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Approximation by Modified Bivariate Bernstein-Durrmeyer and GBS Bivariate Bernstein-Durrmeyer Operators on a Triangular Region

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Abstract

In this paper, the approximation properties and the rate of convergence of modified bivariate Bernstein-Durrmeyer Operators on a triangular region are examined. Furthermore, definitions and some properties of modulus of continuity for functions of two variables are given. Voronovskaya and Grüss Voronovskaja type theorems are used to determine the order of approximation. The GBS (Generalized Boolean Sum) operator of Bivariate Bernstein-Durrmeyer type on a triangular region is studied. Lastly, some numerical examples are given and related graphs are plotted for comparison.

1. Introduction

Classical approximation theory, including polynomial approximation is a fundamental research area in applied mathematics. Development in approximation theory plays an important role in numerical solution of partial differential equations, image processing as well as in data sciences and many other disciplines. For example, radial basis functions and shift-invariant spaces are widely used for geometric modeling in aerospace and automobile industries [1]. In this paper we intend to study the approximation properties of functions of two variables by means of Bernstein-Durrmeyer operator in a triangular domain. Several studies have been conducted on the classical Bernstein operators, as well as using two variables.

From literature , Kingsley [2] proposed the Bernstein operator of two-variables. Pop [3] added some features to the Bernstein operators, defined by Kingsley. Stancu [4] defined two variables of Bernstein operators on the triangular region. Pop and Farcas [5] researched the approximation features of the Bernstein-Kantorovich operators on the triangular region. In [6], authors examined the weighted approximation features of two-variables by Bernstein -Stancu-Chlodowsky polynomials in a triangular region. In 1992, Zhou [7] defined the two variables of Bernstein-Durrmeyer polynomials and obtained the rate of convergence of the functions in L_p spaces.

Some generalization of these polynomials in the one-dimensional case may be found in [8]-[18].

In the light of these studies, we defined the new generalized operator that we think will get better results.

Let $V := \left\{(u,v) \in \mathbb{R}^2 : -1 \le v \le 1, -1 \le u \text{ and } u+v \le 0\right\}$ and $h \in C(V)$, we will examine the Bernstein-Durrmeyer operator of two variables on a triangular region as

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$$H_n(h;u.v)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\varphi_{n,k,l}(u,v)\frac{(n+1)(n+2)}{16}\int_{-1}^{1}\int_{-1}^{-t}\varphi_{n,k,l}(s,t)h(s,t)dsdt$$

in which

$$\varphi_{n,k,l}(u,v)=\binom{n}{k}\binom{n-k}{l}\left(\frac{1+u}{2}\right)^k\left(\frac{1+v}{2}\right)^l\left(1-\frac{1+u}{2}-\frac{1+v}{2}\right)^{n-k-l}$$

In this paper, the approximation features and the speed of approximation of Modified Bivariate Bernstein-Durrmeyer Operators on a Triangular Region will be examined. Furthermore, definitions and some features of moduli of continuity of two variables function are given.We examine the order of approximation by Voronovskaya type theorem and Grüss Voronovskaja type theorem. The GBS (Generalized Boolean Sum) operators of Bivariate Bernstein-Durrmeyer type on a triangular region will be studied. Lastly some numerical examples and the graphics will be drawn.

2. Preliminary results

Theorem 2.1. *For* $e_{i,j}=s^it^j,(i,j)\in\mathbb{N}^0\times\mathbb{N}^0,\mathbb{N}^0=\mathbb{N}\cup\{0\}$, *we have*

i) $H_n(e_{0,0};u,v)=1.$

ii) $H_n(e_{1,0};u,v)=u-\dfrac{3u+1}{n+3}.$

iii) $H_n(e_{0,1};u,v)=v-\dfrac{3v+1}{n+3}.$

iv) $H_n(e_{2,0};u,v)=u^2-\dfrac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)}.$

v) $H_n(e_{0,2};u,v)=v^2-\dfrac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)}.$

vi) $H_n(e_{1,1};u,v)=uv-\dfrac{(8n+3)uv-3n(u+v)-2n}{(n+3)(n+4)}.$

vii) $$H_n(e_{4,0};u,v)=u^4-\frac{(24n^3+108n^2+348n+360)}{(n+3)(n+4)(n+5)(n+6)}u^4+\frac{(-4n^3+12n^2-8n)}{(n+3)(n+4)(n+5)(n+6)}u^3+\frac{(12n^3+12n^2-24n)}{(n+3)(n+4)(n+5)(n+6)}u^2+\frac{(-24n^2-48n)}{(n+3)(n+4)(n+5)(n+6)}u+\frac{(12n^2+60n+72)}{(n+3)(n+4)(n+5)(n+6)}.$$

viii) $$H_n(e_{0,4};u,v)=v^4-\frac{(24n^3+108n^2+348n+360)}{(n+3)(n+4)(n+5)(n+6)}v^4+\frac{(-4n^3+12n^2-8n)}{(n+3)(n+4)(n+5)(n+6)}v^3+\frac{(12n^3+12n^2-24n)}{(n+3)(n+4)(n+5)(n+6)}v^2+\frac{(-24n^2-48n)}{(n+3)(n+4)(n+5)(n+6)}v+\frac{(12n^2+60n+72)}{(n+3)(n+4)(n+5)(n+6)}.$$

Theorem 2.2. *For* $k_{i,j}=(s-u)^i(t-v)^j,(i,j)\in\mathbb{N}^0\times\mathbb{N}^0$, *we have*

i) $H_n(k_{0,0};u,v)=1.$

ii) $H_n(k_{1,0};u,v)=-\dfrac{3u+1}{n+3}.$

iii) $H_n(k_{0,1};u,v)=-\dfrac{3v+1}{n+3}.$

iv) $H_n(k_{2,0};u,v)=-\dfrac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)}.$

v) $H_n(k_{0,2};u,v)=-\dfrac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)}.$

vi) $$H_n(k_{4,0};u,v)=\frac{12n^2-588n-936}{(n+3)(n+4)(n+5)(n+6)}u^4+\frac{144n+480}{(n+3)(n+4)(n+5)(n+6)}u^3+\frac{-24n^2+312n+720}{(n+3)(n+4)(n+5)(n+6)}u^2+\frac{144n+288}{(n+3)(n+4)(n+5)(n+6)}u.$$

vii) $$H_n(h_{0,4};u,v)=\frac{12n^2-588n-936}{(n+3)(n+4)(n+5)(n+6)}v^4+\frac{144n+480}{(n+3)(n+4)(n+5)(n+6)}v^3+\frac{-24n^2+312n+720}{(n+3)(n+4)(n+5)(n+6)}v^2+\frac{144n+288}{(n+3)(n+4)(n+5)(n+6)}v.$$

Theorem 2.3. *For the bivariate operators* $H_n(f;u,v)$, *we have*

i) $\lim\limits_{n\longrightarrow\infty} nH_n((s-u);u,v) = -(3u+1)$.
ii) $\lim\limits_{n\longrightarrow\infty} nH_n((t-v);u,v) = -(3v+1)$.
iii) $\lim\limits_{n\longrightarrow\infty} nH_n((s-u)^2;u,v) = -(8u^2+2u-2)$.
iv) $\lim\limits_{n\longrightarrow\infty} nH_n((t-v)^2;u,v) = -(8v^2+2v-2)$.
v) $\lim\limits_{n\longrightarrow\infty} nH_n((s-u)(t-v);u,v) = -2uv+4(u+v)+2$.

Theorem 2.4. *From Theorem 2.1, we get*

$$H_n((s-u)^2;u,v) \le \frac{3}{n}.$$

$$H_n((t-v)^2;u,v) \le \frac{3}{n}.$$

Proof. For all $u \in [-1,1]$, we write

$$H_n((s-u)^2;u,v) = -\frac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)}.$$

$$H_n((t-v)^2;u,v) = -\frac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)}.$$

If we take the max values of the equations we have obtained, we get $u = \frac{-n}{8n+12}$ and $v = \frac{-n}{8n+12}$. From here

$$\begin{aligned} H_n((s-u)^2;u,v) &= -\frac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)} \\ &= \frac{136n^3+652n^2+1056n+576}{64n^4+640n^3+2112n^2+3456n+1728} \le \frac{3}{n} \end{aligned}$$

and

$$\begin{aligned} H_n((t-v)^2;u,v) &= -\frac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)} \\ &= \frac{136n^3+652n^2+1056n+576}{64n^4+640n^3+2112n^2+3456n+1728} \le \frac{3}{n} \end{aligned}$$

are obtained and proof is completed. □

3. Main results

Basic convergence theorem.

Theorem 3.1. *Let* $V := \{(u,v) : v \le 1, -1 \le u \text{ and } u+v \le 0\}$ *and* $h \in C(V) := \{h : V \to R,\ f \text{ is continuous}\}$; $H_n(h;u,v) : C(V) \to C(R)$ *be linear positive operators.If*
i) $\lim\limits_{n\longrightarrow\infty} H_n(1;u,v) = 1$
ii) $\lim\limits_{n\longrightarrow\infty} H_n(s;u,v) = u$
iii) $\lim\limits_{n\longrightarrow\infty} H_n(t;u,v) = v$
iv) $\lim\limits_{n\longrightarrow\infty} H_n((s^2+t^2;u,v) = u^2+v^2$
H_n *converges to* h*, for* $h \in C(V)$.

3.1. Degree of approximation by $H_n(h;u,v)$

For $h\in C(V)$, the complete moduli of continuity for the two-variable functions is defined as:

$$\omega(h,\delta)=\max_{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2}\leq\delta}|h(u_1,v_1)-h(u_2,v_2)|$$

here $(u_1,v_1),(u_2,v_2)\in V$.
In addition, partial continuity moduli according to u and v are defined as ;

$$\omega^{(1)}(h,\delta)=\max_{|u_1-u_2|\leq\delta}|h(u_1,v)-h(u_2,v)|$$

here $(u_1,v),(u_2,v)\in V$.

$$\omega^{(2)}(h,\delta)=\max_{|v_1-v_2|\leq\delta}|h(u,v_1)-h(u,v_2)|$$

here $(u,v_1),(u,v_2)\in V$.
It is seen that they provide the characteristics of the continuity modulus. In what follows, $\omega(h,\delta)\leq(1+\lambda)\,\omega(h,\delta)$ and $\lim\limits_{\delta\to 0}\omega(h,\delta)=0$.

Theorem 3.2. *Let $h\in C(V)$, we have*

$$\|H_n(h;u,v)-h(u,v)\|_{C(V)}\leq 3\omega\left(h,\frac{1}{\sqrt{n}}\right)$$

Proof. From the well-known features of modulus of continuity, we have

$$|h(s,t)-h(u,v)|\ \leq\ \omega(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2+(t-v)^2\right)^{\frac{1}{2}}\right)$$

Using Cauchy-Schwartz inequality and Theorem 2.2, we obtain

$$\begin{aligned}
|H_n(h;u,v)-h(u,v)| &= |H_n(h(s,t)-h(u,v);u,v)|\\
&\leq H_n(|h(s,t)-h(u,v)|;u,v)\\
&\leq H_n(\omega(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2+(t-v)^2\right)^{\frac{1}{2}}\right);u,v)\\
&\leq \omega(h,\delta)\left(1+\frac{1}{\delta}\left(H_n((s-u)^2+(t-v)^2);u,v\right)^{\frac{1}{2}}\right)\\
&= \omega(h,\delta)\left(1+\frac{1}{\delta}\left(H_n\left((s-u)^2;u,v\right)+H_n\left((t-v)^2;u,v\right)\right)^{\frac{1}{2}}\right)\\
&= \omega(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{(-8n-12)u^2-2nu+2n-4}{(n+3)(n+4)}\right.\right.\\
&\quad\left.\left.+\frac{(-8n-12)v^2-2nv+2n-4}{(n+3)(n+4)}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

Moreover, if we calculate maximum value of the square root and δ=$\frac{1}{\sqrt{n}}$, then we obtain

$$=\omega(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{4}{n}\right)^{\frac{1}{2}}\right)=\omega\left(f,\frac{1}{\sqrt{n}}\right)\left(1+\frac{1}{\delta}\left(\frac{4}{n}\right)^{\frac{1}{2}}\right)=3\omega\left(f,\frac{1}{\sqrt{n}}\right)$$

□

Theorem 3.3. *Let $h\in C(V)$, then the following inequality holds.*

$$\|H_n(h;u,v)-h(u,v)\|_{C(V)}\leq\left(1+\sqrt{3}\right)\left(\omega^{(1)}\left(h,\frac{1}{\sqrt{n}}\right)+\omega^{(2)}\left(h,\frac{1}{\sqrt{n}}\right)\right)$$

Proof. From the well-known features of modulus of continuity, we have

$$|h(s,t)-h(u,v)| \leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2\right)^{\frac{1}{2}}\right)+\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left((t-v)^2\right)^{\frac{1}{2}}\right)$$

Using Cauchy-Schwartz inequality and Theorem 2.2, we obtain

$$\begin{aligned}
|H_n(h;u,v)-h(u,v)| &= |H_n(h(s,t)-h(u,v);u,v)| \\
&\leq H_n(|h(s,t)-h(u,v)|;u,v) \\
&\leq H_n\Bigg(\omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2\right)^{\frac{1}{2}}\right) \\
&\quad +\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left((t-v)^2\right)^{\frac{1}{2}}\right);u,v\Bigg) \\
&\leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left(H_n\left((s-u)^2;u,v\right)\right)^{\frac{1}{2}}\right) \\
&\quad +\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left(H_n\left((t-v)^2;u,v\right)\right)^{\frac{1}{2}}\right) \\
&\leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{(-8n-12)u^2-2nu+2n-4}{(n+3)(n+4)}\right)^{\frac{1}{2}}\right) \\
&\quad +\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{(-8n-12)v^2-2nv+2n-4}{(n+3)(n+4)}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

Moreover, if we calculate maximum value of the square root and $\delta=\frac{1}{\sqrt{n}}$, then we obtain

$$\begin{aligned}
&\leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right)+\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right) \\
&= \omega^{(1)}\left(h,\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{n}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right)+\omega^{(2)}\left(h,\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{n}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right) \\
&= \left(1+\sqrt{3}\right)\left(\omega^{(1)}\left(h,\frac{1}{\sqrt{n}}\right)+\omega^{(2)}\left(h,\frac{1}{\sqrt{n}}\right)\right)
\end{aligned}$$

□

3.2. The Voronovskaja-type result

Theorem 3.4. *For* $\forall\, h \in C^2(V)$*, we have*

$$\begin{aligned}
\lim_{n\to\infty} n.\left(H_n(h;u,v)-h(u,v)\right) &= (-3u-1)\,h_u(u,v)+(-3v-1)\,h_v(u,v) \\
&\quad +\left(-4u^2-u+1\right)h_{u\,u}(u,v)+(-4uv+8(u+v)+4)\,h_{uv}(u,v) \\
&\quad +\left(-4v^2-v+1\right)h_{vv}(u,v)
\end{aligned}$$

uniformly in $(u,v)\in V$.

Proof. If we apply Taylor's formula to $h \in C^2(V)$,

$$\begin{aligned}
h(s,t) &= h(u,v)+h_u(u,v)(s-u)+h_v(u,v)(t-v) \\
&\quad +\frac{1}{2}\left\{h_{uu}(u,v)(s-u)^2+2h_{uv}(u,v)(s-u)(t-v)+h_{vv}(u,v)(t-v)^2\right\} \\
&\quad +\mho(s,t)\left((s-u)^2+(t-v)^2\right)
\end{aligned}$$

here $\mho(.,.;u,v)=\mho(.,.)\in C(V)$ represents the remainder of the Taylor formula. $\mho(.,.)\in C(V)$ is defined in this way

$$\mho(s,t;u,v)=\begin{cases}\dfrac{h(s,t)-h(u,v)-h_u(s-u)-h_v(t-v)-\frac{1}{2}\left\{h_{u\,u}(s-u)^2+2h_{uv}(s-u)(t-v)+h_{vv}(t-v)^2\right\}}{\sqrt{(s-u)^4+(t-v)^4}} & ,(s,t)\neq(u,v) \\ 0 & ,(s,t)=(u,v)\end{cases}$$

Then, H_n is a linear-positive operator, we write

$$\begin{aligned} H_n(h(s,t);u,v) &= h(u,v)+h_u(u,v)H_n((s-u);u,v)+h_v(u,v)H_n((t-v);u,v) \\ &+\frac{1}{2}\left\{h_{uu}(u,v)H_n\left((s-u)^2;u,v\right)+2h_{uv}(u,v)H_n((s-u)(t-v);u,v)\right. \\ &\left.+h_{vv}(u,v)H_n\left((t-v)^2;u,v\right)\right\}+H_n\left(\mho(s,t)\left((s-u)^2+(t-v)^2\right);u,v\right) \end{aligned}$$

Now, let us use the Cauchy-Schwarz inequality in the last term of the last equation,

$$\begin{aligned} & \left|H_n\left(\mho(s,t)\left((s-u)^2+(t-v)^2\right);u,v\right)\right| \\ \leq & \left|H_n\left(\mho(s,t)\sqrt{(s-u)^4+(t-v)^4};u,v\right)\right| \\ \leq & \left\{H_n\left(\mho^2(s,t);u,v\right)\right\}^{\frac{1}{2}}\left\{H_n\left((s-u)^4+(t-v)^4;u,v\right)\right\}^{\frac{1}{2}} \\ = & \left\{H_n\left(\mho^2(s,t);u,v\right)\right\}^{\frac{1}{2}}\left\{H_n\left((s-u)^4;u,v\right)+H_n\left((t-v)^4;u,v\right)\right\}^{\frac{1}{2}} \end{aligned}$$

Since $\mho(.,.;u,v)\in C(V)$ and $\mho(s,t;u,v)\to 0$ as $(s,t)\to(u,v)$ applying Theorem 3.1

$$\lim_{n\to\infty} H_n\left(\mho^2(s,t);u,v\right)=\mho^2(s,t)=0$$

as a result

$$\lim_{n\to\infty} n.\left(H_n\left(\mho(s,t)\sqrt{(s-u)^4+(t-v)^4};u,v\right)\right)=0$$

Then applying Theorem 2.3 and last equality, we have

$$\begin{aligned} \lim_{n\to\infty} n.\left(H_n(h(s,t);u,v)-h(u,v)\right) &= (-3u-1)h_u(u,v)+(-3v-1)h_v(u,v)+\left(-4u^2-u+1\right)h_{uu}(u,v) \\ &+(-4uv+8(u+v)+4)h_{uv}(u,v)+\left(-4v^2-v+1\right)h_{vv}(u,v) \end{aligned}$$

Thus, the proof is completed. □

3.3. The Grüss Voronovskaja-type result

Theorem 3.5. *Let $h''\in C^2(V), w''\in C^2(V)$ then we write*

$$\begin{aligned} \lim_{n\to\infty} n\left\{H_n(hw;u,v)-h(u,v)w(u,v)\right\} &= (2-2u-8u^2)h_u(u,v)w_u(u,v) \\ &+(4(u+v)-2uv+2)\left[h_u(u,v)w_v(u,v)+h_v(u,v)w_u(u,v)\right] \\ &+(2-2v-8v^2)h_v(u,v)w_v(u,v) \end{aligned}$$

Proof. In this study, we examine $n\left\{H_n(hw;u,v)-h(u,v)w(u,v)\right\}$

$$\begin{aligned} &= n\left\{H_n(hw;u,v)-h(u,v)w(u,v)-\left[h(u,v)w_u(u,v)+h_u(u,v)w(u,v)\right]H_n((s-u);u,v)\right. \\ &-\left[h(u,v)w_v(u,v)+h_v(u,v)w(u,v)\right]H_n((t-v);u,v) \\ &-\tfrac{1}{2}\left[h(u,v)w_{uu}(u,v)+2h'_u(u,v)w_u(u,v)+h_{uu}(u,v)w(u,v)\right]H_n\left((s-u)^2;u,v\right) \\ &-\left[h(u,v)w_{uv}(u,v)+h_u(u,v)w_v(u,v)+h_v(u,v)w_u(u,v)+h_{uv}(u,v)w(u,v)\right]H_n((t-v)(s-u);u,v) \\ &-\tfrac{1}{2}\left[h(u,v)w_{vv}(u,v)+2h'_v(u,v)w_v(u,v)+h_{vv}(u,v)w(u,v)\right]H_n\left((t-v)^2;u,v\right) \\ &-w(u,v)\left[H_n(h;u,v)-h(u,v)-h_u(u,v)H_n((s-u);u,v)-h_v(u,v)H_n((t-v);u,v)\right. \\ &\left.-\tfrac{1}{2}h_{uu}(u,v)H_n\left((s-u)^2;u,v\right)-h_{uv}(u,v)H_n((t-v)(s-u);u,v)-\tfrac{1}{2}h_{vv}(u,v)H_n\left((t-v)^2;u,v\right)\right] \\ &-H_n(h;u,v)\left[H_n(w;u,v)-w(u,v)-w_u(u,v)H_n((s-u);u,v)-w_v(u,v)H_n((t-v);u,v)\right. \\ &\left.-\tfrac{1}{2}w_{uu}(u,v)H_n\left((s-u)^2;u,v\right)-w_{uv}(u,v)H_n((s-u)(t-v);u,v)-\tfrac{1}{2}w_{vv}(u,v)H_n\left((s-u)^2;u,v\right)\right] \\ &-w_u(u,v)H_n((s-u);u,v)\left[H_n(h;u,v)-h(u,v)\right]-w_v(u,v)H_n((t-v);u,v)\left[H_n(h;u,v)-h(u,v)\right] \\ &-w_{uu}(u,v)\frac{H_n\left((s-u)^2;u,v\right)}{2}\left[H_n(h;u,v)-h(u,v)\right]-w_{vv}(u,v)\frac{H_n\left((t-v)^2;u,v\right)}{2}\left[H_n(h;u,v)-h(u,v)\right] \\ &-w_{uv}(u,v)H_n((t-v)(s-u);u,v)\left[H_n(h;u,v)-h(u,v)\right]+h_u(u,v)w_u(u,v)H_n\left((s-u)^2;u,v\right) \\ &+h_u(u,v)w_v(u,v)H_n((t-v)(s-u);u,v)+h_v(u,v)w_u(u,v)H_n((t-v)(s-u);u,v) \\ &\left.+h_v(u,v)w_v(u,v)H_n\left((t-v)^2;u,v\right)\right\} \end{aligned}$$

Then, applying Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 3.4, we have

$$\begin{aligned} \lim_{n\to\infty} n\left\{H_n(hw;u,v)-h(u,v)w(u,v)\right\} &= (2-2u-8u^2)h_u(u,v)w_u(u,v) \\ &+(4(u+v)-2uv+2)\left[h_u(u,v)w_v(u,v)+h_v(u,v)w_u(u,v)\right] \\ &+(2-2v-8v^2)h_v(u,v)w_v(u,v) \end{aligned}$$

the proof is completed. □

4. GBS of Bivariate Bernstein-Durrmeyer Operators on a Triangular Region

In [19, 20], the author has defined B-continuous and B-differentiable functions. The expression of Generalized Boolean Sum (GBS) operators was first defined by Badea in [21, 22]. Dobrescu and Matei [23], introduced the approximation features of the two-variable Bernstein GBS operators. Recently, some researchers have made different researches on GBS in the approximation theory [24]-[27]. In this study, we examined the uniform approximation of B-continuous functions using bivariate Bernstein-Durrmeyer GBS operators on a triangular region.
Let $\nabla_{(u,v)}h\,[u_0,v_0;u,v]$ be mixed difference and Φ and Λ be compact real spacing of h defined by

$$\nabla_{(u,v)}h\,[u_0,v_0;u,v] = h(u,v)-h(u,v_0)-h(u_0,v)+h(u_0,v_0)$$

The function $h:\Phi\times\Lambda\to\mathbb{R}$ is called B-continuous function for $(u_0,v_0)\in\Phi\times\Lambda$.

$$\lim_{(u,v)\to(u_0,v_0)}\nabla_{(u,v)}h\,[u_0,v_0;u,v]=0$$

for each $(u,v)\in\Phi\times\Lambda$. Let $C_b(V)$ indicate the space of whole B-continuous functions on V. Here, $C(V)\subset C_b(V)$ [19, 20].

The GBS (Generalized Boolean Sum) associated with $H_n\,(h;u,v)$ defined as

$$E_n(h;u,v)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\varphi_{n,k,l}(u,v)\frac{(n+1)(n+2)}{16}\int_{-1}^{1}\int_{-1}^{-t}\varphi_{n,k,l}(s,t)\left(h(u,t)+h(s,v)-h(s,t)\right)dsdt \tag{4.1}$$

for every $h\in C_b(V)$ at each point $(u,v)\in V$. It is clear that $E_n(h;u,v)$ is a linear and positive operator.

4.1. Approximation by GBS operator $E_n(h;u,v)$

The mixed modulus of smoothness of $h\in C_b(V)$ is defined by

$$\omega_{mixed}\,(h;\delta_1,\delta_2):=\sup\left\{\left|\nabla h\,[(s,t);(u,v)]\right|:|u-s|<\delta_1,|v-t|<\delta_2\right\}$$

for all $(u,v),(s,t)\in V$ and for any $\delta_1,\delta_2\in\mathbb{R}^+$.
The features of mixed moduli of continuity ;

$$\omega_{mixed}\,(h;\lambda_1\delta_1,\lambda_1\delta_2)\le(1+\lambda_1)\,(1+\lambda_2)\,\omega_{mixed}\,(h;\delta_1,\delta_2)$$

we can write,

$$\begin{aligned}|\nabla h[(s,t);(u,v)]| &\le \omega_{mixed}\,(h;|s-u|\,,|t-v|)\\ &\le \left(1+\frac{|s-u|}{\delta_1}\right)\left(1+\frac{|t-u|}{\delta_2}\right)\omega_{mixed}\,(h;\delta_1,\delta_2)\end{aligned}$$

Theorem 4.1. *For $\forall\ h\in C_b\,(V)$ at all point $(u,v)\in V$, the $E_n(h;u,v)$ operator provides the following disparity*

$$|E_n(h;u,v)-h(u,v)|\le 8\ \omega_{mixed}\,(h;\delta_1(n),\delta_2(n))$$

Proof. From the well-known features of mixed moduli of continuity and by the definition of mixed difference, we have

$$\nabla_{(u,v)}h\,[(t,s);u,v]=h(u,t)+h(s,v)-h(s,t)$$

and

$$E_n(h;u,v)-h(u,v)=-H_n(\nabla_{(u,v)}h\,[(s,t);(u,v)]\,;u,v)$$

Then using Cauchy-Schwarz inequality, we have,

$$\begin{aligned}|E_n(h;u,v)-h(u,v)| &\le H_n\left|(\nabla_{(u,v)}h\,[(s,t);u,v]\,;u,v)\right|\\ &\le \Bigg(H_n(e_{00})+\delta_1^{-1}\sqrt{H_n\,((s-u)^2;u,v)}+\delta_2^{-1}\sqrt{H_n\,((t-v)^2;u,v)}\\ &\quad+\delta_1^{-1}\delta_2^{-1}\sqrt{H_n\,((s-u)^2;u,v)\,H_n\,((t-v)^2;u,v)}\Bigg)\,\omega_{mixed}\,(h;\delta_1(n),\delta_2(n))\end{aligned}$$

Then, applying Theorem 2.1 and Theorem 2.4

$$\begin{aligned}|E_n(h;u,v)-h(u,v)| &\leq H_n\left|(\nabla_{(u,v)}h[(s,t);u,v];u,v)\right| \\ &\leq \left(1+\delta_1^{-1}\sqrt{\frac{3}{n}}+\delta_2^{-1}\sqrt{\frac{3}{n}}+\delta_1^{-1}\delta_2^{-1}\sqrt{\frac{3}{n}\frac{3}{n}}\right)\omega_{mixed}(h;\delta_1(n),\delta_2(n))\end{aligned}$$

Therefore, taking $\delta_1=n^{-\frac{1}{2}}$ and $\delta_2=n^{-\frac{1}{2}}$, We achieve the desired result

$$|E_n(h;u,v)-h(u,v)|\leq 8\ \omega_{mixed}(h;\delta_1(n),\delta_2(n))$$

□

4.2. Approximation for the $E_n(h;u,v)$ operators with functions in Lipschitz class

The Lipschitz class $Lip_\beta(\mu,\eta)$ with $\mu,\eta\in(0,1]$ for $h\in C_b(V)$ B-continuous functions is defined as

$$Lip_\beta(\mu,\eta)=\left\{h\in C_b(V):\left|\nabla_{(u,v)}h[(s,t);(u,v)]\right|\leq\beta\ |s-u|^\mu|t-v|^\eta\right\} \tag{4.2}$$

here $(s,t),(u,v)\in V$.

Theorem 4.2. *For $h\in Lip_\beta(\mu,\eta)$, we have*

$$|E_n(h;u,v)-h(u,v)|\leq\beta\ \Psi_n(u)^{\frac{\mu}{2}}\Psi_n(v)^{\frac{\eta}{2}}$$

where $\Psi_n(u)=H_n\left((s-u)^2;u,v\right)$ and $\Psi_n(v)=H_n\left((t-v)^2;u,v\right)$

Proof. From (4.1) and (4.2), we may write

$$\begin{aligned}|E_n(h;u,v)-h(u,v)| &\leq H_n\left(\left|\nabla_{(u,v)}h[(s,t);(u,v)]\right|;u,v\right) \\ &\leq \beta H_n\left(|s-u|^\mu|t-v|^\eta;u,v\right) \\ &= \beta H_n\left(|s-u|^\mu;u,v\right)H_n\left(|t-v|^\eta;u,v\right)\end{aligned}$$

Applying the Hölder's inequality with $(p_1,q_1)=\left(\frac{2}{\mu},\frac{2}{2-\mu}\right)$ and $(p_2,q_2)=\left(\frac{2}{\eta},\frac{2}{2-\eta}\right)$, we get

$$\begin{aligned}|E_n(h;u,v)-h(u,v)| &\leq \beta\left(H_n\left((s-u)^2;u,v\right)^{\frac{\mu}{2}}H_n\left(e_{0,0};u,v\right)^{\frac{2-\mu}{2}}\right. \\ &\qquad\left.\times H_n\left((t-v)^2;u,v\right)^{\frac{\eta}{2}}H_n\left(e_{0,0};u,v\right)^{\frac{2-\eta}{2}}\right) \\ &\leq \beta\Psi_n(u)^{\frac{\mu}{2}}\Psi_n(v)^{\frac{\eta}{2}}\end{aligned}$$

the proof is completed. □

For $(u,v)=(0.05,-0.05)$ in Table 1, we calculated the error in the approximation of $H_n(h;u,v)$ operator and $E_n(h;u,v)$ GBS operator at certain n values. Here $h:V\to R$; $h(u,v)=\left|u^2v^2\right|$

n	$\lvert H_n(h;u,v)-h(u,v)\rvert$	$\lvert E_n(h;u,v)-h(u,v)\rvert$
10	0.008858399150	0.008704518631
25	0.003920956488	0.003831667796
50	0.002201491056	0.002140252626
100	0.001415762625	0.001310766823

Table 1: Error bounds at different n values for $H_n(h;u,v)$ and $E_n(h;u,v)$ GBS operator .

Example 4.3. *The convergence of $H_n(h;u,v)$ operator for n=1 (brown),n=5 (yellow),n=10 (green), n=20 (red) to the function $h(u,v)=\left|u^2v^2\right|$ (blue) is pictorial as shown in Figure 4.1.*

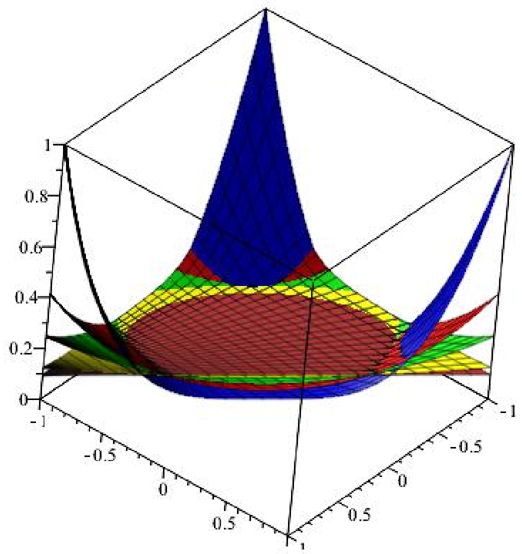

Figure 4.1: The convergence of the $H_n(h;u,v)$ operator to the function $h(u,v)$.

Example 4.4. *The convergence of $E_n(h;u,v)$ GBS operator for n=1 (brown),n=5 (yellow), n=10 (green), n=20 (red) to the function $h(u,v) = \left|u^2v^2\right|$ (blue) is pictorial as shown in Figure 4.2.*

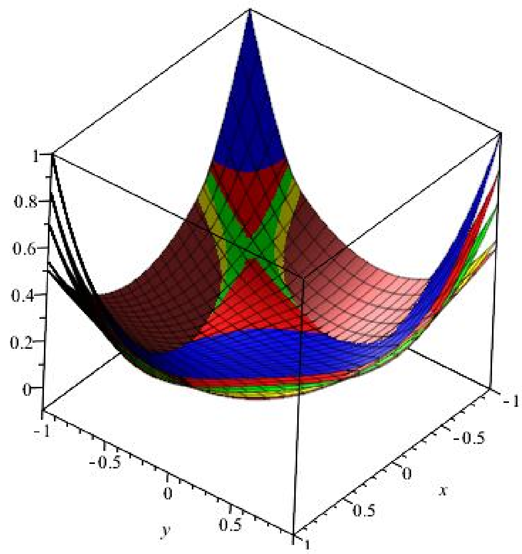

Figure 4.2: The convergence of the $E_n(h;u,v)$ operator to the function $h(u,v)$.

Example 4.5. *For $n = 50$,The convergence of $H_n(h;u,v)$ operator (green) and $E_n(h;u,v)$ GBS operator (red) to the function $h(u,v) = \left|u^2v^2\right|$ (blue) is pictorial as shown in Figure 4.3.*

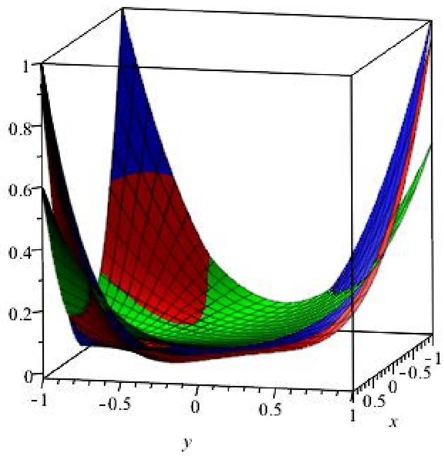

Figure 4.3: The convergence of $H_n(h;u,v)$ operator and $E_n(h;u,v)$ GBS operator to the function $h(u,v) = |u^2v^2|$

(u,v)	$\lvert H_n(h;u,v) - h(u,v)\rvert$	$\lvert E_n(h;u,v) - h(u,v)\rvert$
$(1,-1)$	0.8495540691	0.7663879599
$(0.9,-1)$	0.6861251394	0.6206321071
$(0.9,-0.9)$	0.5459929013	0.5080926891
$(0.8,-0.9)$	0.4291121293	0.4016625079
$(0.8,-1)$	0.5397625418	0.4901649946

Table 2: Error bounds at different (u,v) points for $H_n(h;u,v)$ and $E_n(h;u,v)$ GBS operator .

In table 4.2, we have computed the error in the approximation of $H_n(h;u,v)$ operator and $E_n(h;u,v)$ GBS operator at certain (u,v) points for $n = 200$. It was observed that the convergence rate of $E_n(h;u,v)$ GBS operator to the function $h(u,v)$ is much better than $H_n(h;u,v)$ operator.

5. Conclusion

We proved that bivariate Bernstein-Durrmeyer type operators and GBS form of these operators in a triangular region are better than the classical Bernstein-Durrmeyer type operators.

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