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Research Article

On centrally extended Jordan derivations and related maps in rings

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Abstract

Let R be a ring and $Z(R)$ be the center of R. The aim of this paper is to define the notions of centrally extended Jordan derivations and centrally extended Jordan $*$-derivations, and to prove some results involving these mappings. Precisely, we prove that if a 2-torsion free noncommutative prime ring R admits a centrally extended Jordan derivation (resp. centrally extended Jordan $*$-derivation) $\delta : R \to R$ such that

$$[\delta(x), x] \in Z(R) \quad (resp.\ \ [\delta(x), x^*] \in Z(R)) \text{ for all } x \in R,$$

where $'*'$ is an involution on R, then R is an order in a central simple algebra of dimension at most 4 over its center.

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1. Introduction and notions

Throughout this paper, R denotes an associative ring with center $Z(R)$. The maximal right ring of quotients of R is denoted by $Q_{mr}(R)$ and the center of $Q_{mr}(R)$ is called the extended centroid of R and denoted by C, more information about these object can be found in [5]. For any $x, y \in R$, the symbol $[x, y]$ (resp. $x \circ y$) denotes the commutator (resp. anti-commutator) $xy - yx$ (resp. $xy + yx$). A ring R is called *prime*, if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$, and if $aRa = (0)$ implies $a = 0$, then R is called a *semiprime* ring. For any $n \in \mathbb{Z}^+$, R is called n-torsion free if $nx = 0$ for all $x \in R$, implies $x = 0$. An anti-automorphism $'*'$ of a ring R is called *involution* if it is of period 2. By a *ring with involution*, we mean a ring equipped with an involution $'*'$, it is also called $*$-ring. Let $H(R) := \{x \in R : x^* = x\}$ and $S(R) := \{x \in R : x^* = -x\}$; the elements of $H(R)$ are called *symmetric* and the elements of $S(R)$ are called *skew-symmetric*. Following Herstein [17, Ch. 6], $\overline{H(R)}$ will denote the ring generated by the symmetric elements of R.

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An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For a fixed element $a \in R$, a mapping $x \mapsto [a, x]$ is called *inner derivation* induced by $'a'$. An additive map d is called a *Jordan derivation* if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$. Obviously, every derivation is a Jordan derivation but the converse need not be true (see [4, Example 3.2.1]). Moreover, the question that "when a Jordan derivation is a derivation?" caused a new and significant area of research (see [3,9,10,18,27]). In 1957, Herstein [18] showed that for 2-torsion free prime rings , every Jordan derivation is an ordinary derivation. Later, Brešar and Vukman [9] gave a brief and elegant proof of this result. In the same year, Brešar [10] showed that for a rather wide class of rings, namely semiprime rings with 2-torsion free condition, every Jordan derivation is a derivation. Thenceforth a considerable amount of results has been proved in this direction.

Let R be a $*$-ring. An additive mapping $d : R \to R$ is called a *$*$-derivation* if $d(xy) = d(x)y^* + xd(y)$ for all $x, y \in R$ and is called a *Jordan $*$-derivation* if $d(x^2) = d(x)x^* + xd(x)$ for all $x \in R$. The notions of $*$-derivation and Jordan $*$-derivation are first mentioned in [11]. Note that the mapping $x \to ax^* - xa$, where a is a fixed element of R, is a Jordan $*$-derivation which is known as *inner Jordan $*$-derivation.* The study of Jordan $*$-derivations has been originated from the problem of representability of quadratic forms by bilinear forms (see [25,26]). Since then there has been a significant interest in the study of algebraic structure of Jordan $*$-derivations in rings and algebras, for a good cross-section we refer the reader to [2,12,21,22]. For further generalizations and recent results, see [13].

Let S be a subset of R, a mapping f is called centralizing (resp. commuting) on S, if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) for all $x \in S$. The study of commuting and centralizing mappings goes back to 1955, when Divinsky [16] proved that a simple artinian ring is commutative if it admits a commuting nontrivial automorphism. In this line of investigation, Posner [24] proved another remarkable result which states that if there exists a nonzero centralizing derivation on R, then R must be commutative. Motivated by the centralizing and commuting mappings, Ali and Dar [1] introduced $*$-centralizing and $*$-commuting mappings and defined as follows: a mapping f is called $*$-centralizing (resp. $*$-commuting) on a set S if $[f(x), x^*] \in Z(R)$ (resp. $[f(x), x^*] = 0$) for all $x \in S$.

There has been a rising literature on the investigation of centrally extended mappings in rings under various settings; for e.g. see [6], [14], [15], [23]. Continuing in this line of investigation, in this paper we introduce centrally extended Jordan derivations and give examples to show the existence of these maps in a 2-torsion free prime rings. We also show that there exists no nonzero centrally extended Jordan derivation δ on a 2-torsion free noncommutative prime ring (resp. prime ring with involution $'*'$) satisfying $[\delta(x), x] \in Z(R)$ (resp. $[\delta(x), x^*] \in Z(R)$) for all $x \in R$, unless R is an order in a central simple algebra of dimension 4 over its center. Finally, we give the notion of centrally extended Jordan $*$-derivation and provide the analogous studies.

2. Preliminaries

By s_4, we denote the standard identity in four noncommuting variables, which is defined as follows:

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)},$$

where S_4 is the symmetric group of degree 4 and $(-1)^\sigma$ is the sign of permutation $\sigma \in S_4$. It is known by the standard PI-theory that, a prime ring R satisfying s_4 can be characterized in a number of ways, as follows:

Lemma 2.1. *[8, Lemma 1] Let R be a prime ring with extended centroid C. Then the following statements are equivalent:*

(i) *R satisfies s_4.*
(ii) *R is commutative or R embeds into $M_2(F)$, for a field F.*

(iii) *R is algebraic of bounded degree 2 over C (i.e., for any $a \in R$, there exists a polynomial $x^2 + \alpha x + \beta \in C[x]$ satisfied by a).*
(iv) *R satisfies $[[x^2, y], [x, y]]$.*

Lemma 2.2. *[1, Lemma 2.2] Let R be a 2-torsion free semiprime ring with involution $'*'$. If an additive self-mapping f of R satisfies $[f(x), x^*] \in Z(R)$ for all $x \in R$, then $[f(x), x^*] = 0$ for all $x \in R$.*

Lemma 2.3. *[5, Proposition 2.1.7 (ii)] Let R be a prime ring, $Q_{mr}(R)$ be the maximal right ring of quotients of R and $\mathcal{D}$ be the set of all right dense ideals of R. Then for all $q \in Q_{mr}(R)$, there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$.*

Lemma 2.4. *[7, Proposition 3.1] Let R be a 2-torsion free semiprime ring and U be a Jordan subring of R. If an additive self-mapping $f : R \to R$ satisfies $[f(x), x] \in Z(R)$ for all $x \in U$, then $[f(x), x] = 0$ for all $x \in U$.*

Lemma 2.5. *[7, Theorem 3.2] Let R be a prime ring. If an additive mapping $F : R \to R$ is commuting on R, then there exists $\lambda \in C$ and an additive $\xi : R \to C$, such that $F(x) = \lambda x + \xi(x)$ for all $x \in R$.*

Lemma 2.6. *[17, Theorem 6.5.1] If R is a semiprime ring, then $\overline{H(R)}$ is semiprime, where $\overline{H(R)}$ is the ring generated by all symmetric elements in R.*

Lemma 2.7. *[17, Theorem 6.5.3] If R is a semiprime ring, then $Z(\overline{H(R)}) \subseteq Z(R)$.*

Lemma 2.8. *[19, Lemma 2] If R is a semiprime ring and $[S(R)^2, S(R)^2] = (0)$, then R satisfies s_4.*

Lemma 2.9. *[20, Theorem 3] Let R be a prime ring with involution $'*'$ and center $Z(R)$. If n be a fixed natural number such that $x^n \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4.*

Lemma 2.10. *[20, Theorem 7] Let R be a prime ring with involution $'*'$ and center $Z(R)$. If d is a nonzero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in S(R)$, then R satisfies s_4.*

Lemma 2.11. *[27, Lemma 1.3] Let R be a semiprime ring and $a \in R$ some fixed element. If $a[x, y] = 0$ for all $x, y \in R$, then there exists an ideal I of R such that $a \in I \subseteq Z(R)$ holds.*

3. Results on centrally extended Jordan derivations

This section deals with the study of centrally extended Jordan derivations of rings. In fact, we characterize 2-torsion free noncommutative prime rings admitting $CE-$Jordan derivations. Recently, Bell and Daif [6] introduced *centrally extended derivations* which are obviously a generalization of derivations; and they discussed the existence of these mappings in rings. Accordingly, a self-mapping d of R is called a centrally extended derivation if $d(x + y) - d(x) - d(y) \in Z(R)$ and $d(xy) - d(x)y - xd(y) \in Z(R)$ for all $x, y \in R$. Motivated by this, we now introduce a finer notion than centrally extended derivation, and call a *centrally extended Jordan derivation*, as follows:

Definition 3.1. A mapping $\delta : R \to R$ that satisfies

$$\delta(x + y) - \delta(x) - \delta(y) \in Z(R), \tag{A}$$

$$\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y) \in Z(R) \tag{B}$$

for all $x, y \in R$, is called a *centrally extended Jordan derivation* of R. We shall abbreviate this map as $CE-$Jordan derivation.

Example 3.2. Let $R = M_2(\mathbb{Z}) \times \mathbb{Z}$ be a ring and define a mapping $\delta : R \to R$ by

$$\delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x\right) = \left(\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, 1\right).$$

Then, it is straightforward to check that δ is a $CE-$Jordan derivation of R.

Remark 3.3. If R is a 2-torsion free noncommutative prime ring, then it is not difficult to see that an additive map $\delta : R \to R$ is a $CE-$Jordan derivation if and only if $\delta(x^2) - \delta(x)x - x\delta(x) \in Z(R)$ for all $x \in R$. It is natural to ask whether a $CE-$Jordan derivation can be a $CE-$derivation or a Jordan derivation? In the following counter-example, we show that in case R is a noncommutative prime ring, the above statement is not always true:
Let $\mathbb{Z}$ be the ring of integers and

$$R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}\right\}$$

be a noncommutative prime ring. Then a mapping $\delta : R \to R$ such that

$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

is a $CE-$Jordan derivation, but neither a $CE-$derivation nor a Jordan derivation.

We begin with the following lemma:

Lemma 3.4. *Let R be a 2-torsion free ring with no nonzero central ideal. If δ is a $CE-$Jordan derivation of R, then δ is additive.*

Proof. Let δ be a $CE-$Jordan derivation of R. In view of condition (A), for any $x, y, z \in R$, we have

$$\delta(x + y) = \delta(x) + \delta(y) + c_{\delta(x,y,+)}, \tag{3.1}$$

where $c_{\delta(x,y,+)} \in Z(R)$. There exists some $c_{\delta(z,x+y,\circ)} \in Z(R)$ such that

$$\begin{aligned}\delta(z \circ (x + y)) &= \delta(z) \circ (x + y) + z \circ \delta(x + y) + c_{\delta(z,x+y,\circ)} \\ &= \delta(z) \circ x + \delta(z) \circ y + z \circ (\delta(x) + \delta(y) + c_{\delta(x,y,+)}) + c_{\delta(z,x+y,\circ)} \\ &= \delta(z) \circ x + \delta(z) \circ y + z \circ \delta(x) + z \circ \delta(y) + 2zc_{\delta(x,y,+)} \\ &\quad + c_{\delta(z,x+y,\circ)}.\end{aligned} \tag{3.2}$$

Another way of looking at it is,

$$\begin{aligned}\delta(z \circ (x + y)) &= \delta(z \circ x + z \circ y) \\ &= \delta(z \circ x) + \delta(z \circ y) + c_{\delta(z\circ x,z\circ y,+)} \\ &= \delta(z) \circ x + z \circ \delta(x) + c_{\delta(z,x,\circ)} + \delta(z) \circ y + z \circ \delta(y) \\ &\quad + c_{\delta(z,y,\circ)} + c_{\delta(z\circ x,z\circ y,+)},\end{aligned} \tag{3.3}$$

where $c_{\delta(z\circ x,z\circ y,+)}, c_{\delta(z,x,\circ)}$ and $c_{\delta(z,y,\circ)}$ are the central elements.

Comparing (3.2) and (3.3), we conclude that $2zc_{\delta(x,y,+)} + c_{\delta(z,x+y,\circ)} = c_{\delta(z\circ x,z\circ y,+)} + c_{\delta(z,x,\circ)} + c_{\delta(z,y,\circ)} \in Z(R)$. It forces that $Rc_{\delta(x,y,+)} \subseteq Z(R)$, where $c_{\delta(x,y,+)}$ is a fixed central element in R, but R has no nonzero central ideal, therefore $Rc_{\delta(x,y,+)} = (0)$. Likewise, we get $c_{\delta(x,y,+)}R = (0)$. It implies that $c_{\delta(x,y,+)} \in A(R)$, the annihilator of R. But $A(R)$ is always a central ideal in R, hence our hypothesis forces $A(R) = (0)$ and consequently $c_{\delta(x,y,+)} = 0$. From (3.1), we get $\delta(x + y) = \delta(x) + \delta(y)$ for all $x, y \in R$, as desired. □

Corollary 3.5. *Let R be a 2-torsion free noncommutative prime ring. If δ is a $CE-$Jordan derivation of R, then δ is additive.*

Now, we are in position to state and prove the first result of this paper.

Theorem 3.6. *Let R be a 2-torsion free noncommutative prime ring. If R admits a $CE-$Jordan derivation $\delta : R \to R$ such that $[\delta(x), x] \in Z(R)$ for all $x \in R$, then either $\delta = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Proof. By the hypothesis, we have $[\delta(x), x] \in Z(R)$ for all $x \in R$. In view of Corollary 3.5, δ is additive and hence from Lemma 2.4, it follows that

$$[\delta(x), x] = 0 \text{ for all } x \in R. \tag{3.4}$$

Since δ is an additive and commuting function, by Lemma 2.5, there exists $\lambda \in C$ (extended centroid of R) and an additive mapping $\sigma : R \to C$ such that

$$\delta(x) = \lambda x + \sigma(x) \text{ for all } x \in R. \tag{3.5}$$

Polarizing (3.4), we have

$$[\delta(x), y] + [\delta(y), x] = 0 \text{ for all } x, y \in R.$$

Replacing y by $x \circ y$, we get

$$[\delta(x), x \circ y] + [\delta(x \circ y), x] = 0 \text{ for all } x, y \in R.$$

It implies

$$[\delta(x), x \circ y] + [\delta(x) \circ y, x] + [x \circ \delta(y), x] = 0 \text{ for all } x, y \in R.$$

Using (3.5) in the preceding relation to get

$$[\lambda x + \sigma(x), x \circ y] + [(\lambda x + \sigma(x)) \circ y, x] + [x \circ (\lambda y + \sigma(y)), x] = 0,$$

that is

$$[\lambda x, x \circ y] + [\lambda(x \circ y), x] + [\sigma(x) \circ y, x] + [\lambda(x \circ y), x] + [x \circ \sigma(y), x] = 0 \text{ for all } x, y \in R. \tag{3.6}$$

It follows that R satisfies

$$\lambda[x, x \circ y] + \lambda[x \circ y, x] + 2\sigma(x)[y, x] + \lambda[x \circ y, x] = 0, \tag{3.7}$$

and so

$$2\sigma(x)[y, x] + \lambda[x \circ y, x] = 0 \text{ for all } x, y \in R. \tag{3.8}$$

Further, it implies

$$2\sigma(x)[y, x] + \lambda[y, x^2] = 0 \text{ for all } x, y \in R,$$

and a fortiori

$$\lambda[[y, x^2], [y, x]] = 0 \text{ for all } x, y \in R.$$

It implies that either $\lambda = 0$ or $[[y, x^2], [y, x]] = 0$ for all $x, y \in R$. By Lemma 2.1, the latter case is equivalent to the s_4 identity and R is assumed to be noncommutative, therefore R is an order in a central simple algebra of dimension at most 4 over $Z(R)$.

On the other hand, let us assume that $\lambda = 0$. Then from (3.8), we have

$$2\sigma(x)[y, x] = 0 \text{ for all } x, y \in R.$$

Using the restriction on torsion of R, we have

$$\sigma(x)[y, x] = 0 \text{ for all } x, y \in R.$$

Since R is a prime ring, for each $x \in R$, either $\sigma(x) = 0$ or $[R, x] = (0)$. Put $\mathfrak{U} = \{x \in R : \ \sigma(x) = 0\}$ and $\mathfrak{V} = \{x \in R : \ [R, x] = (0)\}$. Therefore, we note that R can be written as the set-theoretic union of the additive subgroups $\mathfrak{U}$ and $\mathfrak{V}$, which is not possible. Thus, we have either $R = \mathfrak{U}$ or $R = \mathfrak{V}$. It implies that either $\sigma(x) = 0$ for all $x \in R$ or $[R, x] = (0)$ for all $x \in R$. If $\sigma(x) = 0$ for all $x \in R$, then from (3.5), we find $\delta(x) = 0$ for all $x \in R$. In the other case R is a commutative ring; which leads a contradiction. This completes the proof. □

Theorem 3.7. *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$ that admits a $CE-$Jordan derivation $\delta : R \to R$ such that $[\delta(x), x^*] \in Z(R)$ for all $x \in R$. Then either $\delta = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Proof. Let us assume that $[\delta(x), x^*] \in Z(R)$ for all $x \in R$. With the aid of Corollary 3.5 and Lemma 2.2, we have

$$[\delta(x), x^*] = 0 \text{ for all } x \in R. \tag{3.9}$$

Applying involution in (3.9), we get

$$[\delta(x)^*, x] = 0 \text{ for all } x \in R. \tag{3.10}$$

In view of Lemma 2.5, there exists $\lambda \in C$ and an additive mapping $\sigma : R \to C$ such that

$$\delta(x)^* = \lambda x + \sigma(x) \text{ for all } x \in R.$$

It implies

$$\delta(x) = \lambda^* x^* + \sigma(x)^* \text{ for all } x \in R. \tag{3.11}$$

We now split the proof into two parts, as follows:

Case 1. Suppose that there exists a non-zero element $c \in C$ such that $c^* \neq c$. Let $c^* - c = z_c$. Clearly $z_c^* = -z_c \neq 0$. Also $z_c \in C$. By Lemma 2.3, there exists a nonzero ideal J of R such that $z_c J \subseteq R$. Polarizing (3.9), we get

$$[\delta(x), y^*] + [\delta(y), x^*] = 0 \text{ for all } x, y \in R. \tag{3.12}$$

Replacing y by $y \circ r$ in (3.12), where $r \in J$, we get

$$[\delta(x), (y \circ r)^*] + [\delta(y \circ r), x^*] = 0. \tag{3.13}$$

Therefore, we have

$$[\delta(x), y^* \circ r^*] + [\delta(y) \circ r, x^*] + [y \circ \delta(r), x^*] = 0 \text{ for all } x, y \in R, r \in J. \tag{3.14}$$

In particular for $y = h \in H(R)$ in (3.14), we get

$$[\delta(x), r^* \circ h] + [r \circ \delta(h), x^*] + [\delta(r) \circ h, x^*] = 0.$$

Using (3.11) in the above expression, we obtain

$$\lambda^*[x^*, r^* \circ h] + \lambda^*[r^* \circ h, x^*] + \lambda^*[r \circ h, x^*] + 2\sigma(r)^*[h, x^*] + 2\sigma(h)^*[r, x^*] = 0.$$

It implies

$$\lambda^*[r \circ h, x^*] + 2\sigma(r)^*[h, x^*] + 2\sigma(h)^*[r, x^*] = 0 \text{ for all } x \in R, r \in J, h \in H(R). \tag{3.15}$$

Replacing r by $z_c r = r z_c$ in (3.15), we get

$$\lambda^*[r \circ h, x^*]z_c + 2\sigma(r z_c)^*[h, x^*] + 2\sigma(h)^*[r, x^*]z_c = 0 \text{ for all } x \in R, r \in J, h \in H(R). \tag{3.16}$$

Multiply (3.15) by z_c and comparing with (3.16), we obtain

$$(\sigma(r z_c)^* - \sigma(r)^* z_c)[h, x^*] = 0.$$

Therefore primeness of R implies that either $[h, x^*] = 0$ for all $h \in H(R), x \in R$ or $\sigma(r z_c)^* = \sigma(r)^* z_c$. In the former case, R satisfies s_4 identity by Lemma 2.9. But R is noncommutative, therefore R is an order in a central simple algebra of dimension at most 4 over its center. In the latter case, we have

$$\sigma(r z_c)^* = \sigma(r)^* z_c. \tag{3.17}$$

Replacing r by $r z_c$ in (3.14), we obtain

$$[\delta(x), y^* \circ r^*]z_c^* + [\delta(y) \circ r, x^*]z_c + [y \circ \delta(r z_c), x^*] = 0.$$

It implies

$$-[\delta(x), y^* \circ r^*]z_c + [y \circ \delta(r z_c), x^*] + [\delta(y) \circ r, x^*]z_c = 0 \text{ for all } x, y \in R, r \in J. \tag{3.18}$$

Multiplying (3.14) by z_c and then adding it into (3.18), we find

$$[y\circ\delta(rz_c),x^*]+[y\circ\delta(r),x^*]z_c+2[\delta(y)\circ r,x^*]z_c=0 \text{ for all } x,y\in R,r\in J,$$

which by virtue of (3.11), leads to

$$[y\circ(\lambda^*r^*z_c^*+\sigma(rz_c)^*),x^*]+[y\circ(\lambda^*r^*+\sigma(r)^*),x^*]z_c+2[\delta(y)\circ r,x^*]z_c=0.$$

An application of (3.17) yields

$$[y\circ(-\lambda^*r^*+\sigma(r)^*),x^*]z_c+[y\circ(\lambda^*r^*+\sigma(r)^*),x^*]z_c+2[\delta(y)\circ r,x^*]z_c=0.$$

It follows that

$$2[y\circ\sigma(r)^*,x^*]z_c+2[\delta(y)\circ r,x^*]z_c=0 \text{ for all } x,y\in R,r\in J.$$

Since $z_c\neq 0$, it implies that

$$2(2\sigma(r)^*[y,x^*]+[r\circ\delta(y),x^*])=0 \text{ for all } x,y\in R,r\in J.$$

Using 2-torsion freeness of R and (3.11), we have

$$2\sigma(r)^*[y,x^*]+[r\circ(\lambda^*y^*+\sigma(y)^*),x^*]=0.$$

It implies

$$2\sigma(r)^*[y,x^*]+\lambda^*[r\circ y^*,x^*]+2\sigma(y)^*[r,x^*]=0 \text{ for all } x,y\in R,r\in J. \tag{3.19}$$

In particular, for $y\in J$, we replace y by yz_c and get

$$2\sigma(r)^*[y,x^*]z_c-\lambda^*[r\circ y^*,x^*]z_c+2\sigma(yz_c)^*[r,x^*]=0.$$

Using (3.17), we have

$$2\sigma(r)^*[y,x^*]z_c-\lambda^*[r\circ y^*,x^*]z_c+2\sigma(y)^*[r,x^*]z_c=0. \tag{3.20}$$

Multiplying (3.19) by z_c and then subtract it from (3.20), we conclude that

$$\lambda^*[r\circ y^*,x^*]z_c=0 \text{ for all } x\in R,\ r,y\in J.$$

By the primeness of R we have either $[r\circ y,x]=0$ or $\lambda=0$. Since R is noncommutative, the first situation cannot occur; consequently $\lambda=0$. Thus $\delta(x)=\sigma(x)^*$ for all $x\in R$. From relation (B), we have

$$\sigma(x\circ y)^*-\sigma(x)^*\circ y-x\circ\sigma(y)^*\in Z(R) \text{ for all } x,y\in R,$$

that is

$$2\sigma(x)^*[y,x]=0 \text{ for all } x,y\in R.$$

Using 2-torsion freeness hypothesis, we find

$$\sigma(x)^*[y,x]=0 \text{ for all } x,y\in R.$$

Further proceeding as Theorem 3.6, we are done in this case.

Case 2. Let $c^*=c$ for all $c\in C$. Polarizing (3.9), we have

$$[\delta(x),y^*]+[\delta(y),x^*]=0 \text{ for all } x,y\in R. \tag{3.21}$$

Replacing y by $h\circ k$, where $k\in S(R)$ and $h\in H(R)$, we have

$$-[\delta(x),h\circ k]+[\delta(h)\circ k,x^*]+[h\circ\delta(k),x^*]=0. \tag{3.22}$$

In view of (3.11), it follows that

$$-[\lambda x^*,h\circ k]+[(\lambda h+\sigma(h))\circ k,x^*]+[h\circ(-\lambda k+\sigma(k)),x^*]=0.$$

It implies

$$-\lambda[x^*,h\circ k]+2\sigma(h)[k,x^*]+2\sigma(k)[h,x^*]=0 \text{ for all } x\in R,h\in H(R),k\in S(R). \tag{3.23}$$

Replacing x by k in (3.23), we find

$$\lambda[k^2,h]-2\sigma(k)[h,k]=0 \text{ for all } h\in H(R),k\in S(R). \tag{3.24}$$

Taking involution on both side in (3.24) and using $z^* = z$ for all $z \in C$, we find

$$\lambda[k^2, h] + 2\sigma(k)[h, k] = 0 \text{ for all } h \in H(R), k \in S(R). \tag{3.25}$$

Adding (3.25) and (3.24), we obtain

$$2\lambda[k^2, h] = 0 \text{ for all } h \in H(R), k \in S(R). \tag{3.26}$$

Thereby 2-torsion freeness and primeness of R implies that either $\lambda = 0$ or $[k^2, h] = 0$ for all $k \in S(R)$ and $h \in H(R)$. Latter case implies $d_h(k)k + kd_h(k) = 0$ for all $k \in S(R)$, where d_h is the inner derivation induced by h. In view of Lemma 2.10, R satisfies s_4 or $d_h = 0$. Therefore either R satisfies s_4 or $H(R) \subseteq Z(R)$. By Lemma 2.9, in each of the situation, R satisfies s_4. But R is noncommutative, hence R is an order in a central simple algebra of dimension at most 4 over its center.

We now consider $\lambda = 0$. Using (B) and (3.11), we have

$$\sigma(x \circ y) - \sigma(x) \circ y - x \circ \sigma(y) \in Z(R) \text{ for all } x, y \in R.$$

Therefore

$$2\sigma(x)[y, x] = 0 \text{ for all } x, y \in R.$$

Further proceeding as the proof of Theorem 3.6, we get the conclusion. □

4. Results on centrally extended Jordan ∗-derivation

Let R be a ring with involution $'*'$. In [14], El-Deken and Nabiel introduced the notion of *centrally extended ∗-derivation* and investigated the case when centrally extended ∗-derivations are ∗-derivations. More specifically, they established the following result: *If R is a semiprime ∗-ring with no nonzero central ideals, then every centrally extended ∗-derivation d on R is a ∗-derivation.* Motivated by the concept of centrally extended ∗-derivations, we now introduce the notion of centrally extended Jordan ∗-derivation as follows:

Definition 4.1. A mapping $\delta : R \to R$ that satisfies

$$\delta(x + y) - \delta(x) - \delta(y) \in Z(R), \tag{C}$$

$$\delta(x \circ y) - \delta(x)y^* - x\delta(y) - \delta(y)x^* - y\delta(x) \in Z(R) \tag{D}$$

for all $x, y \in R$ is called *centrally extended Jordan ∗-derivation* of R. We shall abbreviate this map as $CE-$Jordan ∗-derivation.

Example 4.2. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ be a ring. Define mapping $\delta, * : R \to R$ by

$$\delta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

where k is a fixed integer. Then, it is straight forward to check that δ is a $CE-$Jordan ∗-derivation of R, with involution $'*'$ of R.

Remark 4.3. If R is a 2-torsion free noncommutative prime ring with involution $'*'$, then an additive map d is a $CE-$Jordan ∗-derivation if and only if $d(x^2) - d(x)x^* - xd(x) \in Z(R)$ for all $x \in R$. It is observed that, in this case, a $CE-$Jordan ∗-derivation is not necessarily a Jordan ∗-derivation or CE ∗-derivation, for example:

Let $\mathbb{Z}$ be the ring of integers and $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$, a prime ring. Define a mapping $* : R \to R$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, which is an involution of R. If

$\delta : R \to R$ be a mapping such that $\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, then δ is a $CE-$Jordan $*$-derivation but neither Jordan $*$-derivation nor CE $*$-derivation.

In this section our focus will be on $CE-$Jordan $*$-derivations acting on 2-torsion free noncommutative prime rings. More specifically, we prove the $CE-$Jordan $*$-derivation analogy of the above results. We begin our discussions with the following lemma.

Lemma 4.4. *Let R be a 2-torsion free semiprime ring with involution $'*'$ and with no nonzero central ideal. If δ is a $CE-$Jordan $*$-derivation of R, then δ is additive.*

Proof. For any $x, y, z \in R$, in view of (C), it follows that there exists $c_{\delta(x,y,+)} \in Z(R)$ such that

$$\delta(x+y) = \delta(x) + \delta(y) + c_{\delta(x,y,+)}, \tag{4.1}$$

Using (D), we have

$$\begin{aligned} \delta(z \circ (x+y)) &= \delta(z)(x^* + y^*) + \delta(x+y)z^* + (x+y)\delta(z) \\ &\quad + z\delta(x+y) + c_{\delta(z,x+y,\circ)} \\ &= \delta(z)(x^* + y^*) + \Big(\delta(x) + \delta(y) + c_{\delta(x,y,+)}\Big)z^* + (x+y) \\ &\quad \delta(z) + z\Big(\delta(x) + \delta(y) + c_{\delta(x,y,+)}\Big) + c_{\delta(z,x+y,\circ)} \\ &= \delta(z)(x^* + y^*) + (\delta(x) + \delta(y))z^* + (x+y)\delta(z) \\ &\quad + z(\delta(x) + \delta(y)) + c_{\delta(x,y,+)}(z + z^*) + c_{\delta(z,x+y,\circ)}, \end{aligned} \tag{4.2}$$

where $c_{\delta(z,x+y,\circ)}, c_{\delta(x,y,+)}$ are central elements. Computing in a different manner, we get

$$\begin{aligned} \delta(z \circ (x+y)) &= \delta(z \circ x + z \circ y) \\ &= \delta(zx + xz) + \delta(zy + yz) + c_{\delta(z \circ x, z \circ y, +)} \\ &= \Big(\delta(z)x^* + \delta(x)z^* + z\delta(x) + x\delta(z) + c_{\delta(z,x,\circ)}\Big) + \Big(\delta(z)y^* + \delta(y)z^* + z \\ &\quad \delta(y) + y\delta(z) + c_{\delta(z,y,\circ)}\Big) + c_{\delta(z \circ x, z \circ y, +)}, \end{aligned} \tag{4.3}$$

where $c_{\delta(z \circ x, z \circ y, +)}, c_{\delta(z,x,\circ)}, c_{\delta(z,y,\circ)}$ are central elements.

Comparing (4.2) and (4.3) to conclude $(z+z^*)c_{\delta(x,y,+)} + c_{\delta(z,x+y,\circ)} = c_{\delta(z,x,\circ)} + c_{\delta(z,y,\circ)} + c_{\delta(z \circ x, z \circ y, +)}$. It forces $(z+z^*)c_{\delta(x,y,+)} \in Z(R)$ for all $z \in R$, where $c_{\delta(x,y,+)}$ is a fixed central element in $Z(R)$. Replacing z by h, where $h \in H(R)$, we have $2hc_{\delta(x,y,+)} \in Z(R)$. Using 2-torsion freeness hypothesis, we get

$$hc_{\delta(x,y,+)} \in Z(R) \text{ for all } h \in H(R). \tag{4.4}$$

It implies

$$c_{\delta(x,y,+)}[h, u] = 0 \text{ for all } h \in H(R),\ u \in R.$$

From the last expression, we can find

$$c_{\delta(x,y,+)}[h, u]h' + c_{\delta(x,y,+)}h[h', u] = 0 \text{ for all } h, h' \in H(R),\ u \in R.$$

Therefore, it follows that

$$c_{\delta(x,y,+)}[\overline{H(R)}, u] = (0) \text{ for all } u \in R, \tag{4.5}$$

where $\overline{H(R)}$ is the ring generated by $H(R)$. In particular

$$c_{\delta(x,y,+)}[\overline{H(R)}, \overline{H(R)}] = (0).$$

In view of Lemma 2.6, $\overline{H(R)}$ is also a semi prime ring. Now, we shall first show that $c_{\delta(x,y,+)} \in H(R)$. Suppose that $c_{\delta(x,y,+)} \notin H(R)$, i.e., $c^*_{\delta(x,y,+)} \neq c_{\delta(x,y,+)}$. Then $(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})k \in H(R)$ for all $k \in S(R)$. Using (4.4), we get

$$c_{\delta(x,y,+)}(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})k \in Z(R) \text{ for all } k \in S(R). \tag{4.6}$$

Applying involution, we have

$$c^*_{\delta(x,y,+)}(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})k \in Z(R) \text{ for all } k \in S(R). \tag{4.7}$$

Subtracting (4.6) from (4.7), we get

$$(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})^2 k \in Z(R) \text{ for all } k \in S(R). \tag{4.8}$$

Using (4.4), we find $c^*_{\delta(x,y,+)}h \in Z(R)$ for all $h \in H(R)$. An easy consequence of this is

$$(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})h \in Z(R) \text{ for all } h \in H(R). \tag{4.9}$$

Since $c_{\delta(x,y,+)} \in Z(R)$, left multiplying (4.9) by $c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)}$ in order to find

$$(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})^2 h \in Z(R) \text{ for all } h \in H(R). \tag{4.10}$$

Adding (4.8) and (4.10), we have

$$(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})^2 (h+k) \in Z(R) \text{ for all } h \in H(R),\ k \in S(R). \tag{4.11}$$

It yields

$$(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})^2 R \subseteq Z(R).$$

Since R does not contain non-zero central ideals, thus we have $(c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)})^2 = 0$, where $c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)} \in Z(R)$. The fact that center of a semiprime ring has no nonzero nilpotent elements, forces $c^*_{\delta(x,y,+)} - c_{\delta(x,y,+)} = 0$, a contradiction to our assumption. Hence $c_{\delta(x,y,+)} \in H(R)$.

Therefore in view of Lemma 2.11, our expression $c_{\delta(x,y,+)}[\overline{H(R)}, \overline{H(R)}] = (0)$ implies that there exists an ideal $I \subseteq \overline{H(R)}$ such that $c_{\delta(x,y,+)} \in I \subseteq Z(\overline{H(R)})$. Using Lemma 2.7, we get $c_{\delta(x,y,+)} \in I \subseteq Z(R)$. As R does not contain non-zero central ideals, we get $c_{\delta(x,y,+)} = 0$; and hence δ is additive, by (4.1). It completes the proof. □

Corollary 4.5. *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$. If δ is a $CE-$Jordan $*$-derivation of R, then δ is additive.*

Theorem 4.6. *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$ that admits a $CE-$Jordan $*$-derivation $\delta : R \to R$ such that $[\delta(x), x] \in Z(R)$ for all $x \in R$. Then either $\delta = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Proof. Let us assume that $[\delta(x), x] \in Z(R)$ for all $x \in R$. By Lemma 2.4 and Lemma 4.4, we conclude that

$$[\delta(x), x] = 0 \text{ for all } x \in R. \tag{4.12}$$

Polarizing (4.12), we have

$$[\delta(x), y] + [\delta(y), x] = 0 \text{ for all } x, y \in R. \tag{4.13}$$

Replacing y by k^2, where $k \in S(R)$ in (4.13), we arrive at

$$[\delta(x), k^2] + [-\delta(k)k + k\delta(k), x] = 0 \text{ for all } x \in R,\ k \in S(R). \tag{4.14}$$

Since δ is a commuting and additive function thereby using Lemma 2.5, there exists $\lambda \in C$ and an additive mapping $\sigma : R \to C$ such that

$$\delta(x) = \lambda x + \sigma(x) \text{ for all } x \in R. \tag{4.15}$$

With this, it follows from (4.14) that

$$[\lambda x, k^2] + [-(\lambda k + \sigma(k))k + k(\lambda k + \sigma(k)), x] = 0 \text{ for all } x \in R,\ k \in S(R).$$

In fact, we have

$$\lambda[x, k^2] = 0 \text{ for all } x \in R,\ k \in S(R). \tag{4.16}$$

Since the center of a prime ring contains no proper zero divisors, it follows that either $\lambda = 0$ or $k^2 \subseteq Z(R)$ for all $k \in S(R)$. In the latter case, we have $0 = [k^2, x] = [k, x]k + k[k, x]$ for all $x \in R$ and $k \in S(R)$. It can be seen as $d_x(k)k + kd_x(k) = 0$ for all $k \in S(R)$, where $d_x(k) = [k, x]$ is the inner derivation induced by $x \in R$. In view of Lemma 2.10, either R satisfies s_4 or $d_x(t) = 0$ for all $t \in R$. Clearly, $d_x(t) = [x, t] = 0$ for all $x, t \in R$ leads a contradiction. Thus we conclude that R is an order in a central simple algebra of dimension at most 4 over its center.

Now, if $\lambda = 0$, then from (4.15), we have $\delta(x) = \sigma(x)$ for all $x \in R$. In view of (D) and torsion condition on R, it implies

$$\sigma(x^2) - \sigma(x)x^* - x\sigma(x) \in Z(R) \text{ for all } x \in R,$$

and so

$$\sigma(x)[x + x^*, y] = 0 \text{ for all } x, y \in R.$$

Using Brauer's trick, we have either $\sigma(x) = 0$ for all $x \in R$ or $x + x^* \in Z(R)$ for all $x \in R$. If $\sigma = 0$, then by (4.15) $\delta = 0$, as desired. Now if $x + x^* \in Z(R)$ for all $x \in R$, then replacing x by $h \in H(R)$, we get $H(R) \subseteq Z(R)$. Hence by Lemma 2.9, we are done. □

Theorem 4.7. *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$ that admits a CE−Jordan $*$-derivation $\delta : R \to R$ such that $[\delta(x), x^*] \in Z(R)$ for all $x \in R$. Then either $\delta = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Proof. Let us consider $[\delta(x), x^*] \in Z(R)$ for all $x \in R$. In light of Lemma 2.2 and Lemma 4.4, we may infer that

$$[\delta(x), x^*] = 0 \text{ for all } x \in R. \tag{4.17}$$

This implies that

$$[\delta(x)^*, x] = 0 \text{ for all } x \in R. \tag{4.18}$$

Using Lemma 2.5 in Eq. (4.18), there exists $\lambda \in C$ and an additive mapping $\sigma : R \to C$ such that

$$\delta(x)^* = \lambda x + \sigma(x) \text{ for all } x \in R,$$

and equivalently we have

$$\delta(x) = \lambda^* x^* + \sigma(x)^* \text{ for all } x \in R. \tag{4.19}$$

Polarizing (4.17), we find

$$[\delta(x), y^*] + [\delta(y), x^*] = 0 \text{ for all } x, y \in R.$$

Replacing y by k^2 in the last expression, where $k \in S(R)$, we get

$$[\delta(x), k^2] + [-\delta(k)k + k\delta(k), x^*] = 0.$$

Using (4.19), it yields

$$[\lambda^* x^*, k^2] + [-(\lambda^* k^* + \sigma(k))k + k(\lambda^* k^* + \sigma(k)), x^*] = 0 \text{ for all } x \in R,\ k \in S(R). \tag{4.20}$$

Since $\lambda \in C$ and $\sigma(R) \subseteq C$, we obtain

$$\lambda^*[x^*, k^2] = 0 \text{ for all } x \in R,\ k \in S(R).$$

Further, a similar demonstration that is given in the proof of Theorem 4.6 ensures the conclusion. □

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