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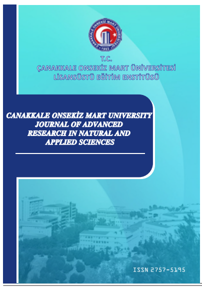

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Approximate Solutions of the Fourth-Order Eigenvalue Problem

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Abstract − In this paper, the differential transformation (DTM) and the Adomian decomposition (ADM) methods are proposed for solving fourth order eigenvalue problem. This fourth order eigenvalue problem has nonstrongly regular boundary conditions. This the fourth order problem has been examined for *p(t) = t, B = 0, a = 0,01* where *p(t)* ≠ 0 is a complex valued and *a* ≠ 0. The differential transformation and the Adomian decomposition methods are briefly described. An approximate solution is obtained by performing seven iterations with the Adomian decomposition method. The same number of iterations have been made in the differential transformation method. The approximation results obtained by both methods have been compared with each other. These data have been presented in table. The ADM and the DTM approximation solutions have been shown by plotting in Figure 1. Here, the approaches obtained by using the two methods are found to be in high agreement. Consequently, highly accurate approximate solutions of fourth order eigenvalue problem are obtained. Such good results also revealed that the Adomian decomposition and the differential transformation methods are fast, economical and motivating. The exact solution of the fourth order eigenvalue problem for nonstrongly regular can not be found in the literature. Therefore, this study will give an important idea to determine approximate solution behavior of this fourth order problem.

Keywords − *Adomian decomposition method, approximate solutions, differential transform method, fourth order eigenvalue problem, nonstrongly regular boundary conditions*

1. Introduction

We examine the problem with nonstrongly regular boundary conditions [1] as

$$u^{(4)} + p(t)u = \lambda u, \ 0<t<1, \tag{1}$$

$$u(1) - (-1)^{\beta} u(0) = 0, \ \ u(1) - (-1)^{\beta} u'(0) = 0, \tag{2}$$

$$u''(1) - (-1)^{\beta} u''(0) = 0, \ \ u'''(1) - (-1)^{\beta} u'''(0) + \alpha u(0) = 0, \tag{3}$$

where λ is spectral parameter; *p(t)* ≠ 0 is a complex valued function; *a* ≠ 0 and *b* = 0,1. We deal with DTM and ADM to solve the above problem at *b* = 0.

Many numerical methods, such as asymptotic formula for eigenfunctions of the considered boundary value problem have been obtained in (Kaya, 2020), the regularized sampling method (Chanane, 2010), the extended sampling method (Chanane, 2010), the α-parameterized differential transform method (Mukhtarov, Yucel, & Aydemir, 2020) variational iteration methods (Syam, & Siyyam, 2009), Sinc-Galerkin method (Alquran, Al-Khaled , 2010), fourth order sturm liouville problem via decomposition method for *p(t)* ≠ 0 (Attili, & Lesnic, 2006), Magnus Method (Alalyani, 2019), differential transform method for high order

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Sturm-Liouville problems (Biazar, Dehghan, & Houlari, 2020), lie group method, FDM and the asymptotic iteration method etc. are implemented to solve this eigenvalue BVP numerically.

Some different authors have worked on the development of numerical methods for solving these differential equations (Gao, Ismael, & Husien, 2020; Baskonus, Sulaiman, & Bulut, 2018). We use ADM and DTM to compare the approximation solutions of the problem that we have suggested as a contribution to the literature.

This paper will continue as follows: In part 2 we give the basic process of the ADM and DTM. In part 3 we present the implementation of the ADM and DTM for computing and comparing the solutions of this eigenvalue problem. The figure of eigenfunctions (approximate solutions) for a found λ (eigenvalue) is plotted. Numerical results are shown in the table.

2. Basic Process of ADM and DTM

Here we will briefly introduce the ADM and DTM as follows:

In the beginning of the 1980s, ADM has been developed by Adomian (Adomian, & Rach, 1993). In these years, the Adomian decomposition method has been implemented for problems arising from physics, biology and engineering. Until now, there has been great interest in DTM and ADM applications to solve various scientific models, you can refer to the references (Adomian et al., 1993; Zhou, 1986; Ayaz, 2004; Abdel-Halim Hassan, 2002; Li et al., 2020; Chakraverty et al., 2019; Adebısı et al., 2021; ***Çakır et al., 2019***; Arslan, 2019a; Arslan, 2018b; Peker et al., 2011; Gubes et al., 2015; Peker et al., 2010).

The equation (1) is rewritten as

$$Fu = g \Rightarrow Lu + Ru + Nu = g(t). \tag{4}$$

F and L are differential operator and fourth order derivative in Equation (4), respectively.

R and N are linear and nonlinear terms in Equation (4), respectively. If the integral operator is applied to each term of Equation (4), we get

$$u(t) = \left(L^{-1}R\right)u - \left(L^{-1}N\right)u + L^{-1}\left(g(t)\right), \tag{5}$$

where $L(\ldots) = \dfrac{d^{(4)}}{dt^{(4)}}(\ldots)$ is the differential operator and $L^{-1}(\ldots) = \int_0^t\int_0^t\int_0^t\int_0^t(\ldots)\,dtdtdtdt$ is integral operator (inverse operator) of L. If we operate on both sides of Equation (5) with the inverse operator of L^{-1}, we obtain

$$u(t) = u_0(t) - \left(L^{-1}R\right)u - \left(L^{-1}N\right)u. \tag{6}$$

After some calculations, the following iteration system is written:

$$u_{k+1}(t) = u_0(t) - \left(L^{-1}R\right)u_k - \left(L^{-1}N\right)u_k, \quad k = 0,1,2, \tag{7}$$

$$u(t) = \sum_{k=0}^{\infty} u_k(t) = u_0 + L^{-1}(Ru) + L^{-1}(Nu), \tag{8}$$

where

$$Ru = \sum_{k=0}^{\infty} u_k(t), \;\; N\mathrm{u} = \sum_{k=0}^{\infty} A_k(u_0 + u_1 + \ldots + u_k).$$

The first approximation $u_0(t)$ can be obtained by using boundary conditions. We have recurrence formula Equation (7) for obtaining other components $u_1(t)$, $u_2(t)$ of the Adomian decomposition Equation (8). Finally, we have the serial solution of problem Equation (1).

$$u(t) = u_0(t) + u_1(t) + \ldots. \tag{9}$$

The DTM is effective in solving most differential equations. The DTM is derived based on the Taylor expansion and was proposed by Zhou for electrical circuits (Zhou, 1986).

The differential transformation $Y(k)$ of function $u(t)$ is defined as (Ayaz, 2004),

$$Y(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}, \tag{10}$$

where $u(t)$is original function and $Y(k)$ is the transformed function.

Differential inverse transform $u(t)$ of $Y(k)$ is defined as (Ayaz, 2004),

$$u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}. \tag{11}$$

If the expansion Equation (11) with Equation (10) is written as follows:

$$u(t) = \sum_{k=0}^{\infty} Y(k) t^k, \tag{12}$$

then it is called series solution of the differential transformation method (Ayaz, 2004).

The following theorems will be used in this study, where $Y(k)$ is differential transformation of $u(t)$ (Ayaz, 2004):

Teorem 1. $u(m) = \dfrac{d^4 w(m)}{dm^4}, \;\; U(h) = \dfrac{(h+4)!}{h!} W(h+4).$

Theorem 2. $u(m) = \alpha w(m), \;\; U(h) = \alpha W(h),$ where α is a reel constant.

Theorem 3. $u(m) = m w(m), \;\; U(h) = \sum_{s=0}^{h} \delta(h-1) W(h-s).$

3. Approximation Solutions by ADM and DTM

By applying ADM and DTM, we will find approximation solutions of eigenvalue problem in series form. The advantages and benefits of the proposed methods on an experiment will be presented.

$$
\begin{aligned}
&u^{(4)} + tu = \lambda u,\ 0<t<1,\\
&u(1) - u(0) = 0,\ u'(1) - u'(0) = 0,\\
&u''(1) - u''(0) = 0,\ u'''(1) - u'''(0) + 0.01u(0) = 0,
\end{aligned} \tag{13}
$$

it is taken as *b = 0, a = 0,01* in this problem.

ADM solution as follows:

$$Lu^{(4)} = \lambda u - .$$

According to Equation (6), $L^{-1}\left(u^{(4)}\right) = L^{-1}(\lambda u) - L^{-1}(tu)$,

$$
\begin{aligned}
&u(t) = A + Bt + \frac{Ct^2}{2} + (D + \alpha A)\frac{t^3}{6} + L^{-1}(\lambda u) - L^{-1}(tu),\\
&u_{k+1} = A + Bt + \frac{Ct^2}{2} + D\frac{t^3}{6} + \alpha A\frac{t^3}{6} + L^{-1}(\lambda u_k - tu_k),\ k = 0,1,2,...,\\
&u_0 = A + Bt + \frac{Ct^2}{2} + (D + \alpha A)\frac{t^3}{6},\ k = 0,
\end{aligned} \tag{14}
$$

u_0 is obtained with the aid of the following boundary conditions *A, B, C, D.*

$$
\begin{aligned}
&u(0) = A,\ u(1) = A,\ u'(0) = B,\ u'(1) = B,\\
&u''(0) = C,\ u''(1) = C,\ u'''(0) = u'''(1) + 0.01u(0) = D.
\end{aligned}
$$

From the recursive relation Equation (14) for k = 0, 1, 2…, we get

$$
\begin{aligned}
&u_0 = 1.000003877 + 0.001412025487t - 0.0004168111954t^2,\\
&u_1 = L^{-1}(\lambda u_0 - tu_0),\\
&u_2 = L^{-1}(\lambda u_1 - tu_1),\\
&u_3 = L^{-1}(\lambda u_2 - tu_2),
\end{aligned}
$$

The solution *u(t)* found by the Adomian decomposition method with seven iterations is obtained as a series and the formula used to normalize the solution (normalized eigenfunction) *u(t)* is as:

$$u(t) = \left(\int_0^1 |u(t)|\,dt\right)^{-1} u(t),$$

with this formula and the ADM method, λ and u_{ADM} are as

$\lambda = 0.4899667963$,

$$
\begin{aligned}
u_{ADM}(t) &= \sum_{k=0}^{\infty} u_k(t)\\
&\approx 1.000003877 + 0.001412025487t - 0.0004168111954t^2\\
&- 0.0130556804t^3 + 0.02040194953t^4 - 0.008318308894t^5\\
&- 0.000007085980934t^6 - 0.000007119093639t^7.
\end{aligned} \tag{15}
$$

DTM solution as follows:

Applying DTM on Equation (14), we reach the following iteration system

$$\delta(s-1)=\begin{cases}1, & s=1,\\ 0, & s\neq 1,\end{cases}$$

$$Y(k+4)=\frac{\lambda Y(k)-\sum_{r=0}^{h}\delta(s-1)Y(k-s)}{(k+1)(k+2)(k+3)(k+4)},\quad k=0,1,2,...,10.$$

Using above recurrence relation and boundary conditions, the following series coefficients *Y(k)* is obtained

$$\lambda = 0.4899666546,$$
$$Y(0)=a,\ \text{ a is real constant},$$
$$Y(1)=0.001388866321a,$$
$$Y(2)=\frac{-0.0008336125306a}{2},$$
$$Y(3)=\frac{-0.08833244692a+0.01a}{6},$$
$$Y(4)=\frac{\lambda a}{24}.$$

Utilizing above calculations *Y(k)* and using Equation (12), we obtain the approximation solutions of the problem Equation (13) with seven iterations.

The formula used to normalize the solution (normalized eigenfunction) is as follows,

$$u(t)=u(t)\left(\int_0^1 |u(t)|\,dt\right)^{-1}.$$

By above formula and DTM, we find the following λ and normalized function u_{DMT}

$$\lambda = 0.4899666546,$$

$$\begin{aligned}u_{DTM}(t) &= 1.000013900+0.001388885626t-0.0004168120589t^2\\ &-0.01305558929t^3+0.02041556105t^4-0.008327778270t^5\\ &-0.000004425304544t^6-0.000007119037321t^7.\end{aligned} \tag{16}$$

Table 1

Calculated results of the normalized eigenfunctions (u_{ADM} and u_{DMT}) Equation (15) and (16)

t	u_{ADM}	u_{DMT}	$\lvert u_{ADM} - u_{DMT} \rvert$
0.0	1.0000139000	1.0000038770	0.0000100230
0.1	1.0001375240	1.0001298130	0.0000077110
0.2	1.0002005600	1.0001951460	0.0000054140
0.3	1.0001856760	1.0001825050	0.0000031710
0.4	1.0001045390	1.0001035030	0.0000010360
0.5	0.9999877962	0.9999887352	0.0000000939
0.6	0.9998750542	0.9998777441	0.0000026899
0.7	0.9998048344	0.9998089895	0.0000041551
0.8	0.9998045015	0.9998097741	0.0000052726
0.9	0.9998801774	0.9998861621	0.0000059847
1.0	1.0000066240	1.0000128460	0.0000062220

The Table 1 shows the comparison of Equation (15) with Equation (16) for different values of t. Next, we plot these results in the Figure 1 to compare the ADM and DTM solutions. In conclusion, it was found that the results obtained by the two methods were in full agreement.

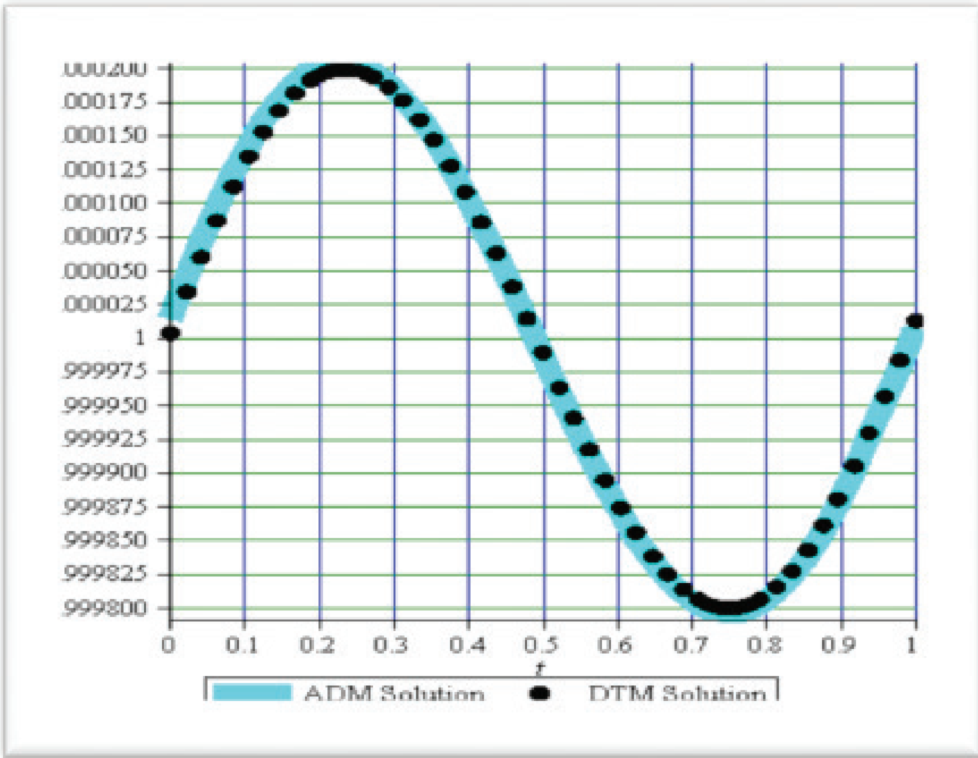

Figure 1. Comparison of ADM and DTM approximate solutions

4. Conclusion

We studied efficient and high accuracy methods for solving fourth order eigenvalue problem with nonstrongly regular boundary conditions. The solutions are very rapidly convergent by utilizing these methods. The numerical results are obtained by mathematics computer programe and are shown in table and figure. The numerical values in all tables and figures prove that we achieved an effective approximation.

Author Contributions

Author has all contributions to this article.

Conflicts of Interest

The author declares no conflict of interest.

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