CONSTRUCTIVE MATHEMATICAL ANALYSIS
4 (2021), No. 4, pp. 400-419
http://dergipark.gov.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Parameterized families of polylog integrals

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ABSTRACT. It is commonly known that integrals containing log-polylog integrands admit representations in terms of special functions such as the Dirichlet eta and Dirichlet beta functions. We investigate two parameterized families of such integrals and in a particular case demonstrate a connection with the Herglotz function. In the process of the investigation, we recover some known Euler sum equalities and discover some new identities.

Keywords: Harmonic number, gamma function, alternating harmonic number, Riemann zeta function, polylogarithm function, polygamma functions, linear Euler sums.

2020 Mathematics Subject Classification: 11M06, 11M35, 26B15, 33B15, 42A70, 65B10.

1. INTRODUCTION, PRELIMINARIES AND NOTATION

In the recent past many books ([5], [20], [35]) have been published whereby the authors describe the connection of the representation of some integrals in terms of Euler sums. Likewise the following papers investigate certain integrals that can be represented by Euler sums [7], [17], [27]. In this paper, we consider two parameterized families of log-polylog integrals that admit solutions dependent on Euler sums, thereby extending the integrals considered by ([3], [6], [14], [23], [36]). We investigate parameterized families of integrals of the type

$$I^{b}_{+,-}(a,p,q,t) = \int_{x} \frac{x^{a} \ln^{p}(x) \operatorname{Li}_{t}(x^{bq})}{1 \pm x^{b}} dx, \tag{1.1}$$

$$K^{b}_{+,-}(a,p,q,t) = \int_{x} \frac{x^{a} \ln^{p}(x) \operatorname{Li}_{t}(-x^{bq})}{1 \pm x^{b}} dx,$$

where $a \geq -2,\ b \in \mathbb{R}^{+},\ p \in \mathbb{N}_0,\ q \in \mathbb{N},\ t \in \mathbb{N}_0$ and for the domain of $x \in (0,1)$. Here and elsewhere, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^{+}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers, positive real numbers, integers and positive integers respectively and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^{-} := \mathbb{Z} \backslash \mathbb{N}_0$. In the case $(a,b) = (0,2)$, we also study the integrals

$$J(p,q,t) = \int_{x} \frac{\ln^{p}(x) \operatorname{Li}_{t}(x^{2q})}{1 - x^{2}} dx, \tag{1.2}$$

$$M(p,q,t) = \int_{x} \frac{\ln^{p}(x) \operatorname{Li}_{t}(-x^{2q})}{1 - x^{2}} dx$$

Received: 08.10.2021; Accepted: 02.12.2021; Published Online: 08.12.2021
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DOI: 10.33205/cma.1006384

in the positive half line $x \geq 0$. In a particular case of the integral $K_+^b(a,p,q,t)$, we make a connection with the Herglotz function [15]. Some other related papers dealing with polylog integrals and Euler sums are [4], [9], [24], [25], [26] and the excellent books [18] and [34]. We describe some notation and special functions, to be used in the following, in the analysis of the integrals (1.1) and (1.2). The generalized harmonic number $H_n^{(t)}(\alpha)$ are defined as

$$H_n^{(t)}(\alpha) = \sum_{j=1}^{n} \frac{1}{(j+\alpha)^t}, \alpha \in \mathbb{C} \backslash \{-1,-2,-3,...\}, t \in \mathbb{C}, n \in \mathbb{N}$$

and when $\alpha = 0$, $H_n^{(t)}(0) = H_n^{(t)}$ are ordinary harmonic numbers of order t, an empty sum is designated as $H_0^{(t)} = 0$. For complex values of z, $z \in \mathbb{C} \backslash \{0,-1,-2,-3,...\}$, $\psi(z)$ is the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the familiar gamma function, (see, e.g. [33], sections 1.1 and 1.3). We know that for $n \geq 1$, $\psi(n+1) - \psi(1) = H_n$ with $\psi(1) = -\gamma$, where γ is the Euler Mascheroni constant and $\psi(n)$ is the digamma function. The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}.$$

The difference of the polygamma functions and generalized harmonic numbers are connected by the zeta function such that for $z \in \mathbb{C} \backslash \{0,-1,-2,-3,...\}$ we have the identity

$$H_z^{(m+1)} - \frac{(-1)^m}{m!} \psi^{(m)}(z+1) = \zeta(m+1). \tag{1.3}$$

The Dirichlet lambda function $\lambda(z)$,

$$\zeta(z) + \eta(z) = 2\lambda(z) \tag{1.4}$$

connects the zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, with the alternating zeta function $\eta(z)$. It is widely known that integrals of the type (1.1) may be represented by Euler sums and therefore in terms of special functions such as the Dirichlet beta function. The following papers [27], [28] and [29] also examined some integrals in terms of Euler sums. Some examples will be given highlighting specific cases of the integrals, some of which cannot be evaluated by a computer mathematical package such as "Mathematica".

2. Polylog Integrals With Positive Argument

Consider the following.

Theorem 2.1. *Let* $(p,q,t) \in \mathbb{N}_0, q \neq 0, a \geq -2,$ *and denote,*

$$I_+(a,p,q,t) = \int_0^1 \frac{x^a \ln^p(x) \operatorname{Li}_t(x^q)}{1+x} dx. \tag{2.5}$$

For an even integer q

$$I_+(a,p,q,t) = (-1)^p p! \sum_{n \geq 1} H_n^{(t)} \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}}, \tag{2.6}$$

for an odd integer q

$$I_+(a,p,q,t) = (-1)^p\, p! \sum_{n\ge1} (-1)^{n+1}\left(H_n^{(t)} - \frac{1}{2^{t-1}} H_{\left[\frac{n}{2}\right]}^{(t)}\right) \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}}, \tag{2.7}$$

and for $q \in \mathbb{R}^+ \setminus \{0\}$

$$I_+(a,p,q,t) = \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\ge1} \frac{1}{n^t}\left(H_{\frac{qn+a}{2}}^{(p+1)} - H_{\frac{qn+a-1}{2}}^{(p+1)}\right), \tag{2.8}$$

where $H_n^{(t)}$ *are harmonic numbers of order* t *and* $[z]$ *denotes the greatest integer that is less than or equal to* z.

Proof. The alternating harmonic numbers $A(n,t)$ of order t are defined by

$$A(n,t) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^t}, \quad n \in \mathbb{N}; t \in \mathbb{C} \tag{2.9}$$

then, see [2],

$$A(n,t) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^t} = H_n^{(t)} - \frac{1}{2^{t-1}} H_{\left[\frac{n}{2}\right]}^{(t)}.$$

The Dirichlet eta function

$$\eta(t) = \lim_{n\to\infty} A(n,t) = \sum_{n\ge1} \frac{(-1)^{n+1}}{n^t}, Re(t) > 0.$$

For $x \in (0,1)$, a Taylor series expansion gives

$$\mathrm{Li}_t(x^q) = \sum_{n\ge1} \frac{x^{qn}}{n^t}, \quad \frac{1}{1+x} = \sum_{n\ge0} (-1)^n x^n.$$

By the Cauchy product of two convergent series, then it follows that for q an even integer

$$\frac{x^a \mathrm{Li}_t(x^q)}{1+x} = \sum_{n\ge1} H_n^{(t)} \sum_{j=1}^{q} (-1)^{j+1} x^{qn+j+a-1}$$

and therefore, for q an even integer

$$\frac{x^a \ln^p(x) \mathrm{Li}_t(x^q)}{1+x} = \sum_{n\ge1} H_n^{(t)} \sum_{j=1}^{q} (-1)^{j+1} x^{qn+j+a-1} \ln^p(x).$$

Integrating both sides for $x \in (0,1)$, we have, after reversing the order of summation and integration, which is justified by the uniform convergence theorem

$$\begin{aligned}\int_0^1 \frac{x^a \ln^p(x) \mathrm{Li}_t(x^q)}{1+x} dx &= \sum_{n\ge1} H_n^{(t)} \sum_{j=0}^{q} (-1)^{j+1} \int_0^1 x^{qn+j+a-1} \ln^p(x)\, dx \\ &= (-1)^p\, p! \sum_{n\ge1} H_n^{(t)} \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}}.\end{aligned}$$

For q an odd integer, we have

$$\frac{x^a \ln^p(x) \mathrm{Li}_t(x^q)}{1+x} = \sum_{n\ge1} (-1)^{n+1} A(n,t) \sum_{j=1}^{q} (-1)^{j+1} x^{qn+j+a-1} \ln^p(x)$$

and

$$\int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(x^q)}{1+x}dx = (-1)^p\, p! \sum_{n\geq 1} (-1)^{n+1} \left(H_n^{(t)} - \frac{1}{2^{t-1}} H_{\left[\frac{n}{2}\right]}^{(t)} \right) \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}}.$$

By simple expansion, we have

$$\sum_{n\geq 1} \frac{(-1)^{n+1} H_{\left[\frac{n}{2}\right]}^{(t)}}{(qn+\alpha)^{p+1}} = \sum_{n\geq 1} \frac{H_n^{(t)}}{(q(2n+1)+\alpha)^{p+1}} - \sum_{n\geq 1} \frac{H_n^{(t)}}{(2qn+\alpha)^{p+1}}$$

and therefore we can also express

$$\begin{aligned} \frac{(-1)^p}{p!} \int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(x^q)}{1+x}dx =& \sum_{n\geq 1} (-1)^{n+1} H_n^{(t)} \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}} + \frac{1}{2^{t-1}} \sum_{n\geq 1} H_n^{(t)} \\ & \times \left(\sum_{j=1}^{q} \frac{(-1)^{j+1}}{(2qn+j+a)^{p+1}} - \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(q(2n+1)+j+a)^{p+1}} \right). \end{aligned}$$

For the representation (2.8), we can write

$$\begin{aligned} \int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(x^q)}{1+x}dx &= (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \sum_{j\geq 0} \frac{(-1)^j}{(qn+j+a+1)^{p+1}} \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(\zeta\left(p+1, \frac{qn+a+1}{2}\right) - \zeta\left(p+1, \frac{qn+a+2}{2}\right) \right) \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(\psi^{(p)}\left(\frac{qn+a+2}{2}\right) - \psi^{(p)}\left(\frac{qn+a+1}{2}\right) \right) \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(H_{\frac{qn+a}{2}}^{(p+1)} - H_{\frac{qn+a-1}{2}}^{(p+1)} \right) \end{aligned}$$

and the proof is finished. □

The next theorem deals with a related integral similar to (2.5).

Theorem 2.2. *For* $(p,t) \in \mathbb{N}$, $a \geq -2$, *and for* q *a positive integer, then*

$$\begin{aligned} I_-(a,p,q,t) &= \int_0^1 \frac{x^a \ln^p(x)}{1-x} \mathrm{Li}_t(x^q) dx \\ &= (-1)^p\, p! \sum_{n\geq 1} H_n^{(t)} \sum_{j=1}^{q} \frac{1}{(qn+j+a)^{p+1}}. \end{aligned} \tag{2.10}$$

For $q \in \mathbb{R}^+ \setminus \{0\}$

$$I_-(a,p,q,t) = (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \left(\zeta(p+1) - H_{nq+a}^{(p+1)} \right), \tag{2.11}$$

where $H_{nq+a}^{(p+1)}$ *are shifted harmonic numbers of order* $p+1$.

Proof. A Taylor series expansion of

$$\mathrm{Li}_t(x^q)=\sum_{n\geq 1}\frac{x^{qn}}{n^t} \text{ and } \frac{1}{1-x}=\sum_{j\geq 0}x^j$$

allows us to write

$$\begin{aligned} I_-(a,p,q,t) &= \sum_{n\geq 1}H_n^{(t)}\sum_{j=1}^{q}\int_0^1 x^{qn+j+a-1}\ln^p(x)\,dx \\ &= (-1)^p\,p!\sum_{n\geq 1}\frac{1}{n^t}\sum_{j=1}^{q}\frac{1}{(qn+j+a)^{p+1}}. \end{aligned}$$

For the representation (2.11), we notice

$$\begin{aligned} I_-(a,p,q,t) &= \sum_{n\geq 1}\frac{1}{n^t}\sum_{j\geq 0}\frac{(-1)^p\,p!}{(qn+j+a+1)^{p+1}} \\ &= (-1)^p\,p!\sum_{n\geq 1}\frac{1}{n^t}\zeta(p+1,qn+a+1) \\ &= (-1)^p\,p!\sum_{n\geq 1}\frac{(-1)^{p+1}}{p!n^t}\psi^{(p)}(qn+a+1). \end{aligned}$$

From the identity (1.3), we obtain the required representation

$$I_-(a,p,q,t)=(-1)^p\,p!\sum_{n\geq 1}\frac{1}{n^t}\left(\zeta(p+1)-H_{nq+a}^{(p+1)}\right).$$

We remark that Coffey [7] obtained solutions of various special cases of $I_-(0,p,1,1)$ in terms of Euler sums. □

Remark 2.1. *For $(p,q)\in\mathbb{N}_0$, we see from (2.10) and (2.11) the remarkable Euler sum identity*

$$\sum_{n\geq 1}H_n^{(t)}\sum_{j=1}^{q}\frac{1}{(qn+j+a)^{p+1}}=\sum_{n\geq 1}\frac{1}{n^t}\left(\zeta(p+1)-H_{nq+a}^{(p+1)}\right). \tag{2.12}$$

Using the notation developed by [13] and generalized by the authors of the paper [1], we define

$$S_{p,q}^{++}(\alpha,\beta)=\sum_{n\geq 1}\frac{H_n^{(p)}(\alpha)}{(n+\beta)^q},\ S_{p,q}^{+-}(\alpha,\beta)=\sum_{n\geq 1}\frac{(-1)^{n+1}H_n^{(p)}(\alpha)}{(n+\beta)^q},$$

where

$$\zeta(p,\alpha)=H_n^{(p)}(\alpha)=\sum_{j=1}^{n}\frac{1}{(n+\alpha)^p},\quad n\in\mathbb{N},p\in\mathbb{C},\alpha\in\mathbb{C}\backslash\mathbb{Z}^-.$$

In the case $\alpha=0,\beta=0$, we write $S_{p,q}^{++}(0,0)=S_{p,q}^{++}$ and $S_{p,q}^{+-}(0,0)=S_{p,q}^{+-}$. For $a=0$, upon rearranging and simplifying we obtain a new Euler identity

$$\sum_{n\geq 1}H_n^{(t)}\sum_{j=1}^{q-1}\frac{1}{(qn+j)^{p+1}}+\sum_{n\geq 1}\frac{H_{qn}^{(p+1)}}{n^t}+\frac{1}{q^{p+1}}S_{t,p+1}^{++}=\zeta(t)\zeta(p+1)+\frac{1}{q^{p+1}}\zeta(t+p+1).$$

If we choose $q=1, a\in\mathbb{R}, a>-1$, then

$$S^{++}_{p+1,t}\left(0,a\right)+S^{++}_{t,p+1}\left(a,0\right)=\zeta\left(t\right)\zeta\left(p+1\right)+\sum_{n\geq1}\frac{1}{n^t\left(n+a\right)^{p+1}},$$

which confirms Theorem 3.1 obtained by [1], using shuffle (or reciprocity) properties of Euler sums. If we now let $a=0$, we recover the well known identity

$$S^{++}_{p+1,t}+S^{++}_{t,p+1}=\zeta\left(t\right)\zeta\left(p+1\right)+\zeta\left(t+p+1\right).$$

In the special case $p+1=t$,

$$\sum_{n\geq1}H^{(t)}_n\sum_{j=1}^{q-1}\frac{1}{\left(qn+j\right)^t}+\sum_{n\geq1}\frac{H^{(t)}_{qn}}{n^t}=\left(1-\frac{1}{q^{t+1}}\right)\zeta^2\left(t\right)+\frac{1}{q^{t+1}}\zeta\left(2t\right).$$

From (2.12) with $q=2, a=0$ (and renaming $p+1$ as p), we have

$$\frac{1}{2^p}S^{++}_{p,t}\left(0,\frac{1}{2}\right)=\zeta\left(t\right)\zeta\left(p\right)-\frac{1}{2^p}\zeta\left(t+p\right)+\left(\frac{1}{2^{p-1}}-2^{t-1}\right)S^{++}_{p,t}-\frac{1}{2^p}S^{++}_{t,p}+2^{t-1}S^{+-}_{p,t}$$

and when $p=t$, we can simplify to obtain the new identity

$$\frac{1}{2^t}S^{++}_{t,t}\left(0,\frac{1}{2}\right)-2^{t-1}S^{+-}_{t,t}=\zeta\left(t\right)\eta\left(t\right)-2^{t-2}\eta\left(2t\right)+\left(\frac{3}{2^{t+1}}-2^{t-2}\right)\zeta^2\left(t\right).$$

In terms of the harmonic numbers at an argument of half integer values we have

$$\frac{1}{2^t}S^{++}_{t,t}\left(0,\frac{1}{2}\right)-2^{t-1}S^{+-}_{t,t}=2^{t-1}S^{++}_{t,t}-2^{t-1}\lambda\left(t\right)\eta\left(t\right)-\frac{1}{2}\sum_{n\geq1}\frac{H^{(t)}_{\frac{n}{2}}}{n^t}-\frac{1}{2}\sum_{n\geq1}\frac{\left(-1\right)^{n+1}H^{(t)}_{\frac{n}{2}}}{n^t},$$

where it has been shown in [29] that

$$\sum_{n\geq1}\frac{\left(-1\right)^{n+1}H^{(t)}_{\frac{n}{2}}}{n^t}=2^{t-1}\left(\eta\left(2t\right)-\eta^2\left(t\right)\right)$$

and

$$\sum_{n\geq1}\frac{H^{(t)}_{\frac{n}{2}}}{n^t}=2^{t-1}\left(\eta\left(2t\right)-\eta^2\left(t\right)\right)+\frac{1}{2^t}\left(\zeta\left(2t\right)+\zeta^2\left(t\right)\right).$$

Some other log-sine-polylog integrals involving alternating Euler sums have recently been investigated by [17].

Remark 2.2. *For the two cases where $b\in\mathbb{R}^+, a+1>-b$*

(2.13)
$$I^b_+\left(a,p,q,t\right)=\int_0^1\frac{x^a\ln^p\left(x\right)\mathrm{Li}_t(x^{bq})}{1+x^b}dx=\frac{1}{b^{p+1}}I_+\left(\frac{a+1-b}{b},p,q,t\right)$$
$$=\frac{\left(-1\right)^{p+1}p!}{\left(2b\right)^{p+1}}\sum_{n\geq1}\frac{1}{n^t}\left(H^{(p+1)}_{\frac{qn}{2}+\frac{a+1-2b}{2b}}-H^{(p+1)}_{\frac{qn}{2}+\frac{a+1-b}{2b}}\right)$$

and

(2.14)
$$I^b_-\left(a,p,q,t\right)=\int_0^1\frac{x^a\ln^p\left(x\right)\mathrm{Li}_t(x^{bq})}{1-x^b}dx=\frac{1}{b^{p+1}}I_-\left(\frac{a+1-b}{b},p,q,t\right).$$

The following theorem applies.

Theorem 2.3. *For* $p,q,t\in\mathbb{N},\ a=0$ *and* $b=2$ *then*

$$J(p,q,t)=\int_0^\infty \frac{\ln^p(x)\,\mathrm{Li}_t(x^{2q})}{1-x^2}dx=\int_0^\infty f(x;p,q,t)\,dx$$

$$=\int_0^1 \ln^p(\tanh\theta)\,\mathrm{Li}_t(\tanh^{2q}\theta)d\theta \tag{2.15}$$

$$=\left(1+(-1)^{p+t}\right)I_-^2(0,p,q,t)+(-1)^{p+t}\frac{(2\pi i)^t}{t!}\int_0^1\frac{\ln^p(x)}{1-x^2}B\left(t,\frac{\ln\left(x^{2q}\right)}{2\pi i}\right)dx, \tag{2.16}$$

where

$$f(x;p,q,t)=\frac{\ln^p(x)}{1-x^2}\mathrm{Li}_t(x^{2q}),$$

$I_-^2(0,p,q,t)$ *is given by* (2.14) *and* $B\left(t,\frac{\ln(x^{2q})}{2\pi i}\right)$ *is the Bernoulli polynomial.*

Proof. We begin with

$$J(p,q,t)=\int_0^\infty \frac{\ln^p(x)\,\mathrm{Li}_t(x^{2q})}{1-x^2}dx=\int_0^\infty f(x;p,q,t)\,dx$$

and put

$$J(p,q,t)=\int_0^\infty f(x;p,q,t)\,dx=\int_0^1 f(x;p,q,t)\,dx+\int_1^\infty f(x;p,q,t)\,dx.$$

We notice that $f(x;p,q,t)$ is continuous, bounded and differentiable on the interval $x\in(0,1]$, with $\lim\limits_{x\to0+}f(x;p,q,t)=\lim\limits_{x\to1}f(x;p,q,t)=0$. Now we make the transformation $xy=1$ in the third integral so that

$$\int_0^\infty f(x;p,q,t)\,dx=\int_0^1 f(x;p,q,t)\,dx+(-1)^p\int_0^1\frac{\ln^p(y)}{1-y^2}\mathrm{Li}_t(y^{-2q})dy. \tag{2.17}$$

From Erdělyi et. al. [11], Jonquiěre's relation states

$$\mathrm{Li}_s(z)+e^{i\pi s}\mathrm{Li}_s\left(\frac{1}{z}\right)=\frac{\left(2\pi e^{i\pi}\right)^s}{\Gamma(s)}\zeta\left(1-s,\frac{\ln z}{2\pi i}\right), \tag{2.18}$$

where $\mathrm{Li}_s(z)$ is a polylogarithm, $i=\sqrt{-1},\ \Gamma(s)$ is the gamma function, $s\in\mathbb{C}$ and $\zeta\left(1-s,\frac{\ln z}{2\pi i}\right)$ is the Hurwitz zeta function and z is not a member of the real interval $[0,1]$. A modified version of (2.18) is given by Crandall [9] as follows. For integer t and $z\in\mathbb{C}$,

$$\mathrm{Li}_t(z)+(-1)^t\,\mathrm{Li}_t\left(\frac{1}{z}\right)=-\frac{(2\pi i)^t}{t!}B\left(t,\frac{\ln(z)}{2\pi i}\right)-2\pi i\Theta(z)\frac{\ln^{t-1}(z)}{(t-1)!}, \tag{2.19}$$

where $B\left(t,\frac{\ln(z)}{2\pi i}\right)$ is the Bernoulli polynomial (see, e.g. [33], sections 1.7), and $\Theta(z)$ is a time dependent step function

$$\Theta(z)=\begin{cases}1, \text{ if } Im(z)<0 \text{ or } z\in[1,\infty)\\ 0, \text{ otherwise}\end{cases}.$$

The function $\Theta(z)$ is intended to provide the conventional behavior in the branch when and only when z is in the lower half plane union with the real cut $[1,\infty)$. For convenience, we list

$$B\left(4,\frac{\ln(z)}{2\pi i}\right)=\frac{1}{16\pi^4}\ln^4 z-\frac{i}{4\pi^3}\ln^3 z-\frac{i}{4\pi^2}\ln^2 z-\frac{1}{30},$$

$$B\left(3,\frac{\ln(z)}{2\pi i}\right)=-\frac{i}{4\pi}\ln z+\frac{3}{8\pi^2}\ln^2 z+\frac{i}{8\pi^3}\ln^3 z.$$

Now, we can substitute (2.19) into (2.17), so that

$$\int_0^\infty f(x;p,q,t)\,dx=\left(1+(-1)^{p+t}\right)\int_0^1 f(x;p,q,t)\,dx+(-1)^{p+t}\frac{(2\pi i)^t}{t!}\int_0^1\frac{\ln^p(x)}{1-x^2}B\left(t,\frac{\ln\left(x^{2q}\right)}{2\pi i}\right)dx.$$

The integral

$$I_-^2(0,p,q,t)=\int_0^1\frac{\ln^p(x)\,\mathrm{Li}_t(x^{2q})}{1-x^2}dx$$

has been evaluated in Theorem 2.1 and therefore

$$J(p,q,t)=\left(1+(-1)^{p+t}\right)I_-^2(0,p,q,t)+(-1)^{p+t}\frac{(2\pi i)^t}{t!}\int_0^1\frac{\ln^p(x)}{1-x^2}B\left(t,\frac{\ln\left(x^{2q}\right)}{2\pi i}\right)dx$$

and the proof is finished. Note that the integral $I_-^2(0,p,q,t)$ does not contribute to $J(p,q,t)$ in the case when $p+t$ is an odd integer. The third integral in (2.15) is obtained by the substitution $x=\tanh\theta$. $\square$

Remark 2.3. *Utilizing (2.19) we are able to evaluate the related integral, from Theorem 2.2, (or from (2.14))*

$$\int_0^1\frac{x^a\ln^p(x)\,\mathrm{Li}_t(x^{-bq})}{1-x^b}dx=(-1)^{t+1}I_-^b(a,p,q,t)+(-1)^{t+1}\frac{(2\pi i)^t}{t!}\int_0^1\frac{x^a\ln^p(x)\,B\left(t,\frac{\ln\left(x^{bq}\right)}{2\pi i}\right)}{1-x^b}dx.$$

Some examples follow. First we record here the following result, given in [27], that will be required for the evaluation of some Euler sums.

Theorem 2.4. *Let α be a real number $\alpha\neq-1,-2,-1,...,$ and assume that $m\in\mathbb{N}\backslash\{1\}$. Then*

$$\sum_{n\geq1}\frac{H_n}{(n+\alpha)^m}=\frac{(-1)^m}{(m-1)!}\begin{bmatrix}(\psi(\alpha)+\gamma)\,\psi^{(m-1)}(\alpha)\\ -\frac{1}{2}\psi^{(m)}(\alpha)+\sum_{j=1}^{m-2}\begin{pmatrix}m-2\\ j\end{pmatrix}\psi^{(j)}(\alpha)\,\psi^{(m-j-1)}(\alpha)\end{bmatrix},$$

where γ is the Euler Mascheroni constant.

Example 2.1.

1. For $a=-1, q=2, p=2m, m\in\mathbb{N}$ and $t+p$ of even weight

$$I_+(-1,2m,2,t)=\int_0^1 \frac{x^{-1}\ln^{2m}(x)\,\mathrm{Li}_t(x^2)}{1+x}dx$$
$$=\frac{(2m)!}{2^{2m+1}}S^{++}_{t,2m+1}-\frac{(2m)!}{2^{2m+1}}S^{++}_{t,2m+1}\left(0,\frac{1}{2}\right)$$

and can be evaluated explicitly in terms of special functions since we have the known Euler sum relations, $S^{++}_{t,2m+1}$ and $S^{++}_{t,2m+1}\left(0,\frac{1}{2}\right)$ defined in Remark 2.1.

2. For $t=p=q=1$ and $a=-\frac{1}{2}$

$$I_+\left(-\frac{1}{2},1,1,1\right)=\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{1}{4}\right)^2}-\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{3}{4}\right)^2}-\sum_{n\geq1}\frac{(-1)^{n+1}H_n}{\left(n+\frac{1}{2}\right)^2},$$

here, the Euler sums $\sum_{n\geq1}\frac{H_n}{(n+x)^m}$ are evaluated using Theorem 2.4, so that

$$I_+\left(-\frac{1}{2},1,1,1\right)=8G\ln2+8Im\left(\mathrm{Li}_3\left(\frac{1\pm i}{2}\right)\right)-\frac{1}{4}\pi\ln^2 2-\frac{5}{16}\pi^3,$$

where $G=\sum_{n\geq0}\frac{(-1)^n}{(2n+1)^2}$. is Catalan's constant. Sofo and Nimbran [32] have shown that the imaginary part of the trilogarithm:

$$W(3):=Im\left(\mathrm{Li}_3\left(\frac{1\pm i}{2}\right)\right)$$
$$=\sum_{n\geq1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\,n^3}$$
$$=\sum_{n\geq1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right)$$

and Lewin ([16], p.164, 296) has also given

$$Re\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)=\frac{1}{48}\ln^3 2+\frac{35}{64}\zeta(3)$$

and therefore

$$I_+\left(-\frac{1}{2},1,1,1\right)=8G\ln2+8W(3)-\frac{1}{4}\pi\ln^2 2-\frac{5}{16}\pi^3.$$

3. For $t=q=1, p=2$ and $a=-\frac{1}{2}$

$$I_+\left(-\frac{1}{2},1,1,2\right)=2\sum_{n\geq1}\frac{(-1)^{n+1}H_n}{\left(n+\frac{1}{2}\right)^3}-\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{1}{4}\right)^3}+\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{3}{4}\right)^3}$$
$$=\frac{63}{8}\pi\zeta(3)+2\pi^2G+\frac{13}{8}\pi^3\ln2-102\beta(4),$$

where the Dirichlet beta function, $\beta(z)$ or Dirichlet L function is given by, see Finch [12],

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z};\ \textit{for}\ \ Re(z)>0,$$

where $\beta(2) = G$ is Catalan's constant. From Remark 2.2

$$I_-^b\left(b-1, t-1, \frac{1}{2}, t\right) = \int_0^1 \frac{x^{b-1}\ln^{t-1}(x)\,\mathrm{Li}_t(x^{b/2})}{1-x^b}dx$$
$$= \frac{(-1)^t\,(t-1)!}{b^t}\left(2^{-t}\zeta(2t) + \left(1-2^{-t}\right)\zeta^2(t) + 2^{t-1}\left(\eta(2t) - \eta^2(t)\right)\right).$$

4. For $a = -\frac{1}{2}, p = 1, q = 1, t = 2$

$$\int_0^1 \frac{x^{-1/2}\ln(x)\,\mathrm{Li}_2(x)}{1-x}dx = 16L(3) - \frac{55}{4}\zeta(4),$$

where, see [13],

$$L(3) = S_{1,3}^{+-} = \frac{11}{4}\zeta(4) - \frac{7}{4}\zeta(3)\ln 2 + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2Li_4\left(\frac{1}{2}\right) \tag{2.20}$$

and

$$\int_0^1 \frac{x^{-1/2}\ln(x)\,\mathrm{Li}_2(\frac{1}{x})}{1-x}dx = \frac{175}{4}\zeta(4) - 16L(3) + i14\pi\zeta(3).$$

5. For $b = 1, p = 1, q = 1, t = 2$

$$I_-^2(-2,1,1,2) = \int_0^1 \frac{x^{-2}\ln(x)\,\mathrm{Li}_2(x^2)}{1-x^2}dx = 8\ln 2 + 4L(3) - 4\zeta(2) - \frac{55}{16}\zeta(4).$$

6. From (2.13)

$$I_+^b(b-1, t-1, 1, t) = \frac{(-1)^t\,(t-1)!}{2b^t}\left(\eta^2(t) - \zeta(2t)\right).$$

7. From (2.16)

$$J(3,2,1) = \int_0^\infty \frac{\ln^3(x)\,\mathrm{Li}_1(x^4)}{1-x^2}dx = \frac{21}{8}\pi^2\zeta(3) + \frac{3}{4}\pi^3 G + \frac{3}{8}\pi^4\ln 2 + 6\pi\beta(4) + i\frac{\pi^5}{16}.$$

In the next section we consider the integral (2.5) with negative polylog argument.

3. Polylog Integrals With Negative Argument

Theorem 3.5. *Let $(p,q,t) \in \mathbb{N}_0, q \neq 0, a \geq -2$, and denote*

$$K_+(a,p,q,t) = \int_0^1 \frac{x^a\ln^p(x)\,\mathrm{Li}_t(-x^q)}{1+x}dx. \tag{3.21}$$

For q an odd integer

$$K_+(a,p,q,t) = (-1)^p\,p!\sum_{n\geq 1}(-1)^{n+1}H_n^{(t)}\sum_{j=1}^q \frac{(-1)^j}{(qn+j+a)^{p+1}}, \tag{3.22}$$

for q an even integer

$$K_+(a,p,q,t) = (-1)^p\,p!\sum_{n\geq 1}\left(H_n^{(t)} - \frac{1}{2^{t-1}}H_{[\frac{n}{2}]}^{(t)}\right)\sum_{j=1}^q \frac{(-1)^j}{(qn+j+a)^{p+1}}, \tag{3.23}$$

and for $q \in \mathbb{R}^+ \setminus \{0\}$

$$K_+(a,p,q,t) = \frac{(-1)^p p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^n}{n^t} \left(H^{(p+1)}_{\frac{qn+a}{2}} - H^{(p+1)}_{\frac{qn+a-1}{2}} \right), \tag{3.24}$$

where $H_n^{(t)}$ are harmonic numbers of order t and $[z]$ denotes the greatest integer that is less than or equal to z.

Proof. The results (3.22), (3.23) and (3.24) can be proven *Mutatis Mutandis* with respect to Theorem 2.1. $\square$

3.1. **Connection to the Herglotz function.** From Theorem 3.5, let

$$\Lambda(q) = -K_+(0,0,q,1) = -\int_0^1 \frac{\mathrm{Li}_1(-x^q)}{1+x} dx = \int_0^1 \frac{\ln(1+x^q)}{1+x} dx.$$

In the paper [22], Zagier stated that Henri Cohen ([8], Ex. 60, p. 902-903) showed him the identity

$$\Lambda\left(1+\sqrt{2}\right) = \frac{1}{2}\ln 2\left(\ln 2 + \ln\left(1+\sqrt{2}\right)\right) - \frac{1}{4}\zeta(2).$$

Radchenko and Zagier [22], evaluated many other cases such as $\Lambda\left(\frac{2}{5}\right)$ and $\Lambda\left(4+\sqrt{17}\right)$ and gave the relation

$$\Lambda(q) = F(2q) - 2F(q) + F\left(\frac{q}{2}\right) + \frac{1}{2q}\zeta(2)$$

in terms of the function

$$F(q) = \sum_{n\geq 1} \frac{1}{n}\left(\psi(nq) - \ln(nq)\right), q \in \mathbb{C}\setminus(-\infty, 0].$$

The function $F(q)$ was introduced and studied by Zagier [37] and he obtained some functional equations that $F(q)$ satisfies, namely, for $q \in \mathbb{C}\setminus(-\infty, 0]$

$$F(q) - F(q+1) - F\left(\frac{q}{1+q}\right) + F(1) = \mathrm{Li}_2\left(\frac{1}{1+q}\right)$$

and

$$F(q) + F\left(\frac{1}{q}\right) - 2F(1) = \frac{1}{2}\ln^2 q - \frac{(q-1)^2}{q}\zeta(2).$$

A similar function to $F(q)$ was also studied by Herglotz in [15] and therefore Radchenko and Zagier [22] named it the Herglotz function. Herglotz [15] also studied the integral $-K_+(0,0,q,1)$ and found explicit values for $\Lambda\left(4+\sqrt{15}\right)$, $\Lambda\left(6+\sqrt{35}\right)$ and $\Lambda\left(12+\sqrt{143}\right)$. Many other identities of this kind were found by Muzzafar and Williams [19], together with some sufficient conditions on q under which one can evaluate $\Lambda\left(q+\sqrt{q^2-1}\right)$. In Section 6, Radchenko and Zaiger [22] give a systematic account, at special values of quadratic units of these identities and list two tables with specific solutions. Radchenko and Zagier [22] study, among other things, the relation of this function with the Dedekind eta-function, functional equations satisfied by $F(q)$ in connection with Hecke operators, the cohomological aspects of $F(q)$ and its special values at positive rationals and quadratic units. Recently, Dixit et al. [10] extended the study to higher Herglotz functionals. From (3.24) we can see that

$$\Lambda(q) = \frac{1}{2}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\left(H_{\frac{qn}{2}} - H_{\frac{qn-1}{2}}\right)$$

and from the identity of the multiple argument of polygamma functions,

$$2H_{qn} - 2\ln 2 = H_{\frac{qn}{2}} + H_{\frac{qn}{2}-\frac{1}{2}}$$

implies

$$\begin{aligned}\Lambda(q) &= \ln^2 2 - \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\left(H_{qn} - H_{\frac{qn}{2}}\right)\\ &= \ln^2 2 - \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\left(\psi(qn+1) - \psi\left(\frac{qn}{2}+1\right)\right).\end{aligned}$$

In the case $q = 1/2$

$$\Lambda\left(\frac{1}{2}\right) = \frac{1}{4}\ln^2 2 + \frac{1}{8}\zeta(2).$$

Consider the case $q = 2m, m \in \mathbb{N}$, then

$$\Lambda(2m) = \ln^2 2 + \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\left(\psi(mn+1) - \psi(2mn+1)\right)$$

and using the known identities, see [31], for the digamma sums, we can write

$$\begin{aligned}\Lambda(2m) = \int_0^1 \frac{\ln(1+x^{2m})}{1+x}dx =& \frac{1}{2}\sum_{j=0}^{2m-1}\ln^2\left(2\sin\left(\frac{(2j+1)\pi}{4m}\right)\right)\\ &+\frac{1-2m^2}{8m}\zeta(2) + \ln^2 2 - \frac{1}{2}\sum_{j=0}^{m-1}\ln^2\left(2\sin\left(\frac{(2j+1)\pi}{2m}\right)\right),\end{aligned}$$

where, in particular

$$\Lambda(6) = 2\ln^2\left(1+\sqrt{3}\right) - 2\ln 2\ln\left(1+\sqrt{3}\right) + \frac{5}{4}\ln^2 2 - \frac{17}{24}\zeta(2).$$

From the functional relationship

$$\ln(1+x^q) - \ln\left(1+x^{-q}\right) = q\ln x$$

we can evaluate the related $\Lambda(-q)$ integral

$$\Lambda(-q) = \int_0^1 \frac{\ln(1+x^{-q})}{1+x}dx = \Lambda(q) - \frac{q}{2}\zeta(2),$$

here

$$\Lambda(-6) = 2\ln^2\left(1+\sqrt{3}\right) - 2\ln 2\ln\left(1+\sqrt{3}\right) + \frac{5}{4}\ln^2 2 + \frac{55}{24}\zeta(2).$$

For the case of q odd, we also have the representation (3.22) and for $q = 3$,

$$\Lambda(3) = \ln 2\ln 3 - \frac{1}{2}\ln^2 2 - \sum_{n\geq 1}\frac{\cos(\pi n/3)}{2^{n-1}n^2}.$$

Theorem 3.6. *Let* $(p,q,t)\in\mathbb{N}, q\neq 0, a\geq -2$, *and denote*

$$K_{-}(a,p,q,t)=\int_0^1\frac{x^a\ln^p(x)\operatorname{Li}_t(-x^q)}{1-x}dx.$$

Then, for $q\in\mathbb{N}$

$$K_{-}(a,p,q,t)=(-1)^p\,p!\sum_{n\geq1}\left(H_n^{(t)}-\frac{1}{2^{t-1}}H_{\left[\frac{n}{2}\right]}^{(t)}\right)\sum_{j=1}^{q}\frac{1}{(qn+j+a)^{p+1}},$$

and for $q\in\mathbb{R}^+\backslash\{0\}$

$$K_{-}(a,p,q,t)=(-1)^p\,p!\zeta(p+1)\,\eta(t)-(-1)^p\,p!\sum_{n\geq1}\frac{(-1)^{n+1}H_{qn+a}^{(p+1)}}{n^t},$$

where $\eta(t)$ *is the Dirichlet eta function, or the alternating zeta function.*

Proof. The proof follows *Mutatis Mutandis* with respect to Theorem 2.1. □

Remark 3.4. *For the two cases where* $b\in\mathbb{R}^+, a+1>-b$

$$K_{+}^{b}(a,p,q,t)=\int_0^1\frac{x^a\ln^p(x)\operatorname{Li}_t(-x^{bq})}{1+x^b}dx=\frac{1}{b^{p+1}}K_{+}\left(\frac{a+1-b}{b},p,q,t\right)$$

and

$$K_{-}^{b}(a,p,q,t)=\int_0^1\frac{x^a\ln^p(x)\operatorname{Li}_t(-x^{bq})}{1-x^b}dx=\frac{1}{b^{p+1}}K_{-}\left(\frac{a+1-b}{b},p,q,t\right).$$

In particular

$$K_{-}^{2}(0,p,q,t)=\int_0^1\frac{\ln^p(x)\operatorname{Li}_t(-x^{2q})}{1-x^2}dx=\frac{1}{2^{p+1}}K_{-}\left(-\frac{1}{2},p,q,t\right)\tag{3.25}$$
$$=\begin{cases}\frac{(-1)^p p!}{2^{p+1}}\sum_{n\geq1}\frac{(-1)^{n+1}H_{qn-\frac{1}{2}}^{(p+1)}}{n^t}-\frac{(-1)^p p!}{2^{p+1}}\zeta(p+1)\,\eta(t)\,, \textit{for } q\in\mathbb{R}^+\\ (-1)^{p+1}p!\sum_{n\geq1}A(n,t)\sum_{j=1}^{q}\frac{1}{(2qn+2j-1)^{p+1}}, \textit{for } q\in\mathbb{N}\end{cases}.$$

Now, we provide a theorem for the representation of a special case of the integral (3.25) in the half plane $x\geq0$.

Theorem 3.7. *For* $b=2, a=0; p,t\in\mathbb{N}, q>0$

$$M(p,q,t)=\int_0^\infty\frac{\ln^p(x)\operatorname{Li}_t(-x^{2q})}{1-x^2}dx=\int_0^\infty g(x;p,q,t)\,dx\tag{3.26}$$
$$=\int_0^1\ln^p(\tanh\theta)\operatorname{Li}_t(-\tanh^{2q}\theta)d\theta$$

$$=\left(1+(-1)^{p+t}\right)K_{-}^{2}\left(0,p,q,t\right)+2\sum_{j=0}^{\left[\frac{t}{2}\right]}(2q)^{t-2j}\,p!\begin{pmatrix}p+t-2j\\ p\end{pmatrix}\eta(2j)\lambda\left(p+t+1-2j\right), \tag{3.27}$$

where

$$g\left(x;p,q,t\right)=\frac{\ln^{p}\left(x\right)\mathrm{Li}_{t}(-x^{2q})}{1-x^{2}}, \tag{3.28}$$

$K_{-}^{2}\left(0,p,q,t\right)$ is given by (3.25), $\eta(2j)$ is the Dirichlet eta function, $\lambda\left(\cdot\right)$ is the Dirichlet lambda function (1.4) and $\left[\frac{t}{2}\right]$ is the Floor function.

Proof. Using the same technique as in Theorem 2.3, we arrive at

$$\int_{0}^{\infty}g\left(x;p,q,t\right)dx=\int_{0}^{1}g\left(x;p,q,t\right)dx+(-1)^{p}\int_{0}^{1}\frac{\ln^{p}\left(y\right)}{1-y^{2}}\mathrm{Li}_{t}(-y^{-2q})dy. \tag{3.29}$$

From Lewin ([16], p.299), Jonquiĕre's relation states

$$\mathrm{Li}_{s}(-z)+(-1)^{t}\,\mathrm{Li}_{s}(-\frac{1}{z})=-2\sum_{j=0}^{\left[\frac{t}{2}\right]}\frac{(\ln z)^{t-2j}}{(t-2j)!}\eta(2j)=2\sum_{j=0}^{\left[\frac{t}{2}\right]}\frac{(\ln z)^{t-2j}}{(t-2j)!}\mathrm{Li}_{2j}(-1), \tag{3.30}$$

where $\mathrm{Li}_{s}(z)$ is a polylogarithm. The relation (3.30) can also be written in terms of Bernoulli numbers so that

$$\mathrm{Li}_{t}(-z)+(-1)^{t}\,\mathrm{Li}_{t}(-\frac{1}{z})=\frac{1}{t!}\sum_{j=0}^{t}\left(1-2^{1-j}\right)\begin{pmatrix}t\\ j\end{pmatrix}B_{j}\left(2\pi i\right)^{j}\left(\ln z\right)^{t-2j},$$

where B_j are the Bernoulli numbers. Now we can substitute (3.30) into (3.29), so that

$$\int_{0}^{\infty}g\left(x;p,q,t\right)dx=\left(1+(-1)^{p+t}\right)\int_{0}^{1}g\left(x;p,q,t\right)dx+2\left(-1\right)^{p+t}\sum_{j=0}^{\left[\frac{t}{2}\right]}\frac{(2q)^{t-2j}}{(t-2j)!}\eta(2j)\int_{0}^{1}\frac{\ln^{p+t-2j}\left(x\right)}{1-x^{2}}dx.$$

The integral

$$K_{-}^{2}\left(0,p,q,t\right)=\int_{0}^{1}\frac{\ln^{p}\left(x\right)\mathrm{Li}_{t}(-x^{2q})}{1-x^{2}}dx$$

and

$$\int_{0}^{1}\frac{\ln^{p+t-2j}\left(x\right)}{1-x^{2}}dx=(-1)^{p+t}\left(p+t-2j\right)!\lambda\left(p+t-1-2j\right).$$

Therefore we obtain (3.27) and the proof is finished. Note that the integral $K_{-}^{2}\left(0,p,q,t\right)$ does not contribute to $M\left(p,q,t\right)$ in the case when $p+t$ is an odd integer. The third integral in (3.26) is obtained by the substitution $x=\tanh\theta$. □

Remark 3.5. *It can be noted, from Jonquière's relation (3.30) and using the integrals in remark 3.4 that we are able to determine the value of the integrals*

$$\int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(-x^{-bq})}{1-x^b}dx = (-1)^{t+1} K_-^b(a,p,q,t) + 2(-1)^{t+1}\sum_{j=0}^{\left[\frac{t}{2}\right]}\frac{\eta(2j)}{(t-2j)!}\int_0^1 \frac{x^a \ln^p(x)\ln^{t-2j}(-x^{bq})}{1-x^b}dx. \tag{3.31}$$

Some examples follow.

Example 3.2.
1. From (3.22) and (3.24) for $q=1, a=0$,

$$\sum_{n\geq 1}\frac{(-1)^n}{(n+1)^{p+1}}H_n^{(t)} = \frac{1}{2^{p+1}}\sum_{n\geq 1}\frac{(-1)^n}{n^t}\left(H_{\frac{n}{2}}^{(p+1)} - H_{\frac{n-1}{2}}^{(p+1)}\right),$$

from the polygamma multiplication formula [30]

$$2^{p+1}H_n^{(p+1)} = 2^{p+1}\eta(p+1) + H_{\frac{n}{2}}^{(p+1)} + H_{\frac{n-1}{2}}^{(p+1)}$$

we can write

$$S_{t,p+1}^{+-} - \eta(p+t+1) = \sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}\left(H_n^{(p+1)} - \eta(p+1) - \frac{1}{2^p}H_{\frac{n}{2}}^{(p+1)}\right)$$

and therefore

$$\frac{1}{2^p}\sum_{n\geq 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(p+1)}}{n^t} = \eta(p+t+1) - \eta(p+1)\eta(t) + S_{p+1,t}^{+-} - S_{t,p+1}^{+-}.$$

If $p+1=t$,

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{\frac{n}{2}}^{(t)}}{n^t} = 2^{t-1}\left(\eta(2t) - \eta^2(t)\right).$$

2. From Theorem 3.6 with $q=2, a=0$, we have

$$\begin{aligned}
&\zeta(p+1)\eta(t) - \sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n}^{(p+1)}}{n^t}\\
=&\frac{1}{2^{p+1}}\left(S_{t,p+1}^{++}\left(0,\frac{1}{2}\right) + S_{t,p+1}^{++}(0,1)\right)\\
-&\frac{1}{2^{1+2p+t}}\left(S_{t,p+1}^{++}\left(0,\frac{1}{4}\right) + S_{t,p+1}^{++}\left(0,\frac{1}{2}\right) + S_{t,p+1}^{++}\left(0,\frac{3}{4}\right) + S_{t,p+1}^{++}(0,1)\right).
\end{aligned}$$

Simplifying we obtain the new identity

$$\begin{aligned}
\sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n}^{(p+1)}}{n^t} &= \zeta(p+1)\eta(t) + 2^{-1-2p-t}\left(S_{t,p+1}^{++}\left(0,\frac{1}{4}\right) + S_{t,p+1}^{++}\left(0,\frac{3}{4}\right)\right)\\
&+\left(2^{-1-2p-t} - 2^{-1-p}\right)\left(S_{t,p+1}^{++} - \zeta(p+t+1) + S_{t,p+1}^{++}\left(0,\frac{1}{2}\right)\right).
\end{aligned}$$

In particular, when $t=1$ we obtain an analogous identity, (to (5.6) in [1])

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n}^{(p+1)}}{n}=\zeta(p+1)\ln 2+2^{-2-2p}\left(S_{1,p+1}^{++}\left(0,\frac{1}{4}\right)+S_{1,p+1}^{++}\left(0,\frac{3}{4}\right)\right)$$
$$+\left(2^{-2-2p}-2^{-1-p}\right)\left(S_{1,p+1}^{++}-\zeta(p+2)+S_{1,p+1}^{++}\left(0,\frac{1}{2}\right)\right),$$

the expression $S_{1,p+1}^{++}(0,\alpha)=\sum_{n\geq 1}\frac{H_n}{(n+\alpha)^{p+1}}$ can be evaluated by Theorem 2.4.

3. For $a=1, p=4, q=2, t=1$

$$K_-(1,4,2,1)=\int_0^1\frac{x\ln^4(x)\,\mathrm{Li}_1(-x^2)}{1-x}dx$$
$$=24\sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n+1}^{(5)}}{n}-24\zeta(5)\ln 2$$
$$=48(1-G)\beta(4)+48G-240+12\pi+24\ln 2+\frac{3}{2}\pi^3$$
$$+\frac{5}{32}\pi^5+\frac{15453}{256}\zeta(6)-\frac{27}{128}\zeta^2(3)-\frac{1581}{64}\zeta(5)\ln 2.$$

4. For $a=-\frac{3}{2}, p=0, q=1, t=2$

$$\int_0^1\frac{x^{-3/2}\mathrm{Li}_2(-x)}{1-x}dx=-2\sum_{n\geq 1}\frac{(-1)^{n+1}H_n^{(2)}}{2n-1}=\frac{11}{45}\zeta(3)+\zeta(2)$$
$$+\frac{\pi}{4}\ln^2 2+4\ln 2-2\pi-4G\ln 2-8W(3).$$

5. For $a=0, p=0, q=\frac{1}{2}, t=3$

$$\int_0^1\frac{\mathrm{Li}_3(-x^{1/2})}{1+x}dx=-2\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^3}\left(H_{\frac{n}{2}}-\ln 2-H_{\frac{n}{4}}\right)$$
$$=\frac{65}{128}\zeta(4)-\eta(3)\ln 2-\frac{3}{8}L(3).$$

6. For $p=t-1, q>0, t\in\mathbb{N}$

$$M(t-1,q,t)=\int_0^\infty\frac{\ln^{t-1}(x)\,\mathrm{Li}_t(-x^{2q})}{1-x^2}dx$$
$$=2(t-1)!\sum_{j=0}^{\left[\frac{t}{2}\right]}(2q)^{t-2j}\binom{2t-2j-1}{t-1}\eta(2j)\lambda(2t-2j),$$

where $\eta(0) = \frac{1}{2}$.
7. For $p = 2m-1$*, where* $m \in \mathbb{N},\ q = 1, t \in \mathbb{N}$

$$\begin{aligned}\frac{1}{(2m-1)!} M\left(2m-1, 1, t\right) &= \frac{1}{(2m-1)!}\int_0^\infty \frac{\ln^{2m-1}(x)\operatorname{Li}_t(-x^2)}{1-x^2}dx \\ &= \left(\frac{\zeta\left(2m\right)}{2^{2m}} + \eta(2m)\right)\eta(t) - \sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}\left(H_{2n}^{(2m)} - \frac{1}{2^{2m}}H_n^{(2m)}\right) \\ &+ 2\sum_{j=0}^{\left[\frac{t}{2}\right]}(2)^{t-2j}\begin{pmatrix} 2m-1+t-2j \\ 2m-1\end{pmatrix}\eta(2j)\lambda\left(2m+t-2j\right).\end{aligned}$$

In particular

$$M\left(7, 1, 1\right) = \frac{427}{64}\pi^7 G + \frac{17}{16}\pi^8\ln 2 + \frac{525}{8}\pi^5\beta\left(4\right) + 630\pi^3\beta\left(6\right) + 5040\pi\beta\left(8\right).$$

8. For $p = t,\ q = \frac{1}{2}, t \in \mathbb{N}$

$$\begin{aligned}\frac{1}{2}M\left(t, \frac{1}{2}, t\right) = \frac{1}{2}\int_0^\infty \frac{\ln^t(x)\operatorname{Li}_t(-x)}{1-x^2}dx &= \sum_{j=0}^{\left[\frac{t}{2}\right]} t!\begin{pmatrix} 2t-2j \\ t\end{pmatrix}\eta(2j)\lambda\left(2t+1-2j\right) \\ &+ (-1)^t t!\left(S_{t+1,t}^{+-} - \frac{1}{2^{t+1}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}H_{\frac{n}{2}}^{(t+1)}\right) \\ &- (-1)^t t!\eta(t)\left(\eta(t+1) + \frac{1}{2^{t+1}}\zeta(t+1)\right)\end{aligned}$$

and the Euler sum $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}H_{\frac{n}{2}}^{(t+1)}$ *can be explicitly evaluated by the techniques developed in* [26], [28] *and* [29]*. Other authors have also evaluated particular case of these integrals, Coffey* [7] *has evaluated, amongst other results,* $K_+\left(0, 1, 1, 2\right)$.
9. For $a = \frac{1}{2}, p = 2, q = 1, t = 2$

$$\begin{aligned}K_-\left(\frac{1}{2}, 2, 1, 2\right) &= \int_0^1 \frac{\sqrt{x}\ln^2(x)\operatorname{Li}_2(-x)}{1-x}dx = 48\pi\beta\left(4\right) + 384 - 128G - 48\pi \\ &+ 8\zeta\left(2\right) + 2\pi^3 G - 2\pi^3 - 96\ln 2 - 7\zeta\left(2\right)\zeta\left(3\right) - 186\zeta\left(5\right)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \frac{\sqrt{x}\ln^2(x)\operatorname{Li}_2(-\frac{1}{x})}{1-x}dx &= 128G - 48\pi\beta\left(4\right) + 48\pi + 8\zeta\left(2\right) \\ &+ 96\ln 2 - 2\pi^3 G - 7\zeta\left(2\right)\zeta\left(3\right) - 186\zeta\left(5\right).\end{aligned}$$

Remark 3.6. *From Theorem 3.7 we can identify a new Euler identity in the case of even weight $p+t$. Consider the case $p=t$, then we can write*

$$2K_{-}^{2}\left(0,t,q,t\right)=M\left(t,q,t\right)-2t!\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(2q\right)^{t-2j}\begin{pmatrix}2t-2j\\t\end{pmatrix}\eta(2j)\lambda\left(2t+1-2j\right)$$

from which we extract the Euler identity

$$\sum_{n\geq1}\frac{\left(-1\right)^{n+1}}{n^{t}}\left(H_{2qn}^{(t+1)}-\frac{1}{2^{t+1}}H_{qn}^{(t+1)}\right)=\frac{\left(-1\right)^{t}}{2t!}M\left(t,q,t\right)$$
$$+\left(-1\right)^{t}\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(2q\right)^{t-2j}\begin{pmatrix}2t-2j\\t\end{pmatrix}\eta(2j)\lambda\left(2t+1-2j\right)+\eta(t)\left(\eta(t+1)+\frac{1}{2^{t+1}}\zeta(t+1)\right).\tag{3.32}$$

Since for $q=1$ Flajolet and Salvy [13] give explicit values for $S_{t+1,t}^{+-}$, then we can obtain an explicit identity for $\sum_{n\geq1}\frac{(-1)^{n+1}H_{2n}^{(t+1)}}{n^{t}}$. Iterating for values $q=1,2,3...$ allows us to obtain new Euler sum identities for $\sum_{n\geq1}\frac{(-1)^{n+1}H_{2qn}^{(t+1)}}{n^{t}}$, $t\in\mathbb{N}$. Let

$$S_{p,t}^{+-}\left(\alpha,\beta;q\right)=\sum_{n\geq1}\frac{\left(-1\right)^{n+1}H_{qn}^{(p)}\left(\alpha\right)}{\left(n+\beta\right)^{t}},$$

then from (3.32) we offer the following examples.

$$S_{2,1}^{+-}\left(0,0;2\right)=2\zeta\left(3\right)-\frac{1}{2}\pi G-\frac{1}{8}\zeta\left(2\right)\ln 2.$$
$$S_{3,2}^{+-}\left(0,0;2\right)=3\pi\beta\left(4\right)+\frac{1}{8}\pi^{3}G-\frac{2997}{256}\zeta\left(5\right)+\frac{3}{32}\zeta\left(2\right)\zeta\left(3\right).$$

$$\begin{aligned}S_{3,2}^{+-}\left(0,0;4\right)&=\frac{\pi^{2}}{512\sqrt{2}}\begin{pmatrix}3\pi\left(\psi'\left(\frac{1}{8}\right)+\psi'\left(\frac{3}{8}\right)-\psi'\left(\frac{5}{8}\right)-\psi'\left(\frac{7}{8}\right)\right)\\-\psi''\left(\frac{1}{8}\right)+\psi''\left(\frac{3}{8}\right)+\psi''\left(\frac{5}{8}\right)-\psi''\left(\frac{7}{8}\right)\end{pmatrix}\\&+\frac{\pi}{512\sqrt{2}}\left(\psi'''\left(\frac{1}{8}\right)+\psi'''\left(\frac{3}{8}\right)-\psi'''\left(\frac{5}{8}\right)-\psi'''\left(\frac{7}{8}\right)\right)\\&+\frac{1}{8}\left(3\pi\beta\left(4\right)+\frac{1}{8}\pi^{3}G-\frac{2997}{256}\zeta\left(2\right)+\frac{3}{32}\zeta\left(2\right)\zeta\left(3\right)\right)-186\zeta\left(5\right).\end{aligned}$$

In the case where $p=2,q=2,t=3$, we can evaluate the result

$$S_{2,3}^{+-}\left(0,0;2\right)=\frac{1973}{128}\zeta\left(5\right)+\frac{61}{32}\zeta\left(2\right)\zeta\left(3\right)-6\pi\beta\left(4\right).$$

The case $p=t+1,t\in\mathbb{N}$ and $\beta=1$ results in

$$S_{t+1,t}^{+-}\left(0,1;q\right)+S_{t+1,t}^{+-}\left(0,0;q\right)=\frac{\eta\left(2t+1\right)}{q^{t+1}}+\sum_{j=1}^{q-1}\sum_{r=0}^{t-1}q^{r}\begin{pmatrix}t+r\\r\end{pmatrix}\frac{\eta\left(t-r\right)}{j^{t+r+1}}$$
$$+\sum_{j=1}^{q-1}\sum_{r=0}^{t}\left(-1\right)^{r}\begin{pmatrix}t+r-1\\r\end{pmatrix}\frac{q^{t}}{j^{t+r}}\sum_{n\geq1}\frac{\left(-1\right)^{n+1}}{\left(qn-j\right)^{t+1-r}}.\tag{3.33}$$

The result (3.33) follows from the consideration

$$S^{+-}_{t+1,t}(0,1;q)+S^{+-}_{t+1,t}(0,0;q)=\sum_{n\geq 1}\frac{(-1)^{n+1}}{q^{t+1}n^{t}\left(n-\frac{j}{q}\right)^{t+1}}$$

and by the known decomposition formula, originally due to Euler ([21], p.48, Eq.(9))

$$\frac{1}{n^{t}(n-\alpha)^{t+1}}=\sum_{r=0}^{t-1}(-1)^{t+1}\binom{t+r}{r}\frac{1}{n^{t+r}\alpha^{t+r+1}}+\sum_{r=0}^{t}(-1)^{r}\binom{t+r-1}{r}\frac{1}{\alpha^{t+r}(n-\alpha)^{t+1-r}}.$$

The classical identity follows, upon putting $q=1$, in which case

$$S^{+-}_{t+1,t}(0,1;1)+S^{+-}_{t+1,t}(0,0;1)=\eta(2t+1).$$

Concluding Remarks. We have extended the current available knowledge for the representation of Euler sums. Moreover, we have demonstrated two parameterized families of log-polylog families that admit solutions dependent on Euler sums and in a particular case have demonstrated a connection with the Herglotz function. As a result of this line of research we expect further studies in the areas of polylog integrals and generalized Herglotz functions.

Acknowledgement. We are indebted to the anonymous referees for their constructive and helpful remarks.

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