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The New Iterative Approximating of Endpoints of Multivalued Nonexpansive Mappings in Banach Spaces

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Abstract

The purpose of this paper is to introduce a modified iteration process to approximate endpoints of multivalued nonexpansive mappings in Banach space. We prove weak and strong convergence theorems of proposed iterative scheme under some suitable assumptions in the framework of a uniformly convex Banach space.

1. Introduction and Preliminaries

In this study, we shall denote by $\mathbb{N}$ the set of natural numbers. Let $(E, \|.\|)$ be a Banach space and C be a nonempty convex subset of E. The distance from a $x \in E$ to a nonempty subset $C \subset E$ is defined by

$$dist\,(x,C) := \inf\{\|x-z\| : z \in C\}.$$

The radius of C relative to x is defined by

$$R(x,C) = \sup\{\|x-z\| : z \in C\}.$$

Definition 1.1. *A Banach space E is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there is a $\delta > 0$ such that for every $x, y \in E$*

$$\left.\begin{array}{c} \|x\| \leq 1 \\ \|y\| \leq 1 \\ \|x-y\| \geq \varepsilon \end{array}\right\} \Rightarrow \frac{\|x+y\|}{2} \leq 1-\delta.$$

We shall denote the family of nonempty compact subsets of C by $K(C)$. The Hausdorff metric H on $K(C)$ is defined as follows:

$$H(A,B) = \max\left\{\sup_{x\in A} dist\,(x,B)\,,\ \sup_{y\in B} dist\,(y,A)\right\} \text{for } A,B \in K(C).$$

A multivalued mapping $T : C \to K(C)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|\,,\ \text{ for each } \ x,y \in C.$$

A point $x \in K$ is a fixed point of a multivalued mapping $T : C \to K(C)$ if $x \in T(x)$. Moreover,if $T(x) = \{x\}$, then x is called an endpoint (or a stationary point) of T . We shall denote the set of all endpoints and the set of all fixed points of T by E_T (or $End(T)$) and F_T, respectively. It is clear that $End(T) \subseteq Fix(T)$. Endpoint for multivalued mappings is an important concept. Many researchers have studied the exsitence of an endpoint of a multivalued mapping. In 1980, Aubin and Siegel [1] proved that every multivalued dissipative mapping on a complete

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metric space has always an endpoint. In 1986, Corley [2] showed that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of certain multivalued mapping. In 2018, Panyanak [3] showed that the modified Ishikawa iteration process converge to an endpoint of a multivalued nonexpansive mapping in Banach spaces. In 2020, Laokul [4] proved Browder's convergence theorem for multivalued mappings in Banach space without the endpoint condition by using the notion of diametrically regular mapping. Abdeljawad et al. [5] introduced the modified S- iteration process for finding endpoints of multivalued nonexpansive mappings in Banach spaces. Ullah et al. [6] proved the strong and Δ-convergence results of endpoints for multivalued generalized nonexpansive in Metric spaces.

Definition 1.2. *A Banach space $(E,\|.\|)$ is said to have Opial property [7] if for each sequence $\{x_n\}$ in E which weakly converges to $x\in E$ and $y\neq x$, it follows that*

$$\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|.$$

Definition 1.3. *[3] A mapping $T:C\to K(C)$ is said to satisfy condition (J) if there exists a nondecreasing function $h:[0,\infty)\to[0,\infty)$ with $h(0)=0$, $h(r)>0$ for $r\in(0,\infty)$ such that*

$$R(x,T(x))\geq h(dist(x,End(T)) \text{ for all } x\in C.$$

Definition 1.4. *[3] The mapping $T:C\to K(C)$ is said to be semicompact if for any sequence $\{x_n\}$ in C such that*

$$\lim_{n\to\infty}R(x_n,T(x_n))=0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q\in C$ such that $\lim_{k\to\infty}x_{n_k}=q$.

Definition 1.5. *A sequence $\{x_n\}$ in E is said to be Fejěr monotone with respect to C if*

$$\|x_{n+1}-p\|\leq\|x_n-p\|$$

for all $p\in C$ and $n\in\mathbb{N}$.

The purpose of this paper is to introduce a modified iteration process to approximate endpoints of multivalued nonexpansive mappings in Banach space.
Let C be a nonempty subset of a Banach space and $T:C\to K(C)$ be a nonexpansive multivalued mapping. Let $\alpha_n,\beta_n,\gamma_n\in[a,b]\subset(0,1)$ are real sequences. We introduce our iteration process as follows: $x_1\in C$

$$z_n=(1-\gamma_n)x_n+\gamma_n v_n,\ \ n\in\mathbb{N}$$

where $v_n\in T(x_n)$ such that $\|x_n-v_n\|=R(x_n,T(x_n))$, and

$$y_n=(1-\beta_n)v_n+\beta_n w_n \tag{1.1}$$

where $w_n\in T(z_n)$ such that $\|z_n-w_n\|=R(z_n,T(z_n))$, and

$$x_{n+1}=(1-\alpha_n)v_n+\alpha_n u_n$$

where $u_n\in T(y_n)$ such that $\|y_n-u_n\|=R(y_n,T(y_n))$.
Following lemmas will be useful to prove our main results.

Lemma 1.6. *[3] For a multivalued mapping $T:C\to K(C)$, the following statements hold.*

(i) $x\in F(T)\Leftrightarrow dist(x,T(x))=0$.
(ii) $x\in End(T)\Leftrightarrow R(x,T(x))=0$.
(iii) If T is nonexpansive, the mapping $g:C\to\mathbb{R}$ defined by $g(x):=R(x,T(x))$ is continuous.

Lemma 1.7. *[8] A Banach space E is uniformly convex if and only if an arbitrary $k>0$, there exists a strictly increasing continuous function $\Psi:[0,\infty)\to[0,\infty)$ with $\Psi(0)=0$ such that*

$$\lim_{n\to\infty}\|\alpha x+(1-\alpha)y\|^2\leq\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\Psi(\|x-y\|),$$

for all $x,y\in B_k(0)=\{x\in X:\|x\|\leq k\}$, and $\alpha\in[0,1]$.

Lemma 1.8. *[9] Let $\{\alpha_n\},\{\beta_n\}$ be two real sequences such that*

(i) $0\leq\alpha_n,\beta_n<1$,
(ii) $\beta_n\to0$ as $n\to\infty$,
(iii) $\sum\alpha_n\beta_n=\infty$,
Let $\{\delta_n\}$ be a nonnegative real sequence such that $\sum\alpha_n\beta_n(1-\beta_n)\delta_n<\infty$. Then $\{\delta_n\}$ has a subsequence which converges to zero.

Definition 1.9. *[10] Let $T:C\to CB(C)$ be a multivalued mapping. A sequence $\{x_n\}$ in C is called an approximate fixed point sequence (resp. an approximate endpoint sequence) for T if $\lim_{n\to\infty}dist(x_n,T(x_n))=0$ (resp. $\lim_{n\to\infty}R(x_n,T(x_n))=0$). The mapping T is said to have the approximate fixed point property (resp. the approximate endpoint property) if it has an approximate fixed point sequence (resp. an approximate endpoint sequence) in C.*

Let C be a nonempty subset of a metric space (X,d) and $\{x_n\}$ be a bounded sequence in X. The asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C,\{x_n\}) = inf\left\{\limsup_{n\to\infty} d(x_n,x) : x \in C\right\}.$$

The asymptotic center of $\{x_n\}$ relative to E is defined by

$$A(C,\{x_n\}) = \left\{x \in C : \limsup_{n\to\infty} d(x_n,x) = r(C,\{x_n\})\right\}.$$

Lemma 1.10. *[11] Let C be a nonempty closed convex subset of a uniformly convex Banach space and $T : C \to K(C)$ be a mutlivalued nonexpansive mapping. Then the following implication holds:*

$$\{x_n\} \subseteq C,\ x_n \rightharpoonup x,\ R(x_n, T(x_n)) \to 0 \Rightarrow x \in End(T).$$

Proposition 1.11. *[10] Let C be a nonempty subset of a metric space (X,d), $\{x_n\}$ be a sequence in E, and $T : C \to K(X)$ be a mapping. Then $R(x_n,T(x_n)) \to 0$ if and only if $dist(x_n,T(x_n)) \to 0$ and $diam(T(x_n)) \to 0$.*

Theorem 1.12. *[10] Let $(X,\|.\|)$ be a uniformly convex Banach space, C be a nonempty bounded closed convex subset of X, and $T : C \to K(C)$ be a nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

2. Main Results

We start with the following lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of an uniformly convex Banach space E and $T : C \to K(C)$ be a multivalued nonexpansive mapping with $E_T \neq \emptyset$. Let $\{x_n\}$ be a sequence as defined in (1.1). Then $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in E_T$.*

Proof. Let $p \in End(T)$. By (1.1), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)v_n + \alpha_n u_n - p\| \\
&\leq (1-\alpha_n)\|v_n - p\| + \alpha_n\|u_n - p\| \\
&= (1-\alpha_n)dist(v_n,T(p)) + \alpha_n dist(u_n,T(p)) \\
&\leq (1-\alpha_n)H(T(x_n),T(p)) + \alpha_n H(T(y_n),T(p)) \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\|.
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)v_n + \beta_n w_n - p\| \\
&\leq (1-\beta_n)\|v_n - p\| + \beta_n\|w_n - p\| \\
&= (1-\beta_n)dist(v_n,T(p)) + \beta_n dist(w_n,T(p)) \\
&\leq (1-\beta_n)H(T(x_n),T(p)) + \beta_n H(T(z_n),T(p)) \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\|
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n v_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|v_n - p\| \\
&= (1-\gamma_n)\|x_n - p\| + \gamma_n dist(v_n,T(p)) \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n H(T(x_n),T(p)) \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{2.3}$$

Using (2.3) and (2.2) ,we obtain

$$\|y_n - p\| \leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| = \|x_n - p\|$$

which implies that

$$\|x_{n+1} - p\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| = \|x_n - p\|.$$

Thus$\{\|x_n - p\|\}$ is nonincreasing sequence and bounded, which implies that $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in E_T$. Also $\{x_n\}$ is bounded. □

Theorem 2.2. *Let E be a uniformly convex Banach space with Opial property, C be a nonempty closed convex subset of E and $T : C \to K(C)$ be a multivalued nonexpansive mapping with $E_T \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (1.1) with $\alpha_n, \beta_n, \gamma_n \in [a,b] \subset (0,1)$ for all n in $\mathbb{N}$, then $\{x_n\}$ converges weakly to an element of E_T.*

Proof. Fix $p \in E_T$. Then, as in the proof of Lemma 2.1,$\{x_n\}$is bounded and so $\{y_n\},\{z_n\}$ are bounded. Therefore, there exists $k>0$ such that $x_n - p,\, y_n - p,\, z_n - p \in B_k(0)$ for all $n \geq 0$. Since E is a uniformly convex, by Lemma 1.7, there exists a strictly increasing continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0)=0$ such that

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\gamma_n)x_n + \gamma_n v_n - p\|^2 \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n \|v_n - p\|^2 - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n dist^2(v_n, T(p)) - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n H^2(T(x_n), T(p)) - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) \\
&\leq \|x_n - p\|^2 - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|).
\end{aligned} \tag{2.4}$$

By Lemma 1.7 and (2.4), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)v_n + \beta_n w_n - p\|^2 \\
&\leq (1-\beta_n)\|v_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n) dist^2(v_n, T(p)) + \beta_n dist^2(w_n, T(p)) - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n) H^2(T(x_n), T(p)) + \beta_n H^2(T(z_n), T(p)) - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|)
\end{aligned} \tag{2.5}$$

from (2.4), (2.5) and by Lemma 1.7 , we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)v_n + \alpha_n u_n - p\|^2 \\
&\leq (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n) dist^2(v_n, T(p)) + \alpha_n dist^2(u_n, T(p)) - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n) H^2(T(x_n), T(p)) + \alpha_n H^2(T(y_n), T(p)) - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|).
\end{aligned} \tag{2.6}$$

so,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|).$$

This implies that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) < \infty$$

By Lemma 1.8, we have $\lim_{n\to\infty}\Psi(\|x_n - v_n\|) = 0$. As Ψ is strictly increasing and continuous, we get $\lim_{n\to\infty}\|x_n - v_n\| = 0$. Hence

$$\lim_{n\to\infty} R(x_n, T(x_n)) = \lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{2.7}$$

We want to show that $\{x_n\}$ converges weakly to an element of E_T. For this, it must be shown that $\{x_n\}$ has unique weak subsequential limit in E_T. Therefore, we assume that there are subsequences $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$ of $\{x_n\}$such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. By (2.7), $\lim_{n_i\to\infty} R(x_{n_i}, T(x_{n_i})) = 0$. It follows from Lemma 1.10 that $u \in E_T$. Similarly, we can be shown that $v \in E_T$. Now, suppose $u \neq v$. By Lemma 2.1 and the Opial property, we get

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{n_i\to\infty}\|x_{n_i} - u\| \\
&< \lim_{n_i\to\infty}\|x_{n_i} - v\| \\
&= \lim_{n\to\infty}\|x - v\| \\
&= \lim_{n_j\to\infty}\left\|x_{n_j} - v\right\| \\
&< \lim_{n_j\to\infty}\left\|x_{n_j} - u\right\| \\
&= \lim_{n\to\infty}\|x_n - u\|
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to an element of E_T.

□

Next, we prove strong convergence theorems in uniformly convex Banach spaces.

Theorem 2.3. *Let E,C and T be as in Theorem 2.2. Let $\{x_n\}$be the sequence defined by (1.1) with $\alpha_n,\beta_n,\gamma_n \in [a,b] \subset (0,1)$. If T is semi-compact, then $\{x_n\}$ converges strongly to an element of E_T.*

Proof. In view of (2.6), we have

$$\alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n-v_n\|)<\infty\,.$$

By Lemma 1.8, there exists subsequence $\{v_{n_k}\}$ and $\{x_{n_k}\}$ of $\{v_n\}$ and $\{x_n\}$, respectively, such that $\lim_{k\to\infty}\Psi(\|x_{n_k}-v_{n_k}\|)=0$. Since Ψ is strictly increasing and continuous, $\lim_{k\to\infty}\|x_{n_k}-v_{n_k}\|=0$. So,

$$\lim_{k\to\infty}R(x_{n_k},T(x_{n_k}))=\lim_{k\to\infty}\|x_{n_k}-v_{n_k}\|=0. \tag{2.8}$$

Conversely, T is semicompact, we may assume, by passing through a subsequence, that $x_{n_k}\to q$ for some $q\in C$. We need show that $q\in E_T$ and $x_n\to q$. By Lemma 1.6 (iii), together with (2.8), we have

$$R(q,T(q))=\lim_{k\to\infty}R(x_{n_k},T(x_{n_k}))=0. \tag{2.9}$$

It follows from Lemma 1.6 (ii) that $q\in E_T$. By Lemma 2.1 $\lim_{n\to\infty}\|x_n-q\|$ exists for each $q\in E_T$ and hence q is the strong limit of $\{x_n\}$. □

Proposition 2.4. *[12] Let C be a nonempty closed subset of a Banach space and $\{x_n\}$ be a Fejer monotone sequence with respect to C. Then $\{x_n\}$ converges strongly to an element of C if and only if $\lim_{n\to\infty}dist(x_n,C)=0$.*

Theorem 2.5. *Let E,C,T and $\{x_n\}$ be as in Theorem 2.2. If T satisfies condition (J), then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. Since T is a nonexpansive mapping, E_T is closed. As T satisfies condition (J), $\lim_{n\to\infty}dist(x_n,E_T)=0$. Lemma 2.1 implies that $\{x_n\}$ is Fejer monotone with to respect E_T. By Proposition 2.4, $\{x_n\}$ converges strongly to an element of E_T. □

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Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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