\left(\theta_1^F, \theta_2^F\right)\right| &\leq \frac{1}{2} M_f \left(1 + \left(\theta_1^F\right)^2 + \left(\theta_2^F\right)^2\right) \left\{\mu_{2,0}^F + 2\mu_{1,1}^F + \mu_{0,2}^F\right\} \\ &\leq M_f \left(1 + \left(\theta_1^F\right)^2 + \left(\theta_2^F\right)^2\right) \left[\mu_{2,0}^F + \mu_{0,2}^F\right]. \end{aligned}$$

□

3. Difference of Bivariate Positive Linear Operators

In this section, we will give estimates for the difference of bivariate positive linear operators, on unbounded set $\mathcal{D}$, in terms of weighted modulus of continuity. Let $\mathbb{K}$ be a set of non-negative integers and consider a family of functions $p_{k,l} : \mathcal{D} \to \mathcal{D}$, $k, l \in \mathbb{K}$. We consider discrete operators given by

$$U(f; x, y) = \sum_{k,l \in \mathbb{K}} F_{k,l}(f)\, p_{k,l}(x, y) \ \ , \ V(f; x, y) = \sum_{k,l \in \mathbb{K}} G_{k,l}(f)\, p_{k,l}(x, y),$$

where $\sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) = 1,\ F_{k,l}, G_{k,l} : \mathbb{D} \to \mathbb{R}$ are positive linear functionals such that $F_{k,l}(\mathbf{1}) = 1,\ G_{k,l}(\mathbf{1}) = 1$. U and V are positive linear operators such that $U, V : \mathbb{D} \to B_\rho(\mathcal{D})$.

Theorem 1. *Let $f \in C_\rho^*(\mathcal{D})$ with all of its partial derivatives of order≤ 2 belong to the space $C_\rho(\mathcal{D})$. Then we have*

$$|(U - V)(f; x, y)| \leq \delta_1 + \delta_2 + 2^8 \Omega_\rho(f, \delta_3, \delta_4) \left(1 + \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y)\, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right)\right),$$

where

$$\delta_1 := M_f \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y)\, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) \left[\mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}\right],$$

$$\delta_2 := M_f \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y)\, \rho\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right) \left[\mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}\right],$$

$$\delta_3^4 = \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y)\, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) \left(\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right)^4,$$

and

$$\delta_4^4 = \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y)\, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) \left(\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right)^4.$$

Proof. We can write

$$\begin{aligned}
\left|(U-V)(f;x,y)\right| &= \Bigg|\sum_{k,l\in\mathbb{K}}\Big\{F_{k,l}(f)-G_{k,l}(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)+f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\\
&\quad -f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)+f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\Big\}p_{k,l}(x,y)\Bigg|\\
&\leq \sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\Big\{\left|F_{k,l}(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right|\\
&\quad +\left|G_{k,l}(f)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\\
&\quad +\left|f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\Big\}.
\end{aligned}$$

Using Lemma 2, (5), (6) and (7), we get

$$\left|F(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right|\leq M_f\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left\{\mu_{2,0}^{F_{k,l}}+2\mu_{1,1}^{F_{k,l}}+\mu_{0,2}^{F_{k,l}}\right\}$$

and

$$\begin{aligned}
&\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left|F_{k,l}(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right|\\
\leq\ & M_f\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left[\mu_{2,0}^{F_{k,l}}+\mu_{0,2}^{F_{k,l}}\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left|G_{k,l}(f)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\\
\leq\ & M_f\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\rho\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\left[\mu_{2,0}^{G_{k,l}}+\mu_{0,2}^{G_{k,l}}\right].
\end{aligned}$$

Using Lemma 1, we get

$$\begin{aligned}
&\left|f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\\
\leq\ & 2^8\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\Omega_\rho(f,\delta_3,\delta_4)\\
&\times\left(1+\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\right)\left(1+\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right)\\
\leq\ & 2^8\Omega_\rho(f,\delta_3,\delta_4)\left\{\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)+\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\right.
\end{aligned}$$

$$+\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}$$
$$\left.+\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right\}$$

and we can write

$$\begin{aligned}
& \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left|f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right| \\
\leq\; & 2^8\Omega_\rho(f,\delta_3,\delta_4)\left\{\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right. \\
& +\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4} \\
& +\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4} \\
+\; & \left.\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right\} \\
=\; & 2^8\Omega_\rho(f,\delta_3,\delta_4)\left\{A_{0,0}+A_{1,0}+A_{0,1}+A_{1,1}\right\},
\end{aligned}$$

where

$$\begin{aligned}
A_{i,j} &= q_{k,l}(x,y)\left[\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\right]^i\left[\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right]^j ;0\leq i,j\leq 1 \\
q_{k,l}(x,y) &= \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right).
\end{aligned}$$

Choosing

$$\delta_3^4=\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4$$

and

$$\delta_4^4=\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4,$$

we reach to the desired result. □

4. Estimate via K-functional

In this section, we give an estimate for the difference of bivariate positive linear operators; in terms of K-functional. For this aim, we firstly recall the definition of K-functional. Let $C_B^2(\mathcal{D}) = \left\{ f \in C_B(\mathcal{D}) ; f^{(p,q)} \in C_B(\mathcal{D}), 1 \leq p, q \leq 2 \right\}$ where $f^{(p,q)}$ is (p,q) th-order partial derivative with respect to x, y of f, equipped with the norm

$$\|f\|_{C_B^2(\mathcal{D})} = \|f\|_{C_B(\mathcal{D})} + \sum_{i=1}^{2} \left\| \frac{\partial^i f}{\partial x^i} \right\|_{C_B(\mathcal{D})} + \sum_{i=1}^{2} \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C_B(\mathcal{D})}.$$

The Peetre K-functional of the function $f \in C_B(\mathcal{D})$ is given by

$$K(f;\delta) = \inf_{g \in C_B^2(\mathcal{D})} \left\{ \|f - g\|_{C_B(\mathcal{D})} + \delta \|g\|_{C_B^2(\mathcal{D})},\ \delta > 0 \right\}.$$

It is known that there is a connection between the second order modulus of smoothness and Peetre's K-functional for all $\delta > 0$ as follows (see [9, p.192] or [7]):

$$K(f;\delta) \leq C_0 \left\{ \omega_2\left(f; \sqrt{\delta}\right) + \min(1,\delta) \|f\|_{C_B(\mathcal{D})} \right\}.$$

Here, the constant C_0 is independent of δ and f, and 2nd order modulus of smoothness of f is a function $\omega_2 : C_B(\mathcal{D}) \times (0,\infty) \to [0,\infty)$ given by

$$\omega_2(f,\delta) = \sup_{0<\|h\|\leq \delta} \sup_{x \in \mathcal{D}} \Delta_h^2 f(x), \quad \delta > 0,$$

where $\|.\|$ is the Euclidean norm in $\mathbb{R}^2$ and $\Delta_h^2 f$ is the 2nd order difference on $\mathcal{D}$ given by

$$\Delta_h^2 f(x) = \sum_{k=0}^{2} (-1)^{2-k} \binom{2}{k} f(x + kh),\ x \in \mathcal{D},\ h \in \mathcal{D}.$$

Now, assume that $C_B^2(\mathcal{D}) \subset \mathbb{D}$, where, as it is mentioned in page 3, $\mathbb{D}$ is the linear subspace of $C(\mathcal{D})$ containing $C_\rho(\mathcal{D})$.

Lemma 3. *Let $f \in \mathbb{D} \cap C_B(\mathcal{D})$. Then*

$$\left| F(f) - f\left(\theta_1^F, \theta_2^F\right) \right| \leq 2K\left(f; \frac{1}{4}\left[\mu_{2,0}^F + \mu_{0,2}^F\right]\right).$$

Proof. Let $g(x,y) \in C_B^2(\mathcal{D})$ and $(t,s) \in \mathcal{D}$. Using Taylor's expansion [8], we have

$$\begin{aligned} g(t,s) - g(x,y) &= \frac{\partial g(x,y)}{\partial x}(t-x) + \frac{\partial g(x,y)}{\partial y}(s-y) \\ &\quad + \int_x^t (t-u) \frac{\partial^2 g(u,y)}{\partial u^2} du + \int_y^s (s-v) \frac{\partial^2 g(x,v)}{\partial v^2} dv. \end{aligned}$$

Application of the functional F on both sides of the last formula gives

$$\left| F(g) - g\left(\theta_1^F, \theta_2^F\right) F(\mathbf{1}) \right|$$

$$
\begin{aligned}
\leq\ & \left|g_x\left(\theta_1^F,\theta_2^F\right)\left(F\left(e_{1,0}\right)-\theta_1^F F\left(\mathbf{1}\right)\right)\right|+\left|g_y\left(\theta_1^F,\theta_2^F\right)\left(F\left(e_{0,1}\right)-\theta_2^F F\left(\mathbf{1}\right)\right)\right| \\
& F\left(\left|\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}\right| du;x,y\right)+F\left(\left|\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv\right|;x,y\right) \\
\leq\ & \frac{1}{2}\left\{\|g_{xx}\|_{C_B(\mathcal{D})}\left(F\left(e_{1,0}\right)-\theta_1^F F\left(\mathbf{1}\right)\right)^2+\|g_{yy}\|_{C_B(\mathcal{D})}\left(F\left(e_{0,1}\right)-\theta_2^F F\left(\mathbf{1}\right)\right)^2\right\}.
\end{aligned}
$$

Taking into account of $F(\mathbf{1})=1$, (4), (5) and (6), we get

$$
\left|F(g)-g\left(\theta_1^F,\theta_2^F\right)\right|\leq\frac{1}{2}\left\{\|g_{xx}\|_{C_B(\mathcal{D})}\mu_{2,0}^F+\|g_{yy}\|_{C_B(\mathcal{D})}\mu_{0,2}^F\right\}.
$$

Now, let $f\in\mathbb{D}\cap C_B(\mathcal{D})$ and $(t,s)\in\mathcal{D}$, then we have

$$
\begin{aligned}
& \left|F(f;x,y)-f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
=\ & \left|F(f-g+g;x,y)-f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})+g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
=\ & \left|F(f-g;x,y)+F(g;x,y)-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right. \\
-\ & \left. f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})+g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
\leq\ & \left|F(f-g;x,y)\right|+\left|F(g;x,y)-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
+\ & \left|f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
\leq\ & 2\|f-g\|_{C_B(\mathcal{D})}+\frac{1}{2}\left\{\|g_{xx}\|_{C_B(\mathcal{D})}\mu_{2,0}^F+\|g_{yy}\|_{C_B(\mathcal{D})}\mu_{0,2}^F\right\} \\
\leq\ & 2\|f-g\|_{C_B(\mathcal{D})}+\frac{1}{2}\|g\|_{C_B^2(\mathcal{D})}\left[\mu_{2,0}^F+\mu_{0,2}^F\right].
\end{aligned}
$$

Therefore, taking the infimum on the right hand side over all $g\in C_B^2(D)$

$$
\begin{aligned}
\left|F(f;x,y)-f\left(\theta_1^F,\theta_2^F\right)\right| &\leq \inf_{g\in C_B^2(D)}\left\{2\|f-g\|_{C_B(\mathcal{D})}+\frac{1}{2}\|g\|_{C_B^2(\mathcal{D})}\left[\mu_{2,0}^F+\mu_{0,2}^F\right]\right\} \\
&= 2K\left(f;\frac{1}{4}\left[\mu_{2,0}^F+\mu_{0,2}^F\right]\right).
\end{aligned}
$$

□

Now, the following theorem can be given.

Theorem 2. *Let $f\in\mathbb{D}\cap C_B(\mathcal{D})$ with all of its first order partial derivatives belong to $C_B(\mathcal{D})$. Then*

$$
|(U-V)(f;x,y)|\leq 4K\left(f,\frac{1}{8}\eta(x,y)\right)+M_f'\mu(x,y),
$$

where $M_f' := \max\left\{\|f_x\|_{C_B(\mathcal{D})}, \|f_x\|_{C_B(\mathcal{D})}\right\}$,

$$\eta(x,y) := \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left(\lambda_{F_{k,l}} + \lambda_{G_{k,l}}\right),$$

with $\lambda_{F_{k,l}} := \mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}$, $\lambda_{G_{k,l}} := \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}$ *and*

$$\mu(x,y) = \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right| + \left|\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right|\right\}.$$

Proof. By the hypothesis, f is differentiable on the line connecting the points $\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right)$ and $\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right)$. From the mean value theorem for function of two variables (see, e.g., [7]), there is a point (c_1, c_2) on this line such that

$$f\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) - f\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right) = f_x(c_1,c_2)\left(\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right) + f_y(c_1,c_2)\left(\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right)$$

holds. For $f \in \mathbb{D} \cap C_B(\mathcal{D})$, using Lemma 3, and the above formula, we have

$$\begin{aligned}
&|(U-V)(f;x,y)| \\
\leq\ & \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left|F_{k,l}(f) - G_{k,l}(f)\right| \\
\leq\ & \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|F_{k,l}(f) - f\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right)\right| + \left|G_{k,l}(f) - f\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right)\right|\right. \\
& \left.\left|f_x(c_1,c_2)\left(\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right) + f_y(c_1,c_2)\left(\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right)\right|\right\} \\
\leq\ & 2\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{K\left(f;\frac{1}{4}\left[\mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}\right]\right) + K\left(f;\frac{1}{4}\left[\mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}\right]\right)\right\} \\
& + \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\|f_x\|_{C_B(\mathcal{D})}\left|\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right| + \|f_y\|_{C_B(\mathcal{D})}\left|\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right|\right\} \\
=\ & 2\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{K\left(f;\frac{1}{4}\lambda_{F_{k,l}}\right) + K\left(f;\frac{1}{4}\lambda_{G_{k,l}}\right)\right\} \\
& + K_f\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right| + \left|\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right|\right\},
\end{aligned}$$

where we denote

$\lambda_{F_{k,l}} := \mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}$, $\lambda_{G_{k,l}} := \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}$ and $M_f' := \max\left\{\|f_x\|_{C_B(\mathcal{D})}, \|f_x\|_{C_B(\mathcal{D})}\right\}$

From the definition of K-functional, for a fixed $g \in C_B^2(\mathcal{D})$, we can write

$$|(U-V)(f;x,y)| \leq 4\|f-g\|_{C(\mathcal{D})}\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)$$

$$
\begin{aligned}
& +\frac{1}{2}\|g\|_{C^2(\mathcal{D})} \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left(\lambda_{F_{k,l}}+\lambda_{G_{k,l}}\right) \\
& +M'_f \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right|+\left|\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right|\right\} \\
= \; & 4K\left(f,\frac{1}{8}\eta(x,y)\right)+M'_f\mu(x,y),
\end{aligned}
$$

where

$$\eta(x,y) := \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left(\lambda_{F_{k,l}}+\lambda_{G_{k,l}}\right)$$

and

$$\mu(x,y) = \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right|+\left|\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right|\right\}.$$

□

Note that using (9), from the above theorem we obtain

$$|(U-V)(f;x,y)| \le C_0\left\{\omega_2\left(f;\sqrt{\frac{1}{8}\eta(x,y)}\right)+\min(1,\lambda)\|f\|_{C_B(\mathcal{D})}\right\}+M'_f\mu(x,y).$$

5. Difference for Chebishev Functionals

For $f,g\in C_\rho$, we take the bivariate positive linear operators U and V defined at the beginning of this section. Assuming that $f,g,\ fg\in C_\rho\ (\mathcal{D})$, we consider the Chebishev functional of U given by $T^U(f,g) := U(fg)-U(f)U(g)$ (similarly for V) (see [5] and references therein). In this part, we give an upper estimate related to the difference $\left|T^U(f,g)-T^V(f,g)\right|$.

Theorem 3. *Let the functions f,g and fg belong to $C_\rho^*(\mathcal{D})$ and all of their partial derivatives of order ≤ 2 belong to $C_\rho(\mathcal{D})$. If*

$$
\begin{aligned}
\theta_1^{F_{k,l}} = \theta_1^{G_{k,l}} = \theta_1,\ \theta_2^{F_{k,l}} = \theta_2^{G_{k,l}} = \theta_2\ , \\
U\left(1+(e_{1,0})^2+(e_{0,1})^2;x,y\right) \le M\rho(x,y)
\end{aligned}
$$

and

$$V\left(1+(e_{1,0})^2+(e_{0,1})^2;x,y\right) \le M\ \rho(x,y),$$

then we have

$$
\begin{aligned}
& \left|T^U(f,g;x,y)-T^V(f,g;x,y)\right| \\
\le \; & (\delta_1+\delta_2)\left[1+M\rho(x,y)\left(\|f\|_\rho+\|g\|_\rho\right)\right]+2^8\left[1+q_{k,l}(x,y)\right] \\
& \times\left\{\Omega_\rho(fg,\delta_3,\delta_4)+M\rho(x,y)\left(\|f\|_\rho\Omega_\rho(g,\delta_3,\delta_4)+\|g\|_\rho\Omega_\rho(f,\delta_3,\delta_4)\right)\right\},
\end{aligned}
$$

where δ_1 and δ_2 are the same as in Theorem 1 and

$$q_{k,l}(x,y)=\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right).$$

Proof. From the definition of Chebyshev functionals, we can write

$$\begin{aligned}
& T^U(f,g;x,y)-T^V(f,g;x,y)\\
=\ & U(fg;x,y)-U(f;x,y)U(g;x,y)-V(fg;x,y)+V(f;x,y)V(g;x,y)\\
=\ & U(fg;x,y)-U(f;x,y)U(g;x,y)-U(f;x,y)V(g;x,y)+U(f;x,y)V(g;x,y)\\
& -V(fg;x,y)+V(f;x,y)V(g;x,y)\\
=\ & U(fg;x,y)-V(fg;x,y)-U(f;x,y)\left[U(g;x,y)-V(g;x,y)\right]\\
& -V(g;x,y)\left[U(f;x,y)-V(f;x,y)\right].
\end{aligned}$$

By taking absolute value of both sides we obtain

$$\begin{aligned}
& \left|T^U(f,g;x,y)-T^V(f,g;x,y)\right|\\
\leq\ & \left|U(fg;x,y)-V(fg;x,y)\right|+\left|U(f;x,y)\right|\left|U(g;x,y)-V(g;x,y)\right|\\
& +\left|V(g;x,y)\right|\left|U(f;x,y)-V(f;x,y)\right|.
\end{aligned}$$

From Theorem 1 we have

$$\begin{aligned}
& \left|U(fg;x,y)-V(fg;x,y)\right|\\
\leq\ & \sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left|F_{k,l}(fg;x,y)-G_{k,l}(fg;x,y)\right|\\
\leq\ & \delta_1+\delta_2+2^8\Omega_\rho(fg,\delta_3,\delta_4)\left(1+q_{k,l}(x,y)\right)
\end{aligned}$$

and

$$\begin{aligned}
& \left|U(f;x,y)\right|\left|U(g;x,y)-V(g;x,y)\right|\\
\leq\ & M\rho(x,y)\left\|f\right\|_\rho\left[\delta_1+\delta_2+2^8\Omega_\rho(g,\delta_3,\delta_4)\left(1+q_{k,l}(x,y)\right)\right]
\end{aligned}$$

$$\begin{aligned}
& \left|V(g;x,y)\right|\left[U(f;x,y)-V(f;x,y)\right]\\
\leq\ & M\rho(x,y)\left\|g\right\|_\rho\left[\delta_1+\delta_2+2^8\Omega_\rho(f,\delta_3,\delta_4)\left(1+q_{k,l}(x,y)\right)\right].
\end{aligned}$$

If necessary arrangements are made, the proof is completed. □

6. Application

If we take the well-known bivariate Szász operator as the operator U and the bivariate Szász-Kantorovich as the operator V given, respectively, by

$$U_{n,m}(f;x,y)=\sum_{k,l=0}^{\infty}e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}f\left(\frac{k}{n},\frac{l}{m}\right)$$

and

$$V_{n,m}(f;x,y)=\sum_{k,l=0}^{\infty}e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}nm\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_{\frac{l}{m}}^{\frac{l+1}{m}}f(t,s)\,dsdt.$$

Theorem 4. *Let $f\in C_\rho^*(\mathcal{D})$ with all of its partial derivatives of order ≤ 2 belong to the space $C_\rho(\mathcal{D})$. Then we have*

$$|(U-V)(f;x,y)|\leq\delta_2+2^8\Omega_\rho(f,\delta_3,\delta_4)\,\psi(x,y),$$

where

$$\delta_2(x,y)=\left\{1+\frac{(1+8nx+4nx^2)}{4n^2}+\frac{(1+8my+4my^2)}{4m^2}\right\}\left\{\frac{1}{3n^2}+\frac{1}{3m^2}\right\},$$

$$\delta_3^4(x,y)=\frac{1}{16n^2}+\frac{nx+4nx^2}{16n^4}+\frac{my+4my^2}{16n^2m^2},$$

$$\delta_4^4(x,y)=\frac{1}{16m^2}+\frac{nx+4nx^2}{16n^2m^2}+\frac{my+4my^2}{16m^4}$$

and

$$\psi(x,y)=2+x^2+y^2+\frac{x}{n}+\frac{y}{m}.$$

Proof. We use Theorem. By making simple calculations for the operators U and V given above, we have

$$F_{k,l}(f)=f\left(\frac{k}{n},\frac{l}{m}\right),$$

$$\theta_1^F=F_{k,l}(e_{1,0})=\frac{k}{n},\qquad\theta_2^F=\frac{l}{m},$$

$$G_{k,l}(f)=nm\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_{\frac{l}{m}}^{\frac{l+1}{m}}f(t,s)\,dsdt,$$

$$\theta_1^G=G_{k,l}(e_{1,0})=\frac{1}{2n}(2k+1),\qquad\theta_2^F=\frac{1}{2m}(2l+1),$$

$$\mu_{2,0}^F=F_{k,l}\left(\left(e_{1,0}-\frac{k}{n}\right)^2\right)=0,\quad\mu_{0,2}^F=F_{k,l}\left(\left(e_{0,1}-\frac{l}{m}\right)^2\right)=0,$$

$$\mu_{2,0}^G=G_{k,l}\left(\left(e_{1,0}-\frac{k}{n}\right)^2\right)=\frac{1}{3n^2},\ \mu_{0,2}^G=G_{k,l}\left(\left(e_{0,1}-\frac{l}{m}\right)^2\right)=\frac{1}{3m^2}.$$

Therefore, we get

$$\begin{aligned}\delta_1(x,y)&=0,\\ \delta_2(x,y)&=\sum_{k,l}^{\infty}e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left\{\left(1+\frac{(2k+1)^2}{4n^2}+\frac{(2l+1)^2}{4m^2}\right)\left\{\frac{1}{3n^2}+\frac{1}{4mn}+\frac{1}{3m^2}\right\}\right\}\\ &=\left\{1+\frac{(1+8nx+4nx^2)}{4n^2}+\frac{(1+8my+4my^2)}{4m^2}\right\}\left\{\frac{1}{3n^2}+\frac{1}{4mn}+\frac{1}{3m^2}\right\},\end{aligned}$$

$$\begin{aligned}
\delta_3^4 &= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4 \\
&= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\rho\left(\frac{k}{n},\frac{l}{m}\right)\left(\frac{k}{n}-\frac{1}{2n}(2k+1)\right)^4 \\
&= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left(1+\frac{k^2}{n^2}+\frac{l^2}{m^2}\right)\left(\frac{1}{2n}\right)^4 \\
&= \frac{1}{16n^2}+\frac{nx+4nx^2}{16n^4}+\frac{my+4my^2}{16n^2m^2}
\end{aligned}$$

and

$$\begin{aligned}
\delta_4^4 &= \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4 \\
&= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left(1+\frac{k^2}{n^2}+\frac{l^2}{m^2}\right)\left(\frac{1}{2m}\right)^4 \\
&= \frac{1}{16m^2}+\frac{nx+4nx^2}{16n^2m^2}+\frac{my+4my^2}{16m^4}.
\end{aligned}$$

$$\begin{aligned}
\psi(x,y) &= 1+\sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\rho\left(\frac{k}{n},\frac{l}{m}\right) \\
&= 1+\sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left(1+\frac{k^2}{n^2}+\frac{l^2}{m^2}\right) \\
&= 2+x^2+y^2+\frac{x}{n}+\frac{y}{m}.
\end{aligned}$$

This completes the proof. □

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