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Several recurrence relations and identities on generalized derangement numbers

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Abstract

In the paper, with aid of generating functions, the authors present several recurrence relations and identities for generalized derangement numbers involving generalized harmonic numbers and the Stirling numbers of the first kind.

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1. Introduction

In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size n is called the derangement number. Derangement numbers can be generated by

$$\frac{\mathrm{e}^{-t}}{1-t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}$$

and possess the closed-form expression

$$d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

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See the monograph [3] and the papers [18, 20, 21, 23, 24, 25, 26].

The generalized derangement numbers $d_{n,r}$ are introduced by Munarini [15] as

$$d_{n,r}=\sum_{k=0}^{n}(-1)^k\binom{r+n-k}{n-k}\frac{n!}{k!}$$

and can be generated by

$$\frac{\mathrm{e}^{-t}}{(1-t)^{r+1}}=\sum_{n=0}^{\infty}d_{n,r}\frac{t^n}{n!}. \tag{1}$$

It is clear that $d_{n,0}=d_n$. The first few generalized derangement numbers $d_{n,r}$ are

$$d_{0,r}=1,\quad d_{1,r}=r,\quad d_{2,r}=r^2+r+1,\quad d_{3,r}=r^3+3r^2+5r+2.$$

Several types of extensions of derangement numbers have been offered and many of remarkable properties and identities have been achieved for them. See, for example, the papers [12, 13, 14, 30, 31, 32].

Harmonic numbers [9] are located in a very important position in combinatorial number theory. Various generalizations for these numbers have been introduced in the literature. One of them, due to [5], is denoted by $H(n,r,\alpha)$ and defined by

$$\sum_{n=0}^{\infty}H(n,r,\alpha)t^n=\frac{\left[-\ln\left(1-\frac{t}{\alpha}\right)\right]^{r+1}}{1-t}. \tag{2}$$

Employing generating functions, the authors of [5] established a number of combinatorial identities and relations for generalized harmonic numbers $H(n,r,\alpha)$ and some important special polynomials and numbers. Setting $r=0$ in (2) leads to another type of generalized harmonic numbers, which are given by

$$H_0(\alpha)=0\quad\text{and}\quad H_n(\alpha)=\sum_{i=1}^{n}\frac{1}{i\alpha^i}$$

for $n\in\mathbb{N}=\{1,2,\dots\}$ and can be generated by

$$\sum_{n=1}^{\infty}H_n(\alpha)t^n=\frac{-\ln\left(1-\frac{t}{\alpha}\right)}{1-t}.$$

Note that $H_n(1)=H_n$ denotes the classical harmonic numbers [6, 16]. Furthermore, a generalization for hyperharmonic numbers was presented in [17] as

$$H_{n,r}(\alpha)=\sum_{i=1}^{n}H_{i,r-1}(\alpha),\quad r,n\ge 1$$

subject to initial condition $H_{n,0}(\alpha)=\frac{1}{n\alpha^n}$, and can be generated by

$$\sum_{n=0}^{\infty}H_{n,r}(\alpha)t^n=\frac{-\ln\left(1-\frac{t}{\alpha}\right)}{(1-t)^r}. \tag{3}$$

For further investigations concerning with generalized harmonic numbers, the readers may consult with [1, 2, 4, 7, 10, 11, 27, 28] and references cited therein.

As another important number in combinatorial analysis, the Stirling numbers of the first kind can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1 \tag{4}$$

and satisfy the (diagonal) recurrence relations

$$\begin{aligned} s(n+1,k) &= s(n,k-1) - ns(n,k), \\ s(n,k) &= (-1)^k \sum_{m=1}^{n} (-1)^m \sum_{\ell=k-m}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell}{k-m} s(n-\ell,k-\ell) \\ &= (-1)^{n-k} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell,k-\ell), \end{aligned}$$

and

$$\frac{s(n+k,k)}{\binom{n+k}{k}} = \sum_{\ell=0}^{n} (-1)^\ell \frac{\langle k\rangle_\ell}{\ell!} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} \frac{s(n+m,m)}{\binom{n+m}{m}}, \tag{5}$$

where the conventions that $\binom{0}{0}=1$, $\binom{-1}{-1}=1$, and $\binom{p}{q}=0$ for $p \ge 0 > q$ are adopted and

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n=0 \end{cases}$$

is the falling factorial. See the book [8] and the papers [19, 22].

The classical Euler's gamma function $\Gamma(z)$ can be defined [29, Chapter 3] by

$$\Gamma(z) = \lim_{n\to+\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}.$$

The extended binomial coefficient $\binom{z}{w}$ for $z, w \in \mathbb{C}$ is defined by

$$\binom{z}{w} = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z \notin \mathbb{N}_-, \quad w, z-w \notin \mathbb{N}_-; \\ 0, & z \notin \mathbb{N}_-, \quad w \in \mathbb{N}_- \text{ or } z-w \in \mathbb{N}_-; \\ \dfrac{\langle z\rangle_w}{w!}, & z \in \mathbb{N}_-, \quad w \in \mathbb{N}_0; \\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_0; \\ 0, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_-; \\ \pm\infty, & z \in \mathbb{N}_-, \quad w \notin \mathbb{Z}; \end{cases}$$

where

$$\begin{aligned} &\mathbb{R} = (-\infty,\infty), && \mathbb{C} = \{x + \mathrm{i}\, y : x, y \in \mathbb{R}, \mathrm{i} = \sqrt{-1}\}, && \mathbb{Z} = \{0, \pm1, \pm2, \dots\}, \\ &\mathbb{N} = \{1, 2, \dots\}, && \mathbb{N}_0 = \{0, 1, 2, \dots\}, && \mathbb{N}_- = \{-1, -2, \dots\}. \end{aligned}$$

In this paper, with help of generating functions, we will present several recurrence relations and identities for generalized derangement numbers $d_{n,r}$ involving generalized harmonic numbers $H(k,r,\alpha)$ and the Stirling numbers of the first kind $s(n,k)$.

2. Main results and their proofs

This section is devoted to demonstrate our main results and their proofs.

Theorem 1. *The generalized derangement numbers $d_{n,r}$ satisfy*

$$\sum_{k=0}^{n} \binom{n}{k} d_{k,r} = n! \binom{n+r}{n}, \quad n \in \mathbb{N}_0.$$

Proof. From the generating function in (1), it is ready that

$$\frac{1}{(1-t)^{r+1}} = \mathrm{e}^t \sum_{n=0}^{\infty} d_{n,r} \frac{t^n}{n!}.$$

Using the binomial theorem and applying the Cauchy product, we have

$$\sum_{n=0}^{\infty} \binom{-r-1}{n} (-t)^n = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right) \sum_{n=0}^{\infty} d_{n,r} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{d_{n-k,r}}{k!(n-k)!}\right] t^n. \tag{6}$$

Comparing the coefficients of t^n on both sides of (6) and using the identity

$$\binom{x}{k} = (-1)^k \binom{-x+k-1}{k}$$

complete the proof. □

Remark 1. For $r=0$, Theorem 1 coincides with the second equality in [26, Theorem 2].

Theorem 2. *Generalized derangement numbers $d_{n,r}$ satisfy*

$$d_{n,r+1} = d_{n,r} + n d_{n-1,r+1}. \tag{7}$$

Proof. From the equation (1), we have

$$\sum_{n=0}^{\infty} (d_{n,r+1} - d_{n,r}) \frac{t^n}{n!} = \frac{\mathrm{e}^{-t}}{(1-t)^{r+2}} - \frac{\mathrm{e}^{-t}}{(1-t)^{r+1}} = t \frac{\mathrm{e}^{-t}}{(1-t)^{r+2}} = \sum_{n=0}^{\infty} d_{n,r+1} \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} n d_{n-1,r+1} \frac{t^n}{n!}.$$

Equating the coefficients $\frac{t^n}{n!}$ gives the desired result. □

Theorem 3. *The generalized derangement numbers $d_{n,r}$ satisfy the recurrence relations*

$$d_{n+1,r} = r d_{n,r+1} + n d_{n-1,r+1} \tag{8}$$

and

$$d_{n+1,r} = (r+1) d_{n,r+1} - d_{n,r}. \tag{9}$$

Proof. From the generating function in (1), we acquire

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[\frac{\mathrm{e}^{-t}}{(1-t)^{r+1}}\right] = \sum_{n=0}^{\infty} d_{n+1,r} \frac{t^n}{n!}. \tag{10}$$

Differentiating on both sides of the equation (1) with respect to t results in

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[\frac{\mathrm{e}^{-t}}{(1-t)^{r+1}}\right] = (r+t) \frac{\mathrm{e}^{-t}}{(1-t)^{r+2}} = (r+t) \sum_{n=0}^{\infty} d_{n,r+1} \frac{t^n}{n!} = r \sum_{n=0}^{\infty} d_{n,r+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} n d_{n-1,r+1} \frac{t^n}{n!}. \tag{11}$$

Combining (10) and (11) lead to the equation (8).

If subtracting (7) from (8), we obtain the relation (9) immediately. □

Theorem 4. *Let $n, r \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} H_{k,r+1}(\alpha) = \sum_{k=0}^{n} \frac{d_{n-k,r-1}}{(n-k)!} H_k(\alpha).$$

Proof. From the generating function in (3), one can write

$$\sum_{n=0}^{\infty} H_{n,r+1}(\alpha)t^n = \frac{-\ln\left(1-\frac{t}{\alpha}\right)}{(1-t)^{r+1}} = \frac{-\ln\left(1-\frac{t}{\alpha}\right)}{1-t}\frac{\mathrm{e}^{-t}}{(1-t)^r}\frac{1}{\mathrm{e}^{-t}},$$

from which it is easy to verify that

$$\sum_{n=0}^{\infty} H_{n,r+1}(\alpha)t^n \sum_{n=0}^{\infty}\frac{(-t)^n}{n!} = \sum_{n=1}^{\infty} H_n(\alpha)t^n \sum_{n=0}^{\infty} d_{n,r-1}\frac{t^n}{n!}.$$

Applying the Cauchy product and equating the coefficients of t^n yield the desired formula. □

Theorem 5. *Let $n, r \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n} k!H(k,r,\alpha)d_{n-k,r-1} = (-1)^{n-r-1}\frac{(r+1)!}{\alpha^n}\sum_{k=0}^{n}(-\alpha)^k s(n-k,r+1)d_{k,r}.$$

Proof. Consider the equality

$$\frac{\left[-\ln\left(1-\frac{t}{\alpha}\right)\right]^{r+1}}{1-t}\frac{\mathrm{e}^{-t}}{(1-t)^r} = \left[-\ln\left(1-\frac{t}{\alpha}\right)\right]^{r+1}\frac{\mathrm{e}^{-t}}{(1-t)^{r+1}}.$$

Utilizing the generating functions in (1), (2), and (4), we find

$$\sum_{n=0}^{\infty} H(n,r,\alpha)t^n \sum_{n=0}^{\infty} d_{n,r-1}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^{n-r-1}(r+1)!s(n,r+1)}{\alpha^n}\frac{t^n}{n!}\sum_{n=0}^{\infty} d_{n,r}\frac{t^n}{n!},$$

from which, we have

$$\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} k!H(k,r,\alpha)d_{n-k,r-1}\right]\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} d_{k,r}\frac{(-1)^{n-k-r-1}(r+1)!s(n-k,r+1)}{\alpha^{n-k}}\right]\frac{t^n}{n!}.$$

Accordingly the desired relation follows immediately. □

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