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Some Results Related to New Jordan Totient Double Sequence Spaces

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Abstract. The 4 dimensional (4d) Jordan totient matrix which is described by the aid of the famous Jordan's function and some new Jordan totient double sequence spaces described as the domain of this aforementioned matrix have been examined by Erdem and Demiriz [10]. In the present paper, first of all we define two new double sequence spaces by using the 4d Jordan totient matrix and we show that this newly described double sequence spaces are Banach spaces with their norm. Then, we give some inclusion relations including this spaces. Moreover, we compute the α-, $\beta(bp)$- and γ-duals and finally, we characterize some new 4d matrix transformation classes and complete this work with some significant results.

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1. Introduction

For the beginning let us give some information about two arithmetic functions that we will use frequently in our study. The Jordans's function $J_t : \mathbb{N} \to \mathbb{N}$, $k \mapsto J_t(k)$ is defined as the number of $t-$tuples of positive integers all less then or equal to k that form a coprime with $(t+1)$-tuples together with k, where $k, t \in \mathbb{N}$ and $\mathbb{N} = \{1, 2, ...\}$. The equation $J_t(k_1k_2) = J_t(k_1)J_t(k_2)$ holds for the coprime numbers $k_1, k_2 \in \mathbb{N}$, that is J_t is multiplicative. If ${a_1}^{b_1}{a_2}^{b_2}{a_3}^{b_3}...{a_i}^{b_i}$ is the prime factorization of $k \in \mathbb{N}$ for $k > 1$, then,

$$J_t(k) = k^t\left(1-\frac{1}{a_1^t}\right)\left(1-\frac{1}{a_2^t}\right)\left(1-\frac{1}{a_3^t}\right)...\left(1-\frac{1}{a_i^t}\right).$$

It should be noted that for $t = 1$, the Jordan's function is reduced to the famous Euler-totient function φ. It is known from [8],

$$\sum_{k|m} J_t(k) = m^t \quad , \quad \sum_{k|m}\frac{\mu(k)}{k^t} = \frac{J_t(m)}{m^t} \quad \text{and} \quad \sum_{k|m}\mu\left(\frac{m}{k}\right)k^t = J_t(m),$$

and the Möbius function μ is defined as follows:

$$\mu(k) := \begin{cases} 1 & , \quad k = 1 \\ (-1)^i & , \quad k = a_1a_2...a_i, \text{ where } a_1, a_2, ..., a_i \text{ are different prime numbers} \\ 0 & , \quad a^2 \mid k \text{ for at least one prime number } a \end{cases}$$

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for $k \in \mathbb{N}$. If ${a_1}^{b_1}{a_2}^{b_2}{a_3}^{b_3}...{a_i}^{b_i}$ is the prime factorization of $k \in \mathbb{N}$ such that $k > 1$, then, $\sum_{k|m} k\mu(k) = (1-a_1)(1-a_2)(1-a_3)...(1-a_i)$. For $m \neq 1$, the equation $\sum_{k|m}\mu(k) = 0$ satisfies and $\mu(k_1k_2) = \mu(k_1)\mu(k_2)$, where $k_1, k_2 \in \mathbb{N}$ are coprime.

Moreover, if $r_1, r_2 \in \mathbb{N}$ are relatively prime, $\mu(r_1r_2) = \mu(r_1)\mu(r_2)$. Therefore, the Möbius function is a multiplicative function, too. Also, the equality $\sum_{t|r}\mu(t) = 0$ satisfies for $r \neq 1$.

A double sequence is the function defined as $f : \mathbb{N}\times\mathbb{N} \to \wp$, $(k,l) \mapsto f(k,l) = x_{kl}$ is called as *double sequence*, where $\wp \neq \emptyset$. $\Omega := \{x = (x_{kl}) : x_{kl} \in \mathbb{C}, \quad \forall k,l \in \mathbb{N}\}$ represents the space of all double sequences and any linear subspace of Ω is entitled as *double sequence space*. Here, $\mathbb{C}$ represents the set of all complex numbers. $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_r$, $\mathcal{L}_q$ $(0 < q < \infty)$ and $\mathcal{L}_u$ are the spaces of all bounded, convergent in the Pringsheim's sense (or shortly p-convergent), regularly convergent, q-absolutely summable and absolutely summable double sequences, respectively. It is worth mentioning that any p-convergent double sequence can be unbounded. For instance, considering the sequence $x = (x_{kl})$ defined as

$$x_{kl} = \begin{pmatrix} 1^2 & 2^2 & 3^2 & \cdots & l^2 & \cdots \\ 2^2 & 0 & 0 & \cdots & 0 & \cdots \\ 3^2 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ k^2 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix},$$

it can be easily seen that $x \in \mathcal{C}_p \setminus \mathcal{M}_u$. The space $\mathcal{C}_{bp}$ is represented as $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$ and Móricz [16] showed that the spaces $\mathcal{M}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$ are Banach spaces with the norm $\|x\|_\infty = \sup_{k,l\in\mathbb{N}}|x_{kl}|$. We denote by $\mathcal{BS}$ and $\mathcal{CS}_\vartheta$ the spaces of all bounded and $\vartheta-$convergent series, respectively.

Assume that $x \in \Omega$ and $R = (r_{mn})$ described as $r_{mn} := \sum_{k=1}^{m}\sum_{l=1}^{n} x_{kl}$, $(m,n \in \mathbb{N})$. Then, the pair $((x_{mn}),(r_{mn}))$ and $R = (r_{mn})$ are called as *double series* and the *sequence of partial sums* of the double series, respectively.

The sum of a double series $\sum_{k,l} x_{kl}$ relating to ϑ-convergence rule is described by $\vartheta-\sum_{k,l} x_{kl} = \vartheta-\lim_{m,n\to\infty}\sum_{k,l}^{m,n} x_{kl}$. In the remainder part of the study, we assume that $\sum_{k,l} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}$, $\vartheta \in \{p, bp, r\}$ and $q' = q/(q-1)$ for $1 < q < \infty$. The sequence $e^{mn} = (e_{kl}^{mn})$ described as $e_{kl}^{mn} = 1$ if $(m,n) = (k,l)$ and $e_{k,l}^{m,n} = 0$ otherwise, and $e = \sum_{m,n} e^{m,n}$ (coordinatewise sum). Let us remember the definition of 4d triangle matrix. If $b_{mnkl} = 0$ for $k > m$ or $l > n$ or both for every $m,n,k,l \in \mathbb{N}$, it is said that $B = (b_{mnkl})$ is a *triangular matrix* and also if $b_{mnmn} \neq 0$ for every $m,n \in \mathbb{N}$, then B is called *triangle*. It should be noted by [3] that, if B is a triangle, then its unique inverse B^{-1} is a triangle, too.

We say that the 4d matrix $B = (b_{mnkl})$ describes a *matrix transformation* from $\Psi \in \Omega$ into $\Lambda \in \Omega$ and it is shown as $B : \Psi \to \Lambda$, if for every $x \in \Psi$, the B-transform $(Bx)_{mn} = \vartheta - \sum_{k,l} b_{mnkl}x_{kl}$ of x exists and is in Λ for each $m,n \in \mathbb{N}$. $(\Psi : \Lambda)$ represents the class of every 4d matrices from Ψ into Λ. Also, $B \in (\Psi : \Lambda)$ if and only if $Bx \in \Lambda$ for all $x \in \Psi$ and $B_{mn} \in \Psi^{\beta(\vartheta)}$, where $B_{mn} = (b_{mnkl})_{k,l\in\mathbb{N}}$, $m,n \in \mathbb{N}$. The set

$$\Psi_B^{(\vartheta)} := \left\{ x = (x_{kl}) \in \Omega : Bx := \left(\vartheta - \sum_{kl} b_{mnkl}x_{kl}\right)_{m,n\in\mathbb{N}} \text{ exists and is in } \Psi \right\}$$

represents ϑ-summability domain.

Recently, several mathematicians have been studied the domains of some 4d triangle matrices and it is listed some of them in Table 1;

Table 1. Domains of some 4d triangle matrices

B	Ψ	Ψ_B	Refer to:
$\Delta(1,-1,1,-1)$	$\mathcal{M}_u, \mathcal{C}_{0p}, \mathcal{C}_p, \mathcal{C}_r, \mathcal{L}_q$	$\mathcal{M}_u(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_r(\Delta), \mathcal{L}_q(\Delta)$	[4]
C	$\mathcal{M}_u, \mathcal{C}_{0p}, \mathcal{C}_p, \mathcal{C}_r, \mathcal{C}_{bp}, \mathcal{L}_q$	$\tilde{\mathcal{M}}_u, \tilde{\mathcal{C}_{0p}}, \tilde{\mathcal{C}_p}, \tilde{\mathcal{C}_r}, \tilde{\mathcal{C}_{bp}}, \tilde{\mathcal{L}_q}$	[19]
C	$\tilde{\mathcal{M}}_u, \tilde{\mathcal{C}_{0p}}, \tilde{\mathcal{C}_p}, \tilde{\mathcal{C}_r}, \tilde{\mathcal{C}_{bp}}, \tilde{\mathcal{L}_q}$	$\tilde{\mathcal{M}}_u(t), \tilde{\mathcal{C}_{0p}}(t), \tilde{\mathcal{C}_p}(t), \tilde{\mathcal{C}_r}(t), \tilde{\mathcal{C}_{bp}}(t), \tilde{\mathcal{L}_q}(t)$	[5]
R^{qt}	$\mathcal{L}_s$	$R^{qt}(\mathcal{L}_s)$	[30]
E(r,s)	$\mathcal{L}_p, \mathcal{M}_u$	$\varepsilon_p^{r,s}, \varepsilon_\infty^{r,s}$	[23]
B(r,s,t,u)	$\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{C}_p, \mathcal{C}_r, \mathcal{L}_q$	$B(\mathcal{M}_u), B(\mathcal{C}_{bp}), B(\mathcal{C}_p), B(\mathcal{C}_r), B(\mathcal{L}_q)$	[24]
B(r,s,t,u)	$\mathcal{C}_f, \mathcal{C}_{f_0}$	$B(\mathcal{C}_f), B(\mathcal{C}_{f_0})$	[25]
$\tilde{B}$	$\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{C}_p, \mathcal{C}_r, \mathcal{L}_q$	$\tilde{B}(\mathcal{M}_u), \tilde{B}(\mathcal{C}_{bp}), \tilde{B}(\mathcal{C}_p), \tilde{B}(\mathcal{C}_r), \tilde{B}(\mathcal{L}_q)$	[27]
$\Phi^\star$	$\mathcal{L}_p$	$\Phi^\star(\mathcal{L}_p)$	[7]
$\Phi^\star$	$\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{C}_p, \mathcal{C}_r$	$\Phi^\star(\mathcal{M}_u), \Phi^\star(\mathcal{C}_{bp}), \Phi^\star(\mathcal{C}_p), \Phi^\star(\mathcal{C}_r)$	[9]
$\mathcal{J}^t$	$\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{C}_p, \mathcal{C}_r, \mathcal{L}_s$	$\mathcal{J}^t_\infty, \mathcal{J}^t_{bp}, \mathcal{J}^t_p, \mathcal{J}^t_r, \mathcal{J}^t_s$	[10]

where $\Delta(1,-1,1,-1)$, C, R^{qt}, E(r,s), B(r,s,t,u), $\tilde{B}$, $\Phi^\star$ and $\mathcal{J}^t$ denote the 4d difference, Cesàro, Riesz, Euler, generalized difference, sequential band, Euler-totient and Jordan totient matrices, respectively. In addition, readers who want to reach the subjects arithmetic functions, summability theory, double sequence spaces and related topics can use the studies [1, 2, 6, 12–15, 20–22, 31, 32].

2. Almost Convergent Jordan Totient Double Sequence Spaces

It is said that $x \in \Omega$ is almost convergent if

$$p-\lim_{\varrho,\varrho'\to\infty}\sup_{m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}x_{kl}-L\right|=0$$

and stated by $f_2-\lim x = L$. We denote by

$$\mathcal{C}_f = \left\{x=(x_{kl})\in\Omega : \exists L\in\mathbb{C}\ni \quad p-\lim_{\varrho,\varrho'\to\infty}\sup_{m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}x_{kl}-L\right|=0,\ \textit{uniformly in } m,n\right\},$$

the space of all almost convergent double sequences. Moreover, the space of all almost null double sequences is represented by $\mathcal{C}_{f_0}$. It is also significant to say that the inclusion $\mathcal{C}_{bp}\subset\mathcal{C}_{f_0}\subset\mathcal{C}_f\subset\mathcal{M}_u$ is valid.

In this section, we describe the sets $\mathcal{J}^t_f$ and $\mathcal{J}^t_{f_0}$ whose elements are double sequences by using domains of 4d Jordan totient matrix on $\mathcal{C}_f$ and $\mathcal{C}_{f_0}$, respectively, show that these aforementioned sets are Banach spaces with their norm. Furthermore, we prove that the spaces $\mathcal{J}^t_f$ and $\mathcal{J}^t_{f_0}$ are linearly norm isomorphic to the spaces $\mathcal{C}_f$ and $\mathcal{C}_{f_0}$, respectively and give inclusion relations related these newly described spaces.

In [10], we have defined the 4d Jordan totient matrix $\mathcal{J}^t=(j^t_{mnkl})$ $(t\in\mathbb{N})$ by

$$j^t_{mnkl} := \begin{cases}\dfrac{J_t(k)J_t(l)}{(mn)^t}, & k\mid m,\ l\mid n,\\[2ex] 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

For $t=1$, the 4d Jordan totient matrix is reduced to the 4d Euler-totient matrix $\Phi^\star$. The $\mathcal{J}^t$-transform of a double sequence $x=(x_{kl})$ is given by

$$y_{mn} := \left(\mathcal{J}^t x\right)_{mn} = \frac{1}{(mn)^t}\sum_{k|m,l|n}J_t(k)J_t(l)x_{kl}. \tag{2.2}$$

The inverse $(\mathcal{J}^t)^{-1}=(j^{t^{-1}}_{mnkl})$ of the triangle matrix $\mathcal{J}^t$ is calculated as

$$j^{t^{-1}}_{mnkl} := \begin{cases}\dfrac{\mu(\frac{m}{k})\mu(\frac{n}{l})}{J_t(m)J_t(n)}(kl)^t, & k\mid m,\ l\mid n,\\[2ex] 0, & \text{otherwise.}\end{cases}$$

It is obtained by applying $(\mathcal{J}^t)^{-1}$ to (2.2) that

$$x_{mn} = \sum_{k|m,l|n} \frac{\mu(\frac{m}{k})\mu(\frac{n}{l})}{J_t(m)J_t(n)}(kl)^t y_{kl}. \tag{2.3}$$

A 4d matrix B is called as RH-regular, if $Bx \in \mathcal{C}_p$ and $bp-\lim x = p-\lim Bx$ for every $x \in \mathcal{C}_{bp}$ [11]. We would like to point out that the 4d Jordan totient matrix defined by (2.1) is RH-regular from Theorem 3 in [10]. Now, we may define the double sequence spaces $\mathcal{J}^t_f$ and $\mathcal{J}^t_{f_0}$ as

$$\mathcal{J}^t_f = \left\{x=(x_{mn})\in\Omega : \exists L\in\mathbb{C} \ni p-\lim_{\varrho,\varrho'\to\infty}\sup_{m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x\right)_{kl} - L\right| = 0,\ \textit{uniformly in } m,n\right\},$$

$$\mathcal{J}^t_{f_0} = \left\{x=(x_{mn})\in\Omega : p-\lim_{\varrho,\varrho'\to\infty}\sup_{m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x\right)_{kl}\right| = 0,\ \textit{uniformly in } m,n\right\}.$$

The sets $\mathcal{J}^t_f$ and $\mathcal{J}^t_{f_0}$ may be rewritten as $\mathcal{J}^t_f = (\mathcal{C}_f)_{\mathcal{J}^t}$ and $\mathcal{J}^t_{f_0} = (\mathcal{C}_{f_0})_{\mathcal{J}^t}$, respectively.

Theorem 2.1. *The sets $\mathcal{J}^t_f$ and $\mathcal{J}^t_{f_0}$ are Banach spaces with the norm defined by*

$$\|x\|_{\mathcal{J}^t_f} = \sup_{\varrho,\varrho',m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x\right)_{kl}\right|. \tag{2.4}$$

Proof. Since it can be similarly proved for the set $\mathcal{J}^t_{f_0}$, we prove the theorem only for the set $\mathcal{J}^t_f$. It is easy to see that the set $\mathcal{J}^t_f$ is a normed linear space. So, we avoid to give the details.

Assume that a Cauchy sequence $x^{(i)} = \left\{x^{(i)}_{kl}\right\}_{k,l\mathbb{N}} \in \mathcal{J}^t_f$. In that case, $\forall\varepsilon>0$, $\exists N\in\mathbb{N}\ni$

$$\|x^{(i)}-x^{(j)}\|_{\mathcal{J}^t_f} = \sup_{\varrho,\varrho',m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left[\left(\mathcal{J}^t x^{(i)}\right)_{kl}-\left(\mathcal{J}^t x^{(j)}\right)_{kl}\right]\right| < \varepsilon \tag{2.5}$$

for all $i,j>N$. It can be known from the inequality (2.5) that, $\left\{\left(\mathcal{J}^t x^{(i)}\right)_{kl}\right\}_{i\in\mathbb{N}}$ is Cauchy sequence in the space $\mathcal{C}_f$. Since, $\mathcal{C}_f$ is a Banach space (see Remark 2.1 in [28]), we can write $\left\{\left(\mathcal{J}^t x^{(i)}\right)_{kl}\right\} \longrightarrow \{(\mathcal{J}^t x)_{kl}\}$ as $i\to\infty$. By using this infinitely many limit points, we can describe the double sequence $\{(\mathcal{J}^t x)_{kl}\}$. Now, by taking the limit as $j\to\infty$ on (2.5), we have

$$\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x^{(i)}\right)_{kl} - \frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x\right)_{kl}\right| < \varepsilon$$

for all $k,l\in\mathbb{N}$. Furthermore, since $\left\{\left(\mathcal{J}^t x^{(i)}\right)_{kl}\right\}\in\mathcal{C}_f$ and $\mathcal{C}_f\subset\mathcal{M}_u$ then for a $M\in\mathbb{R}^+$

$$\sup_{m,n\in\mathbb{N}}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x^{(i)}\right)_{kl}\right| \le M.$$

Thus, we get

$$\begin{aligned}\left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x\right)_{kl}\right| &\le \left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x^{(i)}\right)_{kl}\right| \\ &+ \left|\frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x^{(i)}\right)_{kl} - \frac{1}{(\varrho+1)(\varrho'+1)}\sum_{k=m}^{m+\varrho}\sum_{l=n}^{n+\varrho'}\left(\mathcal{J}^t x\right)_{kl}\right| \\ &< \varepsilon + M.\end{aligned}$$

By taking supremum over $m,n\in\mathbb{N}$ and p-limit as $\varrho,\varrho'\to\infty$ from the inequality above that $\mathcal{J}^t x\in\mathcal{C}_f$, that is $x\in\mathcal{J}^t_f$. We see from this approach that the space $\mathcal{J}^t_f$ is a Banach space with the norm $\|.\|_{\mathcal{J}^t_f}$ described by (2.4). □

Theorem 2.2. *The double sequence spaces $\mathcal{J}^t_f$ and $\mathcal{J}^t_{f_0}$ are linearly norm isomorphic to the spaces C_f and C_{f_0}, respectively.*

Proof. It is seen that the transformation T selected as $Tx = \mathcal{J}^t x$ for all $x \in \mathcal{J}^t_f$ (or $x \in \mathcal{J}^t_{f_0}$) described from the space $\mathcal{J}^t_f$ (or $\mathcal{J}^t_{f_0}$) into the space C_f (or C_{f_0}) is bijective and norm preserving. □

Theorem 2.3. *The inclusion $\mathcal{M}_u \subset \mathcal{J}^t_{f_0}$ holds.*

Proof. Let us select any $x = (x_{kl}) \in \mathcal{M}_u$. In that case, from the following inequality

$$\begin{aligned} \|x\|_{\mathcal{J}^t_{f_0}} &= \sup_{\varrho,\varrho',m,n\in\mathbb{N}} \left| \frac{1}{(\varrho+1)(\varrho'+1)} \sum_{k=m}^{m+\varrho} \sum_{l=n}^{n+\varrho'} \left(\mathcal{J}^t x\right)_{kl} \right| \\ &= \sup_{\varrho,\varrho',m,n\in\mathbb{N}} \left| \frac{1}{(\varrho+1)(\varrho'+1)} \sum_{k=m}^{m+\varrho} \sum_{l=n}^{n+\varrho'} \frac{1}{(kl)^t} \sum_{a|k} \sum_{b|l} J_t(a)J_t(b)x_{ab} \right| \\ &\leq \sup_{a,b\in\mathbb{N}} |x_{ab}| \sup_{\varrho,\varrho',m,n\in\mathbb{N}} \left| \frac{1}{(\varrho+1)(\varrho'+1)} \sum_{k=m}^{m+\varrho} \sum_{l=n}^{n+\varrho'} \frac{1}{(kl)^t} \sum_{a|k} \sum_{b|l} J_t(a)J_t(b) \right| \\ &= \|x\|_\infty , \end{aligned}$$

it is seen that x is in $\mathcal{J}^t_{f_0}$, as desired. □

Theorem 2.4. *The inclusion $\mathcal{J}^t_{f_0} \subset \mathcal{J}^t_f$ holds.*

Proof. If we take $x \in \mathcal{J}^t_{f_0}$ then, $\mathcal{J}^t x \in C_{f_0}$. Since, $C_{f_0} \subset C_f$, we see that $x \in \mathcal{J}^t_f$ and thus the inclusion $\mathcal{J}^t_{f_0} \subset \mathcal{J}^t_f$ holds, as claimed. □

As a consequence of Theorem 2.3 and Theorem 2.4 we reach the following:

Corollary 2.5. *The inclusion $\mathcal{M}_u \subset \mathcal{J}^t_{f_0} \subset \mathcal{J}^t_f$ holds.*

3. Dual Spaces

In this part, we calculate α-, $\beta(bp)$- and γ-duals of the space $\mathcal{J}^t_f$. If Ψ and Λ are two double sequence spaces, then the set $D(\Psi : \Lambda)$ is described as follows:

$$D(\Psi : \Lambda) = \left\{ c = (c_{kl}) \in \Omega : cx = (c_{kl}x_{kl}) \in \Lambda \quad \text{for all} \quad (x_{kl}) \in \Psi \right\}.$$

In that case, α-, $\beta(\vartheta)$- and γ-duals of the space Ψ are described as

$$\Psi^\alpha = D(\Psi : \mathcal{L}_u), \quad \Psi^{\beta(\vartheta)} = D(\Psi : \mathcal{CS}_\vartheta) \quad \text{and} \quad \Psi^\gamma = D(\Psi : \mathcal{BS}).$$

Theorem 3.1. $\left(\mathcal{J}^t_f\right)^\alpha = \mathcal{L}_u$.

Proof. To prove the theorem, we must show the validity of inclusions $\left(\mathcal{J}^t_f\right)^\alpha \subset \mathcal{L}_u$ and $\mathcal{L}_u \subset \left(\mathcal{J}^t_f\right)^\alpha$. To show the inclusion $\left(\mathcal{J}^t_f\right)^\alpha \subset \mathcal{L}_u$, assume the sequence $c = (c_{mn}) \in \left(\mathcal{J}^t_f\right)^\alpha$ but $c \notin \mathcal{L}_u$. Then, $\sum_{m,n} |c_{mn}x_{mn}| < \infty$ for all $x = (x_{mn}) \in \mathcal{J}^t_f$. If we consider $e = \sum_{m,n} e^{mn}$, we see that $e \in \mathcal{J}^t_f$. Since $ce = c \notin \mathcal{L}_u$, i.e, $\sum_{m,n} |c_{mn}| = \infty$, we obtain from $\sum_{m,n} |c_{mn}e| = \sum_{m,n} |c_{mn}| = \infty$ that $c \notin \left(\mathcal{J}^t_f\right)^\alpha$ which is a contradiction. Thus, it must be $c \in \mathcal{L}_u$ and the inclusion $\left(\mathcal{J}^t_f\right)^\alpha \subset \mathcal{L}_u$ is valid.

For the reverse inclusion, let us take the sequences $c \in \mathcal{L}_u$ and $x \in \mathcal{J}^t_f$. Consider the sequence $y \in C_f$ given by relation (2.2). Since $C_f \subset \mathcal{M}_u$, then $y \in \mathcal{M}_u$ and $\sup_{m,n} |y_{mn}| < \xi$, where $\xi \in \mathbb{R}^+$. Therefore,

$$\begin{aligned} \sum_{m,n} |c_{mn}x_{mn}| &= \sum_{m,n} |c_{mn}| \left| \sum_{k|m,l|n} \frac{\mu(\frac{m}{k})\mu(\frac{n}{l})}{J_t(m)J_t(n)} (kl)^t y_{kl} \right| \\ &< \xi \sum_{m,n} |c_{mn}| < \infty. \end{aligned}$$

Thus, we have that $c \in \left(\mathcal{J}_f^t\right)^\alpha$ and $\mathcal{L}_u \subset \left(\mathcal{J}_f^t\right)^\alpha$. Hence, $\left(\mathcal{J}_f^t\right)^\alpha = \mathcal{L}_u$. □

Now, we may give the following conditions that characterize the 4d matrix classes:

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |b_{mnkl}| < \infty, \tag{3.1}$$

$$\exists b_{kl} \in \mathbb{C} \ni \quad bp - \lim_{m,n\to\infty} b_{mnkl} = b_{kl} \quad \text{for every} \quad k,l \in \mathbb{N}, \tag{3.2}$$

$$\exists L \in \mathbb{C} \ni \quad bp - \lim_{m,n\to\infty} \sum_{k,l} b_{mnkl} = L, \tag{3.3}$$

$$\exists k_0 \in \mathbb{N} \ni \quad bp - \lim_{m,n\to\infty} \sum_{l} \left|b_{m,n,k_0,l} - b_{k_0,l}\right| = 0, \quad \forall l \in \mathbb{N}, \tag{3.4}$$

$$\exists l_0 \in \mathbb{N} \ni \quad bp - \lim_{m,n\to\infty} \sum_{k} \left|b_{m,n,k,l_0} - b_{k,l_0}\right| = 0, \quad \forall k \in \mathbb{N}, \tag{3.5}$$

$$bp - \lim_{m,n\to\infty} \sum_k \sum_l |\triangle_{01} b_{mnkl}| = 0, \tag{3.6}$$

$$bp - \lim_{m,n\to\infty} \sum_k \sum_l |\triangle_{10} b_{mnkl}| = 0, \tag{3.7}$$

where $\triangle_{10} b_{mnkl} = b_{mnkl} - b_{mn,k+1,l}$ and $\triangle_{01} b_{mnkl} = b_{mnkl} - b_{mnk,l+1}$ for all $m,n,k,l \in \mathbb{N}$.

Lemma 3.2. *[17, 25]*

(i): *$B = (b_{mnkl}) \in (\mathcal{C}_f : \mathcal{C}_{bp})$ if and only if the conditions (3.1)-(3.7) hold.*

(ii): *$B = (b_{mnkl}) \in (\mathcal{C}_f : \mathcal{M}_u)$ if and only if $B_{mn} \in \left(\mathcal{C}_f\right)^{\beta(\vartheta)}$ and the condition (3.1) holds.*

Now, consider the sets ϖ_f which are defined by

$$\varpi_f = \{c = (c_{mn}) \in \Omega : \text{Condition } (3.f) \text{ holds with } o_{mnkl} \text{ instead of } b_{mnkl}\},$$

where the 4d matrix $O = (o_{mnkl}) = \sum_{a=k,k|a}^{m} \sum_{b=l,l|b}^{n} \frac{\mu\left(\frac{a}{k}\right)\mu\left(\frac{b}{l}\right)}{J_t(a)J_t(b)} (kl)^t c_{ab}$ and $1 \le f \le 7$.

Theorem 3.3. $\left(\mathcal{J}_f^t\right)^{\beta(bp)} = \bigcap_{k=1}^{7} \varpi_k$.

Proof. Suppose that $c = (c_{mn}) \in \Omega$ and $x = (x_{mn}) \in \mathcal{J}_f^t$. Thus, there exists $y = (y_{mn}) \in \mathcal{C}_f$ with $\mathcal{J}^t x = y$. We obtain by the relation (2.3) that

$$\begin{aligned} z_{mn} &= \sum_{k,l=1}^{m,n} c_{kl} x_{kl} \\ &= \sum_{k,l=1}^{m,n} c_{kl} \sum_{a|k,b|l} \frac{\mu\left(\frac{k}{a}\right)\mu\left(\frac{l}{b}\right)}{J_t(k)J_t(l)} (ab)^t y_{ab} \\ &= \sum_{k,l=1}^{m,n} \left[\sum_{a=k,k|a}^{m} \sum_{b=l,l|b}^{n} \frac{\mu\left(\frac{a}{k}\right)\mu\left(\frac{b}{l}\right)}{J_t(a)J_t(b)} (kl)^t c_{ab} \right] y_{kl} \\ &= (Oy)_{mn} \end{aligned} \tag{3.8}$$

for all $m,n \in \mathbb{N}$, where $O = (o_{mnkl})$ defined by

$$o_{mnkl} := \begin{cases} \displaystyle\sum_{a=k,k|a}^{m} \sum_{b=l,l|b}^{n} \frac{\mu\left(\frac{a}{k}\right)\mu\left(\frac{b}{l}\right)}{J_t(a)J_t(b)} (kl)^t c_{ab} & , \quad 1 \le k \le m,\ 1 \le l \le n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for every $m,n,k,l \in \mathbb{N}$. Then, by considering the equality (3.8), we deduce that $cx \in \mathcal{CS}_{bp}$ whenever $x \in \mathcal{J}_f^t$ if and only if $z = (z_{mn}) \in \mathcal{C}_{bp}$ whenever $y \in \mathcal{C}_f$. This implies that $c \in \left(\mathcal{J}_f^t\right)^{\beta(bp)}$ if and only if $O \in \left(\mathcal{C}_f : \mathcal{C}_{bp}\right)$. Hence, we see that $\left(\mathcal{J}_f^t\right)^{\beta(bp)} = \bigcap_{k=1}^{7} \varpi_k$ in view of part (i) of Lemma 3.2.

□

Theorem 3.4. $\left(\mathcal{J}_f^t\right)^{\gamma} = \varpi_1 \cap \mathcal{CS}_\vartheta$.

Proof. Let us choose $c = (c_{mn}) \in \Omega$ and $x = (x_{mn}) \in \mathcal{J}_f^t$. Then, $y = \mathcal{J}^t x \in \mathcal{C}_f$. Therefore, $cx \in \mathcal{BS}$ whenever $x \in \mathcal{J}_f^t$ if and only if $z \in \mathcal{M}_u$ whenever $y \in \mathcal{C}_f$. This means that $c \in \left(\mathcal{J}_f^t\right)^{\gamma}$ if and only if $O \in \left(\mathcal{C}_f : \mathcal{M}_u\right)$, where O and z defined as in Theorem 3.3. In that case, it is achieved from the conditions of the part (ii) of Lemma 3.2 that $O_{mn} \in \left(\mathcal{C}_f\right)^{\beta(\vartheta)}$ for each fixed $m, n \in \mathbb{N}$ and

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} \left| \sum_{a=k,k|a}^{m} \sum_{b=l,l|b}^{n} \frac{\mu\left(\frac{a}{k}\right)\mu\left(\frac{b}{l}\right)}{J_t(a)J_t(b)} (kl)^t c_{ab} \right| < \infty.$$

Therefore, it is obvious that $\left(\mathcal{J}_f^t\right)^{\gamma} = \varpi_1 \cap \mathcal{CS}_\vartheta$, as claimed. □

4. Some Matrix Transformations

Now, we will give the classes $(\mathcal{J}_f^t : \Lambda)$ and $(\Psi : \mathcal{J}_f^t)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{C}_f\}$ and $\Psi \in \{\mathcal{M}_u, \mathcal{C}_{bp}, \mathcal{C}_p, \mathcal{C}_r, \mathcal{C}_f, \mathcal{L}_q\}$. Before these, it is needed to give the following conditions which will be utilized in Lemma 4.1.

$$\exists b_{kl} \in \mathbb{C} \ni bp - \lim_{\varrho,\varrho'\to\infty} \kappa(k,l,\varrho,\varrho',m,n) = b_{kl} \quad \text{uniformly in } m,n \in \mathbb{N} \text{ for each } k,l \in \mathbb{N}, \tag{4.1}$$

$$\exists L \in \mathbb{C} \ni bp - \lim_{\varrho,\varrho'\to\infty} \sum_{k,l} \kappa(k,l,\varrho,\varrho',m,n) = L \text{ uniformly in } m,n \in \mathbb{N}, \tag{4.2}$$

$$\exists b_{kl} \in \mathbb{C} \ni bp - \lim_{\varrho,\varrho'\to\infty} \sum_{k} \left|\kappa(k,l,\varrho,\varrho',m,n) - b_{kl}\right| = 0 \text{ uniformly in } m,n \in \mathbb{N} \text{ for each } l \in \mathbb{N}, \tag{4.3}$$

$$\exists b_{kl} \in \mathbb{C} \ni bp - \lim_{\varrho,\varrho'\to\infty} \sum_{l} \left|\kappa(k,l,\varrho,\varrho',m,n) - b_{kl}\right| = 0 \text{ uniformly in } m,n \in \mathbb{N} \text{ for each } k \in \mathbb{N}, \tag{4.4}$$

$$\lim_{\varrho,\varrho'\to\infty} \sum_{k} \sum_{l} \left|\triangle_{10}\kappa(k,l,\varrho,\varrho',m,n)\right| = 0 \text{ uniformly in } m,n \in \mathbb{N}, \tag{4.5}$$

$$\lim_{\varrho,\varrho'\to\infty} \sum_{l} \sum_{k} \left|\triangle_{01}\kappa(k,l,\varrho,\varrho',m,n)\right| = 0 \text{ uniformly in } m,n \in \mathbb{N}, \tag{4.6}$$

$$\exists l_0 \in \mathbb{N} \ni bp - \lim_{\varrho,\varrho'\to\infty} \sum_{k} \kappa(k,l_0,\varrho,\varrho',m,n) = \lambda_{l_0} \text{ uniformly in } m,n \in \mathbb{N}, \tag{4.7}$$

$$\exists k_0 \in \mathbb{N} \ni bp - \lim_{\varrho,\varrho'\to\infty} \sum_{l} \kappa(k_0,l,\varrho,\varrho',m,n) = \mu_{k_0} \text{ uniformly in } m,n \in \mathbb{N}, \tag{4.8}$$

$$\forall k \in \mathbb{N}, \exists l_0 \in \mathbb{N} \ni \quad b_{mnkl} = 0, \quad \forall l > l_0 \text{ and } m,n \in \mathbb{N}, \tag{4.9}$$

$$\forall l \in \mathbb{N}, \exists k_0 \in \mathbb{N} \ni \quad b_{mnkl} = 0, \quad \forall k > k_0 \text{ and } m,n \in \mathbb{N}, \tag{4.10}$$

$$\exists \lambda_{kl} \in \mathbb{C} \ni \quad f_2 - \lim_{m,n\to\infty} b_{mnkl} = \lambda_{kl} \text{ for all } k,l \in \mathbb{N}, \tag{4.11}$$

$$\forall m,n,l \in \mathbb{N}, \exists \eta_1 \in \mathbb{N} \ni \quad \kappa(k,l,\varrho,\varrho',m,n) = 0, \quad \forall \varrho,\varrho',k > \eta_1, \tag{4.12}$$

$$\forall m,n,k \in \mathbb{N}, \exists \eta_2 \in \mathbb{N} \ni \quad \kappa(k,l,\varrho,\varrho',m,n) = 0, \quad \forall \varrho,\varrho',l > \eta_2, \tag{4.13}$$

$$\sup_{m,n,k,l\in\mathbb{N}} |b_{mnkl}| < \infty, \tag{4.14}$$

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |b_{mnkl}|^q < \infty, \tag{4.15}$$

where $\kappa(k,l,\varrho,\varrho',m,n) = \sum_{r=m}^{m+\varrho} \sum_{s=n}^{n+\varrho'} \frac{b_{rskl}}{(\varrho+1)(\varrho'+1)}$, $\triangle_{10}\kappa(k,l,\varrho,\varrho',m,n) = \kappa(k,l,\varrho,\varrho',m,n) - \kappa(k+1,l,\varrho,\varrho',m,n)$ and $\triangle_{01}\kappa(k,l,\varrho,\varrho',m,n) = \kappa(k,l,\varrho,\varrho',m,n) - \kappa(k,l+1,\varrho,\varrho',m,n)$.

Lemma 4.1. *[18, 26, 29, 33] The following statements hold:*

(i): *$B = (b_{mnkl})$ is almost C_{bp}-conservative, that is, $B \in (C_{bp} : C_f)$ if and only if the conditions (3.1), (4.1)–(4.4) holds.*

(ii): *$B = (b_{mnkl})$ is almost strongly regular, that is, $B \in (C_f : C_f)_{reg}$ if and only if the conditions (3.1) and (4.1)–(4.6) hold whenever $b_{kl} = 0$, $\forall k, l = 1, 2, ...$ and $L = 1$.*

(iii): *$B = (b_{mnkl})$ is almost C_r-conservative, that is, $B \in (C_r : C_f)$ if and only if the conditions (3.1), (4.1), (4.2), (4.7) and (4.8) hold.*

(iv): *$B = (b_{mnkl})$ is almost C_p-conservative, that is, $B \in (C_p : C_f)$ if and only if the conditions (3.1), (4.1), (4.2), (4.9) and (4.10) hold.*

(v): *$B = (b_{mnkl}) \in (\mathcal{M}_u : C_f)$ if and only if the conditions (3.1) and (4.11)–(4.13) hold.*

(vi): *Let $0 < q \leq 1$. Then, $B = (b_{mnkl}) \in (\mathcal{L}_q : C_f)$ if and only if the conditions (4.11) and (4.14) hold.*

(vii): *Let $1 < q < \infty$. Then, $B = (b_{mnkl}) \in (\mathcal{L}_q : C_f)$ if and only if the conditions (4.11) and (4.15) hold.*

Theorem 4.2. *Assume that the elements of 4d matrices $B = (b_{mnkl})$ and $H = (h_{mnkl})$ are connected with the relation*

$$h_{mnkl} = \sum_{a=k,k|a}^{\infty} \sum_{b=l,l|b}^{\infty} \frac{\mu(\frac{a}{k})\mu(\frac{b}{l})}{J_t(a)J_t(b)} (kl)^t b_{mnab}.$$

Then, $B \in (\mathcal{J}_f^t : \mathcal{M}_u)$ if and only if $H \in (C_f : \mathcal{M}_u)$ and

$$B_{mn} \in \left[\mathcal{J}_f^t\right]^{\beta(\vartheta)} \text{ for all } m, n \in \mathbb{N}. \tag{4.16}$$

Proof. Assume that $B \in (\mathcal{J}_f^t : \mathcal{M}_u)$. In that case, Bx exists and $Bx \in \mathcal{M}_u$ for every $x \in \mathcal{J}_f^t$ and it also implies that $B_{mn} \in \left[\mathcal{J}_f^t\right]^{\beta(\vartheta)}$ for every $m, n \in \mathbb{N}$. From partial sums of the series $\sum_{k,l} b_{mnkl}x_{kl}$ with relation (2.3), we have

$$\begin{aligned}\sum_{k,l=1}^{i,j} b_{mnkl}x_{kl} &= \sum_{k,l=1}^{i,j} b_{mnkl}\left[\sum_{a|k,b|l} \frac{\mu(\frac{k}{a})\mu(\frac{l}{b})}{J_t(k)J_t(l)}(ab)^t y_{ab}\right] \\ &= \sum_{k,l=1}^{i,j}\left[\sum_{a=k,k|a}^{i}\sum_{b=l,l|b}^{j} \frac{\mu(\frac{a}{k})\mu(\frac{b}{l})}{J_t(a)J_t(b)}(kl)^t b_{mnab}\right] y_{kl}\end{aligned}$$

for every $i, j \in \mathbb{N}$. Then, when passing to ϑ-limit on the equality above as $i, j \to \infty$, we get $Bx = Hy$. Therefore, we obtain that $Hy \in \mathcal{M}_u$ whenever $y \in C_f$, that is $H \in (C_f : \mathcal{M}_u)$.

Conversely, suppose that $B_{mn} \in \left[\mathcal{J}_f^t\right]^{\beta(\vartheta)}$ for every $m, n \in \mathbb{N}$, $H \in (C_f : \mathcal{M}_u)$ and $x \in \mathcal{J}_f^t$ such that $y = \mathcal{J}^t x$. In that case, Bx exists and therefore, the (ς, τ)th rectangular partial sums of $\sum_{k,l} b_{mnkl}x_{kl}$ obtained as

$$\begin{aligned}(Bx)_{mn}^{[\varsigma,\tau]} &= \sum_{k,l=1}^{\varsigma,\tau} b_{mnkl}x_{kl} \\ &= \sum_{k,l=1}^{\varsigma,\tau} b_{mnkl}\left[\sum_{a|k,b|l} \frac{\mu(\frac{k}{a})\mu(\frac{l}{b})}{J_t(k)J_t(l)}(ab)^t y_{ab}\right] \\ &= \sum_{t,u=1}^{\varsigma,\tau}\left[\sum_{a=k,k|a}^{\varsigma}\sum_{b=l,l|b}^{\tau} \frac{\mu(\frac{a}{k})\mu(\frac{b}{l})}{J_t(a)J_t(b)}(kl)^t b_{mnab}\right] y_{kl}\end{aligned} \tag{4.17}$$

for every $m, n, \varsigma, \tau \in \mathbb{N}$. By taking ϑ-limit on (4.17) while $\varsigma, \tau \to \infty$, it can be easily obtain from the following equality

$$\sum_{k,l} b_{mnkl}x_{kl} = \sum_{k,l} h_{mnkl}y_{kl}$$

for every $m, n \in \mathbb{N}$ that $Bx = Hy$. Thus, $B \in (\mathcal{J}_f^t : \mathcal{M}_u)$. □

Corollary 4.3. *Suppose that $B = (b_{mnkl})$ be a 4d matrix. In that case the following statements hold:*

(i): *$B \in \left(\mathcal{J}_f^t : C_{bp}\right)$ if and only if the conditions (3.1)-(3.7) and (4.16) hold with h_{mnkl} in place of b_{mnkl},*

(ii): $B \in \left(\mathcal{J}_f^t : \mathcal{C}_f\right)_{reg}$ *if and only if the conditions (3.1), (4.1)–(4.6) and (4.16) hold whenever* $b_{kl} = 0$, $\forall k, l = 1, 2, ...$ *and* $L = 1$ *with* h_{mnkl} *in place of* b_{mnkl}.

Lemma 4.4. *[30] Let* $\Psi, \Lambda \in \Omega$, $B = (b_{mnkl})$ *be any 4d matrix and* $F = (f_{mnkl})$ *also be a 4d triangle matrix. In that case,* $B \in (\Psi : \Lambda_F)$ *if and only if* $FB \in (\Psi : \Lambda)$.

Now, let us define the 4d matrix $G = (g_{mnkl})$ by

$$g_{mnkl} = \sum_{i|m, d|n} j^t_{mnid} b_{idkl}$$

for every $m, n, k, l \in \mathbb{N}$ and give following corollary.

Corollary 4.5. *Suppose that* $B = (b_{mnkl})$ *be a 4d matrix. In that case the following statements hold:*

(i): $B \in \left(\mathcal{C}_{bp} : \mathcal{J}_f^t\right)$ *if and only if the conditions (3.1), (4.1)–(4.4) hold with* g_{mnkl} *in place of* b_{mnkl},

(ii): $B \in \left(\mathcal{C}_r : \mathcal{J}_f^t\right)$ *if and only if the conditions (3.1), (4.1), (4.2), (4.7) and (4.8) hold with* g_{mnkl} *in place of* b_{mnkl},

(iii): $B \in \left(\mathcal{C}_p : \mathcal{J}_f^t\right)$ *if and only if the conditions (3.1), (4.1), (4.2), (4.9) and (4.10) hold with* g_{mnkl} *in place of* b_{mnkl},

(iv): $B \in \left(\mathcal{M}_u : \mathcal{J}_f^t\right)$ *if and only if the conditions (3.1) and (4.11)–(4.13) hold with* g_{mnkl} *in place of* b_{mnkl},

(v): $B \in \left(\mathcal{L}_q : \mathcal{J}_f^t\right)$ *if and only if the conditions (4.11) and (4.14) hold for* $0 < q \leq 1$ *with* g_{mnkl} *in place of* b_{mnkl},

(vi): $B \in \left(\mathcal{L}_q : \mathcal{J}_f^t\right)$ *if and only if the conditions (4.11) and (4.15) hold for* $1 < q < \infty$ *with* g_{mnkl} *in place of* b_{mnkl},

(vii): $B \in \left(\mathcal{C}_f : \mathcal{J}_f^t\right)_{reg}$ *if and only if the conditions (3.1) and (4.1)–(4.6) hold whenever* $b_{kl} = 0$, $\forall k, l = 1, 2, ...$ *and* $L = 1$ *hold with* g_{mnkl} *in place of* b_{mnkl}.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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