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Theory of Generalized Compactness in Generalized Topological Spaces: Part I. Basic Properties

Mohammad Irshad Khodabocus [1,2]*; **Noor-Ul-Hacq Sookia** [2]
[1] Université des Mascareignes, Faculty of Sustainable Development and Engineering
Department of Emerging Technologies, Rose Hill Campus, Mauritius
[2] University of Mauritius, Faculty of Science, Department of Mathematics
Réduit, Mauritius, sookian@uom.ac.mu

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Abstract: In this paper, a novel class of generalized compact sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compact sets*) in generalized topological spaces (briefly, $\mathscr{T}_{\mathfrak{g}}$-*spaces*) is studied. The study reveals that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies ordinary compactness (briefly, $\mathfrak{T}_{\mathfrak{g}}$-*compactness*) in $\mathscr{T}_{\mathfrak{g}}$-spaces, and such statement implies its analogue in ordinary topological spaces (briefly, $\mathscr{T}$-*spaces*). Diagrams establish the various relationships amongst these types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness presented here and in relation to other types of $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathscr{T}$-spaces presented in the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces, and a nice application supports the overall theory.

Keywords: Generalized topology ($\mathscr{T}_{\mathfrak{g}}$), generalized topological space ($\mathscr{T}_{\mathfrak{g}}$-*space*), generalized sets ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*), generalized compactness ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*).

1. Introduction

The concepts of $\mathfrak{T}$-compactness and $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathscr{T}$-spaces (ordinary and generalized compactness in ordinary topological spaces) and the concepts of $\mathfrak{T}_{\mathfrak{g}}$-compactness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized compactness in generalized topological spaces) are verily the most important topological invariants [3–5, 7, 15–17, 20, 21, 24, 25, 27, 30–38]. For, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, respectively, are absolute properties of $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets [2, 13, 18, 27, 31, 35, 39]. Typical examples of $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathscr{T}$-spaces are α, β, γ-compactness [10, 19, 28]; examples of $\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces are semi-$^*\alpha$, s, gb-compactness [7, 14, 31], whereas examples of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces are bT^{μ}, μ-rgb, πp-compactness [5, 24, 40], among others.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces, several new classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces, similar in descriptions to $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathscr{T}$-spaces, have been studied [20, 21, 24, 27, 30, 32, 36–38].

*Correspondence: ikhodabocus@udm.ac.mu
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In this paper, a novel class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets in $\mathscr{T}_{\mathfrak{g}}$-spaces is studied.

The paper is organized as follows: In Section 2, preliminary notions are described in Subsection 2.1 and the main results of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathscr{T}_{\mathfrak{g}}$-space are reported in Section 3. In Section 4, the establishment of the relationships among various types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are discussed in Subsection 4.1. To support the work, a nice application of the concept of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Subsection 4.2. Finally, Subsection 4.3 provides concluding remarks and future directions of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathscr{T}_{\mathfrak{g}}$-space.

2. Theory

2.1. Preliminaries

Notations and definitions not presented here are presented in [22, 23].

The set $\mathfrak{U}$ denotes the universe of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness and containing as elements all sets (Λ-sets: $\Lambda \in \{\Omega, \Sigma\}$; $\mathscr{T}_{\Lambda}$, $\mathfrak{g}$-$\mathscr{T}_{\Lambda}$, $\mathfrak{T}_{\Lambda}$, $\mathfrak{g}$-$\mathfrak{T}_{\Lambda}$-sets; $\mathscr{T}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g},\Lambda}$, $\mathfrak{T}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-sets, to name a few) considered in this theory, and $I_{n}^{0} \stackrel{\text{def}}{=} \{\nu \in \mathbb{N}^{0} : \nu \leq n\}$; index sets I_{∞}^{0}, I_{n}^{*}, I_{∞}^{*} are defined similarly [22, 23]. Every one-valued map of the type $\mathscr{T}_{\mathfrak{g},\Lambda} : \mathscr{P}(\Lambda) \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g},\nu} : \mathscr{O}_{\mathfrak{g},\nu} \subseteq \Lambda\} \longrightarrow \mathscr{P}(\Lambda)$, satisfying $\mathscr{T}_{\mathfrak{g},\Lambda}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g},\Lambda}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ and $\mathscr{T}_{\mathfrak{g},\Lambda}\big(\bigcup_{\nu \in I_{\infty}^{*}} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_{\infty}^{*}} \mathscr{T}_{\mathfrak{g},\Lambda}(\mathscr{O}_{\mathfrak{g},\nu})$ is called an absolute $\mathfrak{g}$-topology on Λ while $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g},\nu} : \mathscr{O}_{\mathfrak{g},\nu} \subset \Gamma \subseteq \Lambda\} \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Lambda}\}$ defines a relative $\mathfrak{g}$-topology on Γ, and the structures $\mathfrak{T}_{\mathfrak{g},\Lambda} \stackrel{\text{def}}{=} (\Lambda, \mathscr{T}_{\mathfrak{g},\Lambda})$ and $\mathfrak{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$, respectively, are called a $\mathscr{T}_{\mathfrak{g},\Lambda}$-space and a $\mathscr{T}_{\mathfrak{g},\Gamma}$-subspace [22, 23], on which no separation axioms are assumed unless otherwise mentioned [11, 12, 29].

The classes of $\mathscr{T}_{\mathfrak{g},\Lambda}$-open and $\mathscr{T}_{\mathfrak{g},\Lambda}$-closed sets are $\mathscr{T}_{\mathfrak{g},\Lambda} \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Lambda}\}$ and $\neg\mathscr{T}_{\mathfrak{g},\Lambda} \stackrel{\text{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \complement_{\Lambda}(\mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g},\Lambda}\}$, respectively; $\mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g},\Lambda}}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Lambda} : \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g},\Lambda}}[\mathscr{S}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g},\Lambda} : \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}}\}$, respectively, are the classes of $\mathscr{T}_{\mathfrak{g},\Lambda}$-open subsets and $\mathscr{T}_{\mathfrak{g},\Lambda}$-closed supersets (complements of the $\mathscr{T}_{\mathfrak{g},\Lambda}$-open subsets) of the $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ [22, 23].

The operator $\mathrm{cl}_{\mathfrak{g},\Lambda} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$ carrying $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ into its closure $\mathrm{cl}_{\mathfrak{g},\Lambda}(\mathscr{S}_{\mathfrak{g}})$ is called a $\mathfrak{g}$-closure operator and the operator $\mathrm{int}_{\mathfrak{g},\Lambda} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$ carrying it into its interior $\mathrm{int}_{\mathfrak{g},\Lambda}(\mathscr{S}_{\mathfrak{g}})$ is called a $\mathfrak{g}$-interior operator [6, 22, 23], where:

$$\mathrm{int}_{\mathfrak{g},\Lambda}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g},\Lambda}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \quad \mathrm{cl}_{\mathfrak{g},\Lambda}(\mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g},\Lambda}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}. \tag{1}$$

For convenience of notation, let $\mathscr{P}^*(\Omega) = \mathscr{P}(\Omega) \setminus \{\emptyset\}$, $\mathscr{T}_{\mathfrak{g}}^* = \mathscr{T}_{\mathfrak{g}} \setminus \{\emptyset\}$, and $\neg\mathscr{T}_{\mathfrak{g}}^* = \neg\mathscr{T}_{\mathfrak{g}} \setminus \{\emptyset\}$. The mapping $\mathrm{op}_{\mathfrak{g}} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$ is called a $\mathfrak{g}$-operation on $\mathscr{P}(\Lambda)$ if it holds that:

$$\left(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}^*(\Lambda)\right)\left(\exists (\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g},\Lambda}^* \times \neg\mathscr{T}_{\mathfrak{g},\Lambda}^*\right)\left[\left(\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)\right], \quad (2)$$

where $\neg\,\mathrm{op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is called the " complementary $\mathfrak{g}$-operation"on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ and, for all $(\mathscr{S}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g},\mu}, \mathscr{V}_{\mathfrak{g},\nu}) \in_{\alpha\in I_{3^*}} \mathscr{P}^*(\Omega)$ such that $\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g},\mu} \cup \mathscr{V}_{\mathfrak{g},\nu}$ and $\left(\hat{\mathscr{W}}_{\mathfrak{g}}, \neg\hat{\mathscr{W}}_{\mathfrak{g}}\right) = \left(\mathrm{op}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}), \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)$, the following axioms are satisfied:

- Ax. I. $\left(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)$,
- Ax. II. $\left(\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)$,
- Ax. III. $\left(\hat{\mathscr{W}}_{\mathfrak{g}} \subseteq \bigcup_{\sigma=\mu,\nu} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})\right) \bigvee \left(\neg\hat{\mathscr{W}}_{\mathfrak{g}} \supseteq \bigcup_{\sigma=\mu,\nu} \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})\right)$,
- Ax. IV. $\left(\mathscr{U}_{\mathfrak{g},\mu} \subseteq \mathscr{V}_{\mathfrak{g},\nu} \longrightarrow \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\right) \vee \left(\mathscr{U}_{\mathfrak{g},\mu} \supseteq \mathscr{V}_{\mathfrak{g},\nu} \longleftarrow \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu})\right)$

for some $(\mathscr{O}_{\mathfrak{g}}, \mathscr{O}_{\mathfrak{g},\mu}, \mathscr{O}_{\mathfrak{g},\nu}) \in_{\alpha\in I_{3^*}} \mathscr{T}_{\mathfrak{g},\Lambda}^*$ and $(\mathscr{K}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g},\mu}, \mathscr{K}_{\mathfrak{g},\nu}) \in_{\alpha\in I_{3^*}} \neg\mathscr{T}_{\mathfrak{g},\Lambda}^*$ [8, 26]. The class of all possible $\mathfrak{g}$-operators and their complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{g},\Lambda}$-space $\mathfrak{T}_{\mathfrak{g},\Lambda}$ are $\mathscr{L}_{\mathfrak{g}}[\Lambda] \stackrel{\text{def}}{=} \left\{\mathbf{op}_{\mathfrak{g},\nu\mu}(\cdot) = \left(\mathrm{op}_{\mathfrak{g},\nu}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g},\mu}(\cdot)\right) : (\nu,\mu) \in I_3^0 \times I_3^0\right\} = \mathscr{L}_{\mathfrak{g}}^{\omega}[\Lambda] \times \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Lambda]$, where:

$$\begin{aligned}
\mathrm{op}_{\mathfrak{g}}(\cdot) \quad &\in \quad \mathscr{L}_{\mathfrak{g}}^{\omega}[\Lambda] \stackrel{\text{def}}{=} \left\{\mathrm{op}_{\mathfrak{g},0}(\cdot),\ \mathrm{op}_{\mathfrak{g},1}(\cdot),\ \mathrm{op}_{\mathfrak{g},2}(\cdot),\ \mathrm{op}_{\mathfrak{g},3}(\cdot)\right\} \\
&= \quad \left\{\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)\right\}; \\
\neg\,\mathrm{op}_{\mathfrak{g}}(\cdot) \quad &\in \quad \mathscr{L}_{\mathfrak{g}}^{\kappa}[\Lambda] \stackrel{\text{def}}{=} \left\{\neg\,\mathrm{op}_{\mathfrak{g},0}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},1}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},2}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},3}(\cdot)\right\} \\
&= \quad \left\{\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot)\right\}. \qquad (3)
\end{aligned}$$

A $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathscr{S}_{\mathfrak{g},\Lambda} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g},\Lambda}$-space is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-set if and only if $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g},\Lambda} \times \neg\mathscr{T}_{\mathfrak{g},\Lambda}$ and $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathscr{L}_{\mathfrak{g}}[\Lambda]$ exist such that the following statement holds:

$$(\exists \xi)\left[(\xi \in \mathscr{S}_{\mathfrak{g}}) \wedge \left(\left(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right)\right)\right]. \qquad (4)$$

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ is of category ν if and only if is in the class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-sets:

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \stackrel{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \left(\exists \mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \left[\left(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\right) \vee \left(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\right)\right]\right\}. \qquad (5)$$

The classes of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-closed sets, respectively, are defined by

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g},\Lambda}\big] \stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \big(\exists \mathscr{O}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\big)\big[\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big]\big\},$$

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g},\Lambda}\big] \stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \big(\exists \mathscr{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\big)\big[\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big]\big\} \quad (6)$$

and $\mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g},\Lambda}\big] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g},\Lambda}\big] = \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}\big[\mathfrak{T}_{\mathfrak{g}}\big] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ [22, 23].

By adding a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation axiom of type H, called $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$*-axiom*, to the axioms for a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ to obtain a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} \stackrel{\text{def}}{=} \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$ is meant that, for every disjoint pair $(\xi,\zeta) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ of points in $\mathfrak{T}_{\mathfrak{g}}$, there exists a disjoint pair $(\mathscr{O}_{\mathfrak{g},\xi}, \mathscr{O}_{\mathfrak{g},\zeta}) \in \mathscr{T}_{\mathfrak{g}} \times \mathscr{T}_{\mathfrak{g}}$ of $\mathscr{T}_{\mathfrak{g}}$-open sets such that $(\xi,\zeta) \in \big(\mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi}), \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\zeta})\big)$ [23]. The definition follows:

Definition 2.1 *[23][$\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-Space] A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ endowed with a $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-axiom is called a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} \stackrel{\text{def}}{=} \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$.*

By omitting the subscript $\mathfrak{g}$ in almost all symbols of the above definitions, we obtain very similar definitions but in a $\mathscr{T}$-space; see [22, 23].

Definition 2.2 *[23][$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Sets Sequence] Let $\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{T}_{\mathfrak{g}}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category ν in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. The symbol $\big\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\big\rangle_{\alpha\in I_\sigma^*}$ denotes a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category ν in $\mathfrak{T}_{\mathfrak{g}}$ that has been indexed by $I_\sigma^* \subseteq I_\infty^*$, inheriting its order from I_σ^*, and the corresponding index mapping $\phi : \alpha \mapsto \mathscr{S}_{\mathfrak{g},\alpha}$ denotes the α^{th} term of the sequence.*

Throughout, the relation $\big\langle \mathscr{R}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in I_\infty^*} \prec \big\langle \mathscr{S}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in I_\infty^*}$ means that the one preceding " $\prec$" is a subsequence of the other following " $\prec$". Suppose a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is related to a sequence $\big\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\big\rangle_{\alpha\in I_\sigma^*}$ by the relation $\mathscr{R}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\alpha}$, then $\mathscr{R}_{\mathfrak{g}}$ is said to be *covered* by a sequence $\big\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\big\rangle_{\alpha\in I_\sigma^*}$ whose *cardinality is at most* $\sigma \in I_\infty^*$. The definition follows:

Definition 2.3 *[23][$\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Covering] Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\nu \in I_3^0$:*

- *I. $\mathscr{S}_{\mathfrak{g}}$ is said to be " covered" by a sequence $\big\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\big\rangle_{\alpha\in I_\sigma^*}$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open sets whose cardinality is at most $\sigma \in I_\infty^*$ if and only if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_\sigma^*} \mathscr{U}_{\mathfrak{g},\alpha}$.*

- *II. $\mathscr{S}_{\mathfrak{g}}$ is said to be " covered" by a sequence $\big\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\big\rangle_{\alpha\in I_\sigma^*}$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets whose cardinality is at most $\sigma \in I_\infty^*$ if and only if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_\sigma^*} \mathscr{V}_{\mathfrak{g},\alpha}$.*

Accordingly, $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$, $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$, and $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$, respectively, are simply said to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*covering*, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open covering* and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closed covering* of $\mathscr{S}_{\mathfrak{g}}$ whose cardinality is at most $\sigma \in I_{\infty}^{*}$.

Definition 2.4 *[23][$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Subcovering] Let $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ and let $\vartheta : I_{\sigma}^{*} \longrightarrow I_{\vartheta(\sigma)}^{*} \subseteq I_{\sigma}^{*}$ be an index mapping. Then the map*

$$\vartheta : \langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}} \longrightarrow \langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \tag{7}$$

is said to realise a " $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering" $\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ from the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathscr{S}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ if and only if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)}$.

Thus, $\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{S}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ is equivalent to this definition, meaning that, for every $\vartheta(\alpha) \in I_{\vartheta(\sigma)}^{*} \subseteq I_{\sigma}^{*}$, there exists $\alpha \in I_{\sigma}^{*} \subseteq I_{\infty}^{*}$ such that $\mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} = \mathscr{S}_{\mathfrak{g},\alpha}$. It is plain that, for every $\sigma \in I_{\infty}^{*}$, $\vartheta(\sigma) = \mathrm{card}(I_{\vartheta(\sigma)}^{*}) \leq \mathrm{card}(I_{\sigma}^{*}) = \sigma$.

Definition 2.5 *[23][$\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Compact Set] A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is said to be $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compact if and only if, for every $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$,*

$$\exists \langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} : \quad \mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}, \tag{8}$$

where $\vartheta(\sigma) = \mathrm{card}(I_{\vartheta(\sigma)}^{*}) \leq \mathrm{card}(I_{\sigma}^{*}) = \sigma$. The class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets of category $\nu \in I_{3}^{0}$ is:

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \Big\{ \mathscr{S}_{\mathfrak{g}} : \big[\forall \langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}\big]\big[\exists \langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\big] \\ \Big(\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \Big) \Big\}.\end{aligned} \tag{9}$$

Thus, by a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compact set is meant a type of set $\mathfrak{T}_{\mathfrak{g}}$-set every $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open covering of which has a finite $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering [27, 36, 37]. Further, it is clear from the context that, $\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu\in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$; its elements, then, are simply called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets. Stated differently, the above definition says that, given any sequence $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that every point $\xi \in \mathscr{S}_{\mathfrak{g}}$ belongs to at least one $\mathscr{U}_{\mathfrak{g},\alpha}$, $\alpha \in I_{\sigma}^{*}$, it is possible to select from $\langle \mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ a finite number of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\mathscr{U}_{\mathfrak{g},\vartheta(1)}, \mathscr{U}_{\mathfrak{g},\vartheta(2)}, \ldots, \mathscr{U}_{\mathfrak{g},\vartheta(\sigma)}$ whose union covers all of $\mathscr{S}_{\mathfrak{g}}$.

Remark 2.6 *Since* $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compactness of a* $\mathfrak{T}_{\mathfrak{g}}$*-set is defined in terms of* relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets.

Definition 2.7 *[23][*$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Refinement] A* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-covering* $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ *of a* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is a "* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-refinement" of another* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-covering* $\langle \mathscr{R}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\beta\in I_{\mu}^{*}}$ *of the same* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}}$ *if and only if:*

$$\left(\forall\alpha \in I_{\sigma}^{*}\right)\left(\exists\beta \in I_{\mu}^{*}\right)\left[\mathscr{S}_{\mathfrak{g},\alpha} \subseteq \mathscr{R}_{\mathfrak{g},\beta}\right]. \tag{10}$$

In the event that $\mathscr{S}_{\mathfrak{g}} = \Omega$, $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-covering of $\mathfrak{T}_{\mathfrak{g}}$ if $\Omega = \bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\alpha}$. Accordingly, $\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ is a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-subcovering of $\mathfrak{T}_{\mathfrak{g}}$ if the relation $\Omega = \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)}$ holds, where $\vartheta(\sigma) = \operatorname{card}\left(I_{\vartheta(\sigma)}^{*}\right) < \operatorname{card}(I_{\sigma}^{*}) < \infty$. The definition follows.

Definition 2.8 *[23][*$\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-Space] A* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is called a* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space denoted* $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ *if and only if each* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-open covering* $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ *of* $\mathfrak{T}_{\mathfrak{g}}$ *has a finite* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-open subcovering.*

In the sequel, by a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ is meant $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \bigvee_{\nu\in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \bigvee_{\nu\in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}) = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$.

3. Main Results

The main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are presented in this section.

Theorem 3.1 *A* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})})$ *is a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ *if and only if every sequence* $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ *of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed sets which has the finite intersection property has a non-empty intersection.*

Proof *Necessity.* Let the $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})})$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$, and let $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ be a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ such that $\bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$. For every $\alpha \in I_{\sigma}^{*}$, set $\mathscr{U}_{\mathfrak{g},\alpha} = \complement(\mathscr{V}_{\mathfrak{g},\alpha})$ and consider the sequence $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets. Since $\bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\alpha} = \bigcup_{\alpha\in I_{\sigma}^{*}} \complement(\mathscr{V}_{\mathfrak{g},\alpha}) = \complement\left(\bigcap_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha}\right) = \Omega$, it follows that $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. But $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space

$\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ and, thus, there exists a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\left\langle \mathscr{U}_{\mathfrak{g},\beta(\alpha)} \right\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \prec \left\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ such that

$$\Omega = \bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{U}_{\mathfrak{g},\beta(\alpha)} = \bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \complement\left(\mathscr{V}_{\mathfrak{g},\beta(\alpha)}\right) = \complement\Bigg(\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)}\Bigg).$$

This implies that $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} = \emptyset$. Hence, if a sequence $\left\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ has the finite intersection property, then $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} \neq \emptyset$.

Sufficiency. Conversely, suppose that $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space in which every sequence $\left\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets which has the finite intersection property has a non-empty intersection. Then, for every subsequence $\left\langle \mathscr{V}_{\mathfrak{g},\beta(\alpha)} \right\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \prec \left\langle \mathscr{V}_{\mathfrak{g},\alpha} \right\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets, the relation $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} \neq \emptyset$ holds. Consequently, $\bigcap_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} \neq \emptyset$. In other words, $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} \neq \emptyset$ for every $I_{n}^{*} \subseteq I_{\sigma}^{*}$ implies $\bigcap_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} \neq \emptyset$. But this is the contrapositive statement of $\bigcap_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$ implies that there exists $I_{n}^{*} \subseteq I_{\sigma}^{*}$ such that $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} = \emptyset$. It results that, every sequence $\left\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets of $\mathfrak{T}_{\mathfrak{g}}$, $\bigcap_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$ implies $\left\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ contains a finite subsequence $\left\langle \mathscr{V}_{\mathfrak{g},\beta(\alpha)} \right\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}$ of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closed sets with $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} = \emptyset$. Hence, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$. □

An interesting remark may well be given at this stage.

Remark 3.2 *In particular, if the* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ *is a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ *and the elements of* $\left\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ *forms a descending sequence* $\mathscr{V}_{\mathfrak{g},1} \supset \mathscr{V}_{\mathfrak{g},2} \supset \cdots \supset \mathscr{V}_{\mathfrak{g},\alpha} \supset \cdots$ *of non-empty* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closed sets, then* $\bigcap_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} \neq \emptyset$*. Such property in its own right is weaker than* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-compactness. In fact, it indicates the sense in which* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-compactness asserts that the* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ *has enough points, namely, at least enough points to yield one point in each such intersection of a descending sequence* $\mathscr{V}_{\mathfrak{g},1} \supset \mathscr{V}_{\mathfrak{g},2} \supset \cdots \supset \mathscr{V}_{\mathfrak{g},\alpha} \supset \cdots$ *of non-empty* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closed sets.*

Theorem 3.3 ($\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-Refinement) *In a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, any* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-subcovering of the type* $\left\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ *derived from a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-covering* $\left\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \right\rangle_{\alpha\in I_{\sigma}^{*}}$ *is a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-refinement.*

Proof Let $\left\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ be any $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-subcovering derived from a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-covering

$\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, it results, consequently, that the relation $\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{S}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ holds true. Thus,

$$\big(\forall \vartheta(\alpha) \in I_{\vartheta(\sigma)}^{*}\big)\big(\exists \alpha \in I_{\sigma}^{*}\big)\big[\mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \subseteq \mathscr{S}_{\mathfrak{g},\alpha}\big].$$

Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering $\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ derived from the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is therefore a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-refinement. This completes the proof of the theorem. □

Theorem 3.4 *Let* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. *Then,* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *if and only if, for each* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open covering* $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ *of* $\mathscr{S}_{\mathfrak{g}}$*, there is a finite* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open subcovering* $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ *of* $\mathscr{S}_{\mathfrak{g}}$*:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \Leftrightarrow \big(\forall \langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}\big)\big(\exists \langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}\big)$$
$$\Big[\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\Big]. \tag{11}$$

Proof *Necessity.* Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$, and let $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$. Then, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha \in I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\alpha}$ and, consequently, $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\alpha \in I_{\sigma}^{*}} (\mathscr{U}_{\mathfrak{g},\alpha} \cap \mathscr{S}_{\mathfrak{g}})$. Therefore, $\langle \mathscr{U}_{\mathfrak{g},\alpha} \cap \mathscr{S}_{\mathfrak{g}} \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$ by relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\mathscr{U}_{\mathfrak{g},1} \cap \mathscr{S}_{\mathfrak{g}}$, $\mathscr{U}_{\mathfrak{g},2} \cap \mathscr{S}_{\mathfrak{g}}$, ..., $\mathscr{U}_{\mathfrak{g},\sigma} \cap \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ such that $\mathscr{S}_{\mathfrak{g}} = \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} (\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{S}_{\mathfrak{g}})$. Thus, it follows that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{S}_{\mathfrak{g}}$.

Sufficiency. Conversely, suppose that, for every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$, $\langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ has a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of the type $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$. It must be shown that, given a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \hat{\mathscr{U}}_{\mathfrak{g},\beta} \rangle_{\beta \in I_{\mu}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ by relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\hat{\mathscr{U}}_{\mathfrak{g},1}$, $\hat{\mathscr{U}}_{\mathfrak{g},2}$, ..., $\hat{\mathscr{U}}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\beta)} \rangle_{(\beta,\vartheta(\beta)) \in I_{\mu}^{*} \times I_{\vartheta(\mu)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ such that $\mathscr{S}_{\mathfrak{g}} = \bigcup_{(\beta,\vartheta(\beta)) \in I_{\mu}^{*} \times I_{\vartheta(\mu)}^{*}} \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\beta)}$. For every $\beta \in I_{\mu}^{*}$, since $\hat{\mathscr{U}}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ is a relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set in $\mathscr{S}_{\mathfrak{g}}$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\hat{\mathscr{U}}_{\mathfrak{g},\beta} = \mathscr{U}_{\mathfrak{g},\beta} \cap \mathscr{S}_{\mathfrak{g}}$. But $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\beta \in I_{\mu}^{*}} \hat{\mathscr{U}}_{\mathfrak{g},\beta} = \bigcup_{\beta \in I_{\mu}^{*}} (\mathscr{U}_{\mathfrak{g},\beta} \cap \mathscr{S}_{\mathfrak{g}}) \subseteq \bigcup_{\beta \in I_{\mu}^{*}} \mathscr{U}_{\mathfrak{g},\beta}$ and, consequently, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\beta \in I_{\mu}^{*}} \mathscr{U}_{\mathfrak{g},\beta}$, implying that $\langle \mathscr{U}_{\mathfrak{g},\beta} \rangle_{\beta \in I_{\mu}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$ by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\mathscr{U}_{\mathfrak{g},1}$, $\mathscr{U}_{\mathfrak{g},2}$, ..., $\mathscr{U}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. By hypothesis, there exists a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering

$\langle\mathscr{U}_{\mathfrak{g},\vartheta(\beta)}\rangle_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ such that $\mathscr{S}_{\mathfrak{g}}\subseteq\bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\beta)}$. Thus,

$$\mathscr{S}_{\mathfrak{g}}=\Big(\bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\beta)}\Big)\cap\mathscr{S}_{\mathfrak{g}} = \bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}\big(\mathscr{U}_{\mathfrak{g},\vartheta(\beta)}\cap\mathscr{S}_{\mathfrak{g}}\big) = \bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\beta)}.$$

Hence, it results that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle\hat{\mathscr{U}}_{\mathfrak{g},\beta}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\beta\in I_{\mu}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ by relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\hat{\mathscr{U}}_{\mathfrak{g},1}, \hat{\mathscr{U}}_{\mathfrak{g},2}, \ldots, \hat{\mathscr{U}}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ has a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\beta)}\rangle_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$. □

Theorem 3.5 *If $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be $\mu\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\alpha\in I_{\mu}^{*}}\mathscr{S}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\bigwedge\nolimits_{\alpha\in I_{\mu}^{*}}\big(\mathscr{S}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]\big)\ \Rightarrow\ \bigcup\nolimits_{\alpha\in I_{\mu}^{*}}\mathscr{S}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]. \tag{12}$$

Proof Let $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be $\mu\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets in $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\alpha\in I_{\mu}^{*}$, there exists $\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}\rangle_{(\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}}\prec\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\vartheta(\alpha)\in I_{\sigma}^{*}}$, where $I_{\beta(\sigma)}^{*}\subseteq I_{\sigma}^{*}$, such that $\mathscr{S}_{\mathfrak{g},\alpha}\subseteq\bigcup_{(\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}$ holds. Consequently,

$$\bigcup\nolimits_{\alpha\in I_{\mu}^{*}}\mathscr{S}_{\mathfrak{g},\alpha}\subseteq\bigcup_{\alpha\in I_{\mu}^{*}}\Big(\bigcup_{(\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}\Big)\subseteq\bigcup_{(\alpha,\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\mu}^{*}\times I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}.$$

Hence, it follows that, $\bigcup_{\alpha\in I_{\mu}^{*}}\mathscr{S}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. □

Theorem 3.6 *If $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be any finite $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$:*

$$\big(\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}\big)\wedge\big(\mathrm{card}(\mathscr{S}_{\mathfrak{g}})<\infty\big)\ \Rightarrow\ \mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]. \tag{13}$$

Proof Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be any finite $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, there exist $\langle\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\langle\mathscr{O}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ such that $\bigcup_{\xi\in\mathscr{S}_{\mathfrak{g}}}\{\xi\}\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}$ holds. Since $\mathscr{O}_{\mathfrak{g},\alpha}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})$ for every $\alpha\in I_{\sigma}^{*}$ and $\bigcup_{\xi\in\mathscr{S}_{\mathfrak{g}}}\{\xi\}=\mathscr{S}_{\mathfrak{g}}$, it results that,

$$\mathscr{S}_{\mathfrak{g}}\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \subseteq \mathrm{op}_{\mathfrak{g}}\Big(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\Big) \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathrm{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\big)\subseteq\bigcup_{\alpha\in I_{\sigma}^{*}}\mathrm{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},\alpha}\big).$$

Therefore, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right)$. But, for every pair $(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}$, $\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Consequently, for every $(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}$, there exists $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ such that $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} = \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right)$. Thus, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$ and hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. This completes the proof of the theorem. □

Corollary 3.7 *Let* $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set of a discrete* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$. *Then,* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *if and only if it is a finite* $\mathfrak{T}_{\mathfrak{g}}$*-set.*

Proposition 3.8 *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ *is a finite strong* $\mathscr{T}_{\mathfrak{g}}$*-space, then it is a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$*:*

$$\left(\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})\right)\wedge\left(\mathrm{card}\,(\Omega)<\infty\right) \;\Rightarrow\; \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right). \tag{14}$$

Proof Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ be a finite strong $\mathscr{T}_{\mathfrak{g}}$-space with $\Omega = \left\{\xi_{\alpha}:\ \alpha\in I_{\mu}^{*}\right\}$ and $\mu<\infty$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a finite strong $\mathscr{T}_{\mathfrak{g}}$-space, if $\left\langle\mathscr{O}_{\mathfrak{g},\alpha}\right\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω, then, for every $\alpha\in I_{\mu}^{*}$, there exists a $\vartheta(\alpha)\in I_{\sigma}^{*}$ such that $\xi_{\alpha}\in\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}$. Thus, $\Omega = \bigcup_{\alpha\in I_{\mu}^{*}}\{\xi_{\alpha}\} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}} \mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}$ and consequently, $\left\langle\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open subcovering of Ω. But, for every $(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}$, $\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right) \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Consequently, for each $(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}$, there corresponds a $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ such that $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} = \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right)$. Thus, $\Omega \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$. Hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$. The proof of the proposition is complete. □

Proposition 3.9 *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ *be a* $\mathscr{T}_{\mathfrak{g}}$*-space generated by unit* $\mathfrak{T}_{\mathfrak{g}}$*-sets of* Ω*, then any infinite* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *is not* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-compact.*

Proof Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any infinite $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ generated by unit $\mathfrak{T}_{\mathfrak{g}}$-sets of Ω. Then, since $\{\xi\}\in\mathfrak{T}_{\mathfrak{g}}$ and $\{\xi\}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\{\xi\}\right)$ hold for every $\{\xi\}\subset\mathscr{S}_{\mathfrak{g}}$, it follows that, for every $\xi\in\mathscr{S}_{\mathfrak{g}}$, $\{\xi\}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\{\xi\}\right)$. Consequently, $\mathscr{S}_{\mathfrak{g}} = \bigcup_{\xi\in\mathscr{S}_{\mathfrak{g}}}\{\xi\} \subseteq \bigcup_{\xi\in\mathscr{S}_{\mathfrak{g}}}\mathrm{op}_{\mathfrak{g}}\left(\{\xi\}\right)$. Clearly, $\mathrm{op}_{\mathfrak{g}}\left(\{\xi\}\right)\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ for every $\xi\in\mathscr{S}_{\mathfrak{g}}$ and therefore, there exists, for each $\xi\in\mathscr{S}_{\mathfrak{g}}$, a $\mathscr{U}_{\mathfrak{g},\xi}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ such that $\mathscr{U}_{\mathfrak{g},\xi} = \mathrm{op}_{\mathfrak{g}}\left(\{\xi\}\right)$. Hence, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\xi\in\mathscr{S}_{\mathfrak{g}}}\mathscr{U}_{\mathfrak{g},\xi}$, implying that $\left\langle\mathscr{U}_{\mathfrak{g},\xi}\right\rangle_{\xi\in\mathscr{S}_{\mathfrak{g}}}$ is an infinite $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$. Consequently, there exists no finite $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\left\langle\mathscr{U}_{\mathfrak{g},\vartheta(\xi)}\right\rangle_{(\xi,\vartheta(\xi))\in\mathscr{S}_{\mathfrak{g}}\times I_{\sigma}^{*}} \prec \left\langle\mathscr{U}_{\mathfrak{g},\xi}\right\rangle_{\xi\in\mathscr{S}_{\mathfrak{g}}}$ of $\mathscr{S}_{\mathfrak{g}}$ such that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi))\in\mathscr{S}_{\mathfrak{g}}\times I_{\sigma}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\xi)}$. Hence, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. This completes the proof of the theorem. □

Corollary 3.10 *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *be a* $\mathscr{T}_{\mathfrak{g}}$*-space generated by unit* $\mathfrak{T}_{\mathfrak{g}}$*-sets of* Ω *and* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$*, then* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *if and only if it is a finite* $\mathfrak{T}_{\mathfrak{g}}$*-set in* $\mathfrak{T}_{\mathfrak{g}}$*.*

Theorem 3.11 *Let* $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. If* $\mathscr{S}_{\mathfrak{g}}$ *be* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact, then it is also* $\mathfrak{T}_{\mathfrak{g}}$*-compact:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \;\Rightarrow\; \mathscr{S}_{\mathfrak{g}} \in \text{A}[\mathfrak{T}_{\mathfrak{g}}]. \tag{15}$$

Proof Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathscr{S}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha} \in I_{\sigma}^{*}$ of $\mathscr{S}_{\mathfrak{g}}$ which has a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ such that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$. The assertion that, $\mathscr{U}_{\mathfrak{g},\vartheta(\xi)} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\alpha, \vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$ implies the existence of $\mathscr{O}_{\mathfrak{g},\vartheta(\xi)} \in \mathscr{T}_{\mathfrak{g}}$ such that, $\mathscr{U}_{\mathfrak{g},\vartheta(\xi)} \subseteq \text{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\vartheta(\xi)})$ for every $(\alpha, \vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$. Consequently,

$$\begin{aligned} \mathscr{S}_{\mathfrak{g}} &= \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{S}_{\mathfrak{g}}\right) \\ &\subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \cap \text{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\vartheta(\xi)})\right) \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathscr{O}_{\mathfrak{g},\vartheta(\xi)}, \end{aligned}$$

thereby implying, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathscr{O}_{\mathfrak{g},\vartheta(\xi)}$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{S}_{\mathfrak{g}} \in \text{A}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. □

Proposition 3.12 *If* $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ *be any infinite* $\mathfrak{T}_{\mathfrak{g}}$*-set of a discrete* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then* $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$*.*

Proof Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a discrete $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ if and only if it is a finite $\mathfrak{T}_{\mathfrak{g}}$-set. Since $\mathfrak{T}_{\mathfrak{g}}$ is a discrete $\mathscr{T}_{\mathfrak{g}}$-space, consider the class $\{\{\xi\} : \xi \in \mathscr{S}_{\mathfrak{g}}\}$ of unit $\mathfrak{T}_{\mathfrak{g}}$-sets of $\mathscr{S}_{\mathfrak{g}}$. Clearly, the relation $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\xi \in \mathscr{S}_{\mathfrak{g}}} \{\xi\} \subseteq \bigcup_{\xi \in \mathscr{S}_{\mathfrak{g}}} \text{op}_{\mathfrak{g}}(\{\xi\})$ holds and, for every $\xi \in \mathscr{S}_{\mathfrak{g}}$, $\text{op}_{\mathfrak{g}}(\{\xi\}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Accordingly, for every $\xi \in \mathscr{S}_{\mathfrak{g}}$, set $\text{op}_{\mathfrak{g}}(\{\xi\}) = \mathscr{U}_{\mathfrak{g},\xi}$. Then, $\langle \mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\xi \in \mathscr{S}_{\mathfrak{g}}}$ is an infinite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$. Consequently, $\langle \mathscr{U}_{\mathfrak{g},\xi} \rangle_{\xi \in \mathscr{S}_{\mathfrak{g}}}$ contains no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\xi)} \rangle_{(\xi,\vartheta(\xi)) \in \mathscr{S}_{\mathfrak{g}} \times I_{\sigma}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\xi} \rangle_{\xi \in \mathscr{S}_{\mathfrak{g}}}$ of $\mathscr{S}_{\mathfrak{g}}$ such that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi)) \in \mathscr{S}_{\mathfrak{g}} \times I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\xi)}$. Hence, $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. □

Corollary 3.13 *Let* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *to be a* $\mathscr{T}_{\mathfrak{g}}$*-space. If* $\mathfrak{T}_{\mathfrak{g}}$ *is a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\text{A}]}$*-space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\text{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\text{A}]})$*, then it is also a* $\mathscr{T}_{\mathfrak{g}}^{[\text{A}]}$*-space* $\mathfrak{T}_{\mathfrak{g}}^{[\text{A}]} = (\Omega, \mathscr{T}_{\mathfrak{g}}^{[\text{A}]})$*.*

Theorem 3.14 *A necessary and sufficient conditions for a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ to be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ is that, whenever a sequence $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets is such that $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$, then there exists $\langle \mathscr{V}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \prec \langle \mathscr{V}_{\mathfrak{g},\alpha}\rangle_{\alpha \in I_{\sigma}^{*}}$ such that the relation $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$ holds.*

Proof *Necessity.* Suppose $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ and a sequence $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets is given such that $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$. Then, $\bigcup_{\alpha \in I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\alpha} = \bigcup_{\alpha \in I_{\sigma}^{*}} \complement(\mathscr{V}_{\mathfrak{g},\alpha}) = \complement(\bigcap_{\alpha \in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha}) = \Omega$, so that $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Thus, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha \in I_{\sigma}^{*}}$ and, thus, $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\beta(\alpha)} = \complement(\bigcup_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathscr{U}_{\mathfrak{g},\beta(\alpha)}) = \emptyset$.

Sufficiency. Conversely, suppose that, for every $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha} = \emptyset$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering given by $\langle \mathscr{V}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \prec \langle \mathscr{V}_{\mathfrak{g},\alpha}\rangle_{\alpha \in I_{\sigma}^{*}}$ such that $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathscr{V}_{\mathfrak{g},\alpha}$. Further, let $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\mu}^{*}}$ stand for a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Then $\langle \complement(\mathscr{U}_{\mathfrak{g},\alpha}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\mu}^{*}}$ is a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that $\bigcap_{\alpha \in I_{\mu}^{*}} \complement(\mathscr{U}_{\mathfrak{g},\alpha}) = \emptyset$. Thus $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\mu}^{*} \times I_{n}^{*}} \complement(\mathscr{U}_{\mathfrak{g},\beta(\alpha)}) = \emptyset$ and $\langle \mathscr{U}_{\mathfrak{g},\beta(\alpha)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{(\alpha,\beta(\alpha)) \in I_{\mu}^{*} \times I_{n}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathfrak{T}_{\mathfrak{g}}$. □

If $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ be a $\mathscr{T}_{\mathfrak{g}}$-space such that $(\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma}) \subseteq (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $(\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma}) \subseteq (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ are two $\mathscr{T}_{\mathfrak{g}}$-spaces satisfying $(\Omega, \mathscr{T}_{\mathfrak{g},\Omega}) \neq (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$, then $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longrightarrow \mathscr{P}(\Gamma)$ is the same whether $\mathfrak{T}_{\mathfrak{g},\Gamma} \subseteq \mathfrak{T}_{\mathfrak{g},\Omega}$ or $\mathfrak{T}_{\mathfrak{g},\Gamma} \subseteq \mathfrak{T}_{\mathfrak{g},\Sigma}$ and, hence, the assertion that, $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Gamma}^{[\mathrm{A}]} = (\Gamma, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Gamma}^{[\mathrm{A}]})$ depends only on the elements forming the structure $(\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ may be related to $\mathscr{T}_{\mathfrak{g},\Omega} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ by virtue of the following theorem.

Theorem 3.15 *Let $\Gamma \subset \Omega$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, the following statements are equivalent:*

- *I.* $\Gamma \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *with respect to the absolute* $\mathfrak{g}$*-topology* $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}_{\mathfrak{g}}(\Omega) \longrightarrow \mathscr{P}_{\mathfrak{g}}(\Omega)$.

- *II.* $\Gamma \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *with respect to the relative* $\mathfrak{g}$*-topology* $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}_{\mathfrak{g}}(\Gamma) \mapsto \mathscr{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}\}$.

Proof *I.* $\longrightarrow$ *II.* Let $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Γ with respect to the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma}:\mathscr{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathscr{T}_{\mathfrak{g},\Gamma}$. The relative $\mathfrak{g}$-topology being $\mathscr{T}_{\mathfrak{g},\Gamma}:\mathscr{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathscr{T}_{\mathfrak{g},\Gamma}\stackrel{\text{def}}{=}\{\mathscr{O}_{\mathfrak{g}}\cap\Gamma:\ \mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}\}$, it consequently follows that, for every $\alpha\in I_{\sigma}^{*}$, there exists $\hat{\mathscr{O}}_{\mathfrak{g},\alpha}\in\mathscr{T}_{\mathfrak{g}}$ such that $\mathscr{U}_{\mathfrak{g},\alpha}\subseteq\operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})=\operatorname{op}_{\mathfrak{g}}(\hat{\mathscr{O}}_{\mathfrak{g},\alpha}\cap\Gamma)\subseteq\operatorname{op}_{\mathfrak{g}}(\hat{\mathscr{O}}_{\mathfrak{g},\alpha})$. For every $\alpha\in I_{\sigma}^{*}$, set $\hat{\mathscr{U}}_{\mathfrak{g},\alpha}=\operatorname{op}_{\mathfrak{g}}(\hat{\mathscr{O}}_{\mathfrak{g},\alpha}\cap\Gamma)$. Thus, $\Gamma\subseteq\bigcup_{\alpha\in I_{\sigma}^{*}}\hat{\mathscr{U}}_{\mathfrak{g},\alpha}$ and therefore, $\langle\hat{\mathscr{U}}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Γ with respect to the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}_{\mathfrak{g}}(\Omega)\longrightarrow\mathscr{P}_{\mathfrak{g}}(\Omega)$. By virtue of I., $\Gamma\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ with respect to $\mathscr{T}_{\mathfrak{g}}$ and consequently, a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\langle\hat{\mathscr{U}}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ exists where, for every $(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}$, $\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}=\operatorname{op}_{\mathfrak{g}}(\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\cap\Gamma)$. But then

$$\begin{aligned}\Gamma\subseteq\Gamma\cap\Bigg(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\Bigg) &= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\big(\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\cap\Gamma\big)\\ &= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}.\end{aligned}$$

Thus, it follows that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle\mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ contains a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of Γ with respect to the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma}:\mathscr{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathscr{T}_{\mathfrak{g},\Gamma}$. Hence, $(\Gamma,\mathscr{T}_{\mathfrak{g},\Gamma})$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space. This proves that I. implies II.

I. $\leftarrow$ *II.* Let $\langle\hat{\mathscr{U}}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Γ with respect to the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}_{\mathfrak{g}}(\Omega)\longrightarrow\mathscr{P}_{\mathfrak{g}}(\Omega)$. For every $\alpha\in I_{\sigma}^{*}$, there exists, then, $\hat{\mathscr{O}}_{\mathfrak{g},\alpha}\in\mathscr{T}_{\mathfrak{g}}$ such that $\hat{\mathscr{U}}_{\mathfrak{g},\alpha}=\operatorname{op}_{\mathfrak{g}}(\hat{\mathscr{O}}_{\mathfrak{g},\alpha})$. For every $\alpha\in I_{\sigma}^{*}$, set $\mathscr{O}_{\mathfrak{g},\alpha}=\hat{\mathscr{O}}_{\mathfrak{g},\alpha}\cap\Gamma$. Consequently, $\Gamma\subseteq\bigcup_{\alpha\in I_{\sigma}^{*}}\hat{\mathscr{U}}_{\mathfrak{g},\alpha}$ implies

$$\begin{aligned}\Gamma\subseteq\Gamma\cap\Bigg(\bigcup_{\alpha\in I_{\sigma}^{*}}\hat{\mathscr{U}}_{\mathfrak{g},\alpha}\Bigg)=\bigcup_{\alpha\in I_{\sigma}^{*}}\big(\Gamma\cap\hat{\mathscr{U}}_{\mathfrak{g},\alpha}\big) &= \bigcup_{\alpha\in I_{\sigma}^{*}}\big(\Gamma\cap\operatorname{op}_{\mathfrak{g}}(\hat{\mathscr{O}}_{\mathfrak{g},\alpha})\big)\\ &= \bigcup_{\alpha\in I_{\sigma}^{*}}\operatorname{op}_{\mathfrak{g}}\big(\hat{\mathscr{O}}_{\mathfrak{g},\alpha}\cap\Gamma\big)=\bigcup_{\alpha\in I_{\sigma}^{*}}\operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})\end{aligned}$$

and from which it results that, $\Gamma\subseteq\bigcup_{\alpha\in I_{\sigma}^{*}}\operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})$. Since $\mathscr{O}_{\mathfrak{g},\alpha}\in\mathscr{T}_{\mathfrak{g},\Gamma}$ and $\operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I_{\sigma}^{*}$, set $\mathscr{U}_{\mathfrak{g},\alpha}=\operatorname{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})$. Then, $\langle\mathscr{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Γ with respect to the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma}:\mathscr{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathscr{T}_{\mathfrak{g},\Gamma}$. But, by hypothesis, $\Gamma\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ with respect to the relative $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g},\Gamma}:\mathscr{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathscr{T}_{\mathfrak{g},\Gamma}$ and, therefore, a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ exists. Accordingly,

$$\begin{aligned}
\Gamma &\subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} = \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\right) \\
&= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\left(\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\cap\Gamma\right) = \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \left(\Gamma\cap\mathrm{op}_{\mathfrak{g}}\left(\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\right)\right) \\
&= \Gamma\cap\left(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\left(\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\right)\right) \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\left(\hat{\mathscr{O}}_{\mathfrak{g},\vartheta(\alpha)}\right) \\
&= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}.
\end{aligned}$$

Thus, $\langle \hat{\mathscr{U}}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ is reducible to a a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ with respect to the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}_{\mathfrak{g}}(\Omega)\longrightarrow\mathscr{P}_{\mathfrak{g}}(\Omega)$. Hence, $\Gamma\in\mathfrak{g}\text{-A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ with respect to the absolute $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}}:\mathscr{P}_{\mathfrak{g}}(\Omega)\longrightarrow\mathscr{P}_{\mathfrak{g}}(\Omega)$. Thus proves that I. is implied by II. □

Theorem 3.16 *Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Then, the following statements are equivalent:*

- *I. $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$.*
- *II. For every sequence $\langle \mathscr{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of $\mathfrak{T}_{\mathfrak{g}}$, the equality relation $\bigcap_{\alpha\in I_{\sigma}^{*}}\mathscr{V}_{\mathfrak{g},\alpha}=\emptyset$ implies that the sequence $\langle \mathscr{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ contains a finite subsequence $\langle \mathscr{V}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets with $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{V}_{\mathfrak{g},\beta(\alpha)}=\emptyset$.*

Proof *I.* $\longrightarrow$ *II.* Suppose $\bigcap_{\alpha\in I_{\sigma}^{*}}\mathscr{V}_{\mathfrak{g},\alpha}=\emptyset$. Then, by virtue of De Morgan's Law, it follows that $\Omega=\complement(\emptyset)=\complement\big(\bigcap_{\alpha\in I_{\sigma}^{*}}\mathscr{V}_{\mathfrak{g},\alpha}\big)=\bigcup_{\alpha\in I_{\sigma}^{*}}\complement(\mathscr{V}_{\mathfrak{g},\alpha})=\bigcup_{\alpha\in I_{\sigma}^{*}}\mathscr{U}_{\mathfrak{g},\alpha}$. Therefore, $\langle \mathscr{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. But since $\mathfrak{T}_{\mathfrak{g}}$ is, by hypothesis, a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$, there exists a finite subsequence $\langle \mathscr{U}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets such that $\Omega=\bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{U}_{\mathfrak{g},\beta(\alpha)}$. Thus, by De Morgan's Law, it follows that $\emptyset=\complement(\Omega)=\complement\big(\bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{U}_{\mathfrak{g},\beta(\alpha)}\big)=\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\complement\left(\mathscr{U}_{\mathfrak{g},\beta(\alpha)}\right)=\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{V}_{\mathfrak{g},\beta(\alpha)}$. This proves that I. implies II.

I. $\leftarrow$ *II.* Let $\langle \mathscr{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Then, $\Omega=\bigcup_{\alpha\in I_{\sigma}^{*}}\mathscr{U}_{\mathfrak{g},\alpha}$. Moreover, by De Morgan's Law, $\emptyset=\complement(\Omega)=\complement\big(\bigcup_{\alpha\in I_{\sigma}^{*}}\mathscr{U}_{\mathfrak{g},\alpha}\big)=\bigcap_{\alpha\in I_{\sigma}^{*}}\complement(\mathscr{U}_{\mathfrak{g},\alpha})=\bigcap_{\alpha\in I_{\sigma}^{*}}\mathscr{V}_{\mathfrak{g},\alpha}$. Thus, $\langle \mathscr{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets and, by above, has an empty intersection. By hypothesis, it follows, then, that there exists a finite subsequence $\langle \mathscr{V}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}$

of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{V}_{\mathfrak{g},\beta(\alpha)}=\emptyset$. Thus, by virtue of De Morgan's Law, it results that $\Omega=\complement(\emptyset)=\complement\big(\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{V}_{\mathfrak{g},\beta(\alpha)}\big)=\bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\complement\left(\mathscr{V}_{\mathfrak{g},\beta(\alpha)}\right)=\bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathscr{U}_{\mathfrak{g},\beta(\alpha)}$. Accordingly, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ and, hence, I. is implied by II. □

Proposition 3.17 *If* $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed set of a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$, *then* $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$*:*

$$\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\;\Rightarrow\;\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{A}\left[\mathfrak{T}_{\mathfrak{g}}\right].\tag{16}$$

Proof Let it be assumed that $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$. Then, $\complement\left(\mathscr{S}_{\mathfrak{g}}\right)\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$; that is, $\Omega\setminus\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$. Let $\left\langle\mathscr{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right]\right\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$ in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$ and, for every $\alpha\in I_{\sigma}^{*}$, set $\hat{\mathscr{U}}_{\mathfrak{g},\alpha}=\mathscr{U}_{\mathfrak{g},\alpha}\cup\complement\left(\mathscr{S}_{\mathfrak{g}}\right)$. Then, $\left\langle\hat{\mathscr{U}}_{\mathfrak{g},\alpha}\right\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω. But since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space, there exists a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\left\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\left\langle\mathscr{U}_{\mathfrak{g},\alpha}\right\rangle_{\alpha\in I_{\sigma}^{*}}$ such that $\Omega\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}$, where $\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\alpha)}=\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\cup\complement\left(\mathscr{S}_{\mathfrak{g}}\right)$ for every $(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}$. Therefore, $\left\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathscr{S}_{\mathfrak{g}}$. Hence, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of the proposition is complete. □

Theorem 3.18 *A necessary and sufficient conditions for a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *to be a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ *is that, whenever for each* $\xi\in\mathfrak{T}_{\mathfrak{g}}$ *a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$*-open neighborhood of* ξ *is given, there is a finite collection* $\mathscr{C}_{\xi}=\{\xi_{\eta}:\eta\in I_{n}^{*}\}$ *of points* $\xi_1,\ \xi_2,\ \ldots,\ \xi_n\in\mathfrak{T}_{\mathfrak{g}}$ *such that* $\Omega=\bigcup_{\xi\in\mathscr{C}_{\xi}}\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\xi}\right)$.

Proof *Necessity.* Suppose $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$. Let there be given, for each $\xi\in\mathfrak{T}_{\mathfrak{g}}$, a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open neighborhood of ξ. For each $\xi\in\mathfrak{T}_{\mathfrak{g}}$, there is a $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi}\subset\mathfrak{T}_{\mathfrak{g}}$ satisfying $\xi\in\mathscr{U}_{\mathfrak{g},\xi}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\xi}\right)$. Thus, for every $\xi\in\mathfrak{T}_{\mathfrak{g}}$, $\mathscr{U}_{\mathfrak{g},\xi}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and, consequently, $\left\langle\mathscr{U}_{\mathfrak{g},\xi}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\rangle_{\xi\in\mathfrak{T}_{\mathfrak{g}}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\left\langle\mathscr{U}_{\mathfrak{g},\xi_{\mu}}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\rangle_{\mu\in I_{n}^{*}}$. But, for every $\mu\in I_{n}^{*}$, $\xi_{\mu}\in\mathscr{U}_{\mathfrak{g},\xi_{\mu}}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\xi_{\mu}}\right)$, whence $\Omega=\bigcup_{\mu\in I_{n}^{*}}\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\xi_{\mu}}\right)=\bigcup_{\xi\in\mathscr{C}_{\xi}}\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\xi}\right)$.

Sufficiency. Conversely, suppose that whenever, for each $\xi\in\mathfrak{T}_{\mathfrak{g}}$, a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open neighborhood of ξ is given, there is a finite collection $\mathscr{C}_{\xi}=\{\xi_{\eta}:\eta\in I_{n}^{*}\}$ of points $\xi_1,\ \xi_2,\ \ldots,\ \xi_n\in\mathfrak{T}_{\mathfrak{g}}$ such that $\Omega = \bigcup_{\xi \in \mathscr{C}_{\xi}} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi})$. Let $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Then, for each $\xi \in \mathfrak{T}_{\mathfrak{g}}$, there exists an $\alpha = \alpha(\xi)$ such that $\xi \in \mathscr{U}_{\mathfrak{g},\alpha(\xi)}$, and hence, $\mathscr{U}_{\mathfrak{g},\alpha(\xi)} = \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi})$ for every $(\xi, \alpha(\xi)) \in \mathfrak{T}_{\mathfrak{g}} \times I_{n}^{*}$. By hypothesis, there is, then, a finite collection $\mathscr{C}_{\xi} = \{\xi_{\eta} : \eta \in I_{n}^{*}\}$ of points ξ_1, ξ_2, $\ldots$, $\xi_n \in \mathfrak{T}_{\mathfrak{g}}$ such that $\Omega = \bigcup_{\xi \in \mathscr{C}_{\xi}} \mathscr{U}_{\mathfrak{g},\alpha(\xi)}$, and thus, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$. □

4. Discussion

4.1. Categorical Classifications

Having adopted a categorical approach in the classifications of the $\mathscr{T}_{\mathfrak{g}}$-property called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the dual purposes of the this section are firstly, to establish the various relationships amongst the elements of the sequences $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}) \rangle_{\nu \in I_3^0}$, and $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}^{[\mathrm{A}]}) \rangle_{\nu \in I_3^0}$ of $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-spaces and $\mathfrak{g}$-ν-$\mathscr{T}^{[\mathrm{A}]}$-spaces, respectively, and secondly, to illustrate them through a so-called *categorical compactness diagram*.

Let $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ be any $\mathscr{T}_{\mathfrak{g}}$-open set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, for every $\nu \in I_3^0$, let there exist a $\mu \in I_3^0$ such that the relation $\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},\mu}(\mathscr{O}_{\mathfrak{g}})$ holds. Then, since $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})$ for every $\nu \in I_3^0$, it follows that $\mathfrak{T}_{\mathfrak{g}}$-openness implies $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-openness and the latter, in turn, implies $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-openness. But since the statement that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness is a consequence of the statement that $\mathfrak{T}_{\mathfrak{g}}$-openness implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness, it evidently follows that, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compactness and the latter, in turn, implies $\mathfrak{T}_{\mathfrak{g}}$-compactness. On the other hand, for every $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds [22, 23]. Consequently,

$$\mathrm{op}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}.$$

Therefore, for each $(\mu, \nu) \in \{(0,1), (1,3), (2,3)\}$, from $\mathfrak{g}$-μ-$\mathfrak{T}_{\mathfrak{g}}$-openness implies $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-openness, it results that $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{g}$-μ-$\mathfrak{T}_{\mathfrak{g}}$-compactness. Thus, if $\mathscr{U}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set then, with respect to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness, the following left-hand side system of implications holds:

$$\begin{array}{ccc} \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-}0\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] & \Longrightarrow & \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-}1\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \\ & & \Downarrow \\ \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-}2\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] & \Longrightarrow & \mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-}3\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]; \end{array} \qquad \begin{array}{ccc} \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}0\text{-}\mathrm{A}[\mathfrak{T}_{\mathfrak{g}}] & \Longleftarrow & \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}1\text{-}\mathrm{A}[\mathfrak{T}_{\mathfrak{g}}] \\ & & \Uparrow \\ \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}2\text{-}\mathrm{A}[\mathfrak{T}_{\mathfrak{g}}] & \Longleftarrow & \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}3\text{-}\mathrm{A}[\mathfrak{T}_{\mathfrak{g}}]. \end{array}$$

Such left-hand side system with respect to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, in turn, implies the right-hand side system of implications. For visualization, a so-called *categorical compactness diagram*,

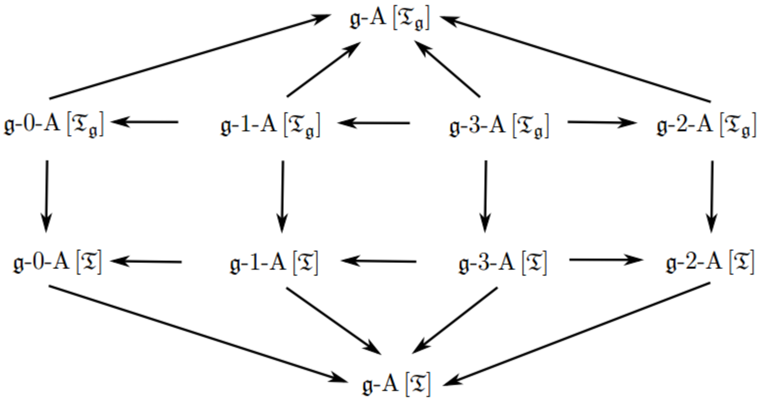

Figure 1: Relationships: classes of $\mathfrak{g}$-$\mathfrak{T}$-compact and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets

expressing the various relationships amongst the classes of $\mathfrak{g}$-$\mathfrak{T}$-compact and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets, is presented in Figure 1. The notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-spaces of category $\nu \in I_3^0$ is exemplified below.

4.2. A Nice Application

A nice application is now presented. Let $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the $\mathfrak{g}$-topology on $\Omega = (0,1) \subset \mathbb{R}$ (set of real numbers) generated by $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets belonging to:

$$\mathscr{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{ \mathscr{O}_{\mathfrak{g},\mu} : \; (\forall \mu \in I_{\infty}^{*} \setminus I_{2}^{*}) \Big(\big[\mathscr{O}_{\mathfrak{g},\mu} = \emptyset \big] \vee \Big[\mathscr{O}_{\mathfrak{g},\mu} = \Big(\frac{1}{\mu}, 1 - \frac{1}{\mu} \Big) \Big] \Big) \right\};$$

$$\neg\mathscr{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{ \mathscr{K}_{\mathfrak{g},\mu} : \; (\forall \mu \in I_{\infty}^{*} \setminus I_{2}^{*}) \Big(\big[\mathscr{K}_{\mathfrak{g},\mu} = \Omega \big] \vee \Big[\mathscr{K}_{\mathfrak{g},\mu} = \complement \Big(\frac{1}{\mu}, 1 - \frac{1}{\mu} \Big) \Big] \Big) \right\},$$

respectively. Clearly, the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the relations $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu}) \subseteq \left(\frac{1}{\mu}, 1 - \frac{1}{\mu} \right) = \mathscr{O}_{\mathfrak{g},\mu}$ and, moreover, $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\mu \in I_{\sigma}^{*} \setminus I_{2}^{*}} \mathscr{O}_{\mathfrak{g},\mu} \big) = \bigcap_{\mu \in I_{\sigma}^{*} \setminus I_{2}^{*}} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu})$ and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\mu \in I_{\infty}^{*} \setminus I_{2}^{*}} \mathscr{O}_{\mathfrak{g},\mu} \big) = \bigcup_{\mu \in I_{\infty}^{*} \setminus I_{2}^{*}} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\mu})$ are also satisfied, since $\bigcap_{\mu \in I_{\sigma}^{*} \setminus I_{2}^{*}} \mathscr{O}_{\mathfrak{g},\mu} = \mathscr{O}_{\mathfrak{g},3} \in \mathscr{T}_{\mathfrak{g}}$ and $\bigcup_{\mu \in I_{\infty}^{*} \setminus I_{2}^{*}} \mathscr{O}_{\mathfrak{g},\mu} = \Omega \in \mathscr{T}_{\mathfrak{g}}$, respectively. Thus, $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega)$ is a $\mathscr{T}_{\mathfrak{g}}$-space and, since $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega) = (\mathscr{T}, \Omega) = \mathfrak{T}$, it is also a $\mathscr{T}$-space. Observe that $\langle \mathscr{O}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\infty}^{*} \setminus I_{2}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω, since $\mathscr{O}_{\mathfrak{g},\alpha} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha \in I_{\infty}^{*} \setminus I_{2}^{*}$ and, moreover, it is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω, since $\mathscr{O}_{\mathfrak{g},\alpha} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha \in I_{\infty}^{*} \setminus I_{2}^{*}$. On the other hand, for each $\sigma > 3$, the relation $\frac{1}{\sigma} \in \bigcup_{\mu \in I_{\sigma}^{*} \setminus I_{2}^{*}} \mathscr{O}_{\mathfrak{g},\mu} = \left(\frac{1}{\sigma}, 1 - \frac{1}{\sigma} \right)$. Hence, from every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in J_{\infty}^{*} \times J_{\vartheta(\infty)}^{*}} \prec \langle \mathscr{O}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\infty}^{*} \setminus I_{2}^{*}}$, where $J_{\infty}^{*} = I_{\infty}^{*} \setminus I_{2}^{*}$ and $J_{\vartheta(\infty)}^{*} = I_{\vartheta(\infty)}^{*} \setminus I_{2}^{*}$, the union $\bigcup_{(\alpha,\vartheta(\alpha)) \in J_{\infty}^{*} \times J_{\vartheta(\infty)}^{*}} \mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}$ must fail to contain some point of Ω and, hence, there exist no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\langle \mathscr{O}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\infty}^{*} \setminus I_{2}^{*}}$. This proves that $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega)$, where $\Omega = (0,1)$, is not a $\mathfrak{T}_{\mathfrak{g}}^{[A]}$-space. Since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness, it follows, consequently, that it is also not a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]}$-space. Finally, from this case, it results that, not every $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]}$-space is itself $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact.

4.3. Concluding Remarks

In this paper, a new theory called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Compactness* has been presented, the foundation of which was based on the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets [22, 23]. The theory holds equally well when $(\Omega, \mathscr{T}_{\mathfrak{g}}) = (\Omega, \mathscr{T})$, and other characteristics adapted on this ground, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Connectedness*.

Thus, it follows that in a $\mathscr{T}_{\mathfrak{g}}$-space the theoretical framework categorises such statements as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively open $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively semi-open $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively preopen $\mathfrak{T}_{\mathfrak{g}}$-sets, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively semi-preopen $\mathfrak{T}_{\mathfrak{g}}$-sets as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of categories 0, 1, 2 and 3, respectively, and theorises the concepts in a unified way; in a $\mathscr{T}$-space it categorises such statements as $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively open $\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively semi-open $\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively preopen $\mathfrak{T}$-sets, and $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively semi-preopen $\mathfrak{T}$-sets as $\mathfrak{g}$-$\mathfrak{T}$-compactness of categories 0, 1, 2 and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to study other derived concepts called countable, sequential, and local generalized compactness (*countable, sequential, local* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compactness*) in $\mathscr{T}_{\mathfrak{g}}$-spaces. Such a study will be considered in a next paper, and this paper ends here.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Mohammad Irshad Khodabocus]: Thought and designed the research/problem, collected the data, contribution to completing the research and solving the problem, wrote the manuscript (%70).

Author [Noor-Ul-Hacq Sookia]: Contributed to research method or evaluation of data (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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