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Research Article

Fekete-Szegö problem for q-starlike functions in connected with k-Fibonacci numbers

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Abstract

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{U}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}).$$

The coefficient functional $\phi_\lambda(f) = a_3 - \lambda a_2^2$ on $f \in \mathcal{A}$ represents various geometric quantities. For example, $\phi_1(f) = a_3 - a_2^2 = S_f(0)/6$, where S_f is the Schwarzian derivative. The problem of maximizing the absolute value of the functional $\phi_\lambda(f)$ is called the Fekete-Szegö problem.

In a very recent paper, Shafiq *et al.* [Symmetry 12:1043, 2020] defined a new subclass $\mathcal{SL}(k,q), (k > 0,\ 0 < q < 1)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination:

$$\frac{z\,D_q f(z)}{f(z)} \prec \frac{2\tilde{p}_k(z)}{(1+q) + (1-q)\,\tilde{p}_k(z)} \qquad (z \in \mathbb{U}),$$

where

$$\tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k - \sqrt{k^2+4}}{2},$$

and investigated the Fekete-Szegö problem for functions belong to the class $\mathcal{SL}(k,q)$. This class is connected with k-Fibonacci numbers.

The main purpose of this paper is to obtain sharp bounds on $\phi_\lambda(f)$ for functions f belong to the class $\mathcal{SL}(k,q)$ when both $\lambda \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, and to improve the result given in the above mentioned paper.

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\}$$

be the set of positive integers.

Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and the class $\mathcal{P}$ was defined by

$$\mathcal{P} = \{p \in \mathcal{H} : \ p(0) = 1 \quad \text{and} \quad \Re\,(p(z)) > 0, \ z \in \mathbb{U}\} .$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f\,(z) \prec g\,(z) \qquad (z \in \mathbb{U}) ,$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega\,(z)| < 1 \ \ (z \in \mathbb{U})\} ,$$

such that

$$f\,(z) = g\,(\omega\,(z)) \quad (z \in \mathbb{U}) .$$

Indeed, it is known that

$$f\,(z) \prec g\,(z) \quad (z \in \mathbb{U}) \Rightarrow f\,(0) = g\,(0) \quad \text{and} \quad f\,(\mathbb{U}) \subset g\,(\mathbb{U}) .$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f\,(z) \prec g\,(z) \quad (z \in \mathbb{U}) \Leftrightarrow f\,(0) = g\,(0) \quad \text{and} \quad f\,(\mathbb{U}) \subset g\,(\mathbb{U}) .$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}) . \tag{1.1}$$

We also denote by $\mathcal{S}$ the class of univalent functions in $\mathcal{A}$.

It is well-known that the class of starlike functions of order α is defined by

$$\mathcal{S}^*\,(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'\,(z)}{f\,(z)}\right) > \alpha \quad (0 \leq \alpha < 1; \ z \in \mathbb{U}) \right\}$$

and $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

Quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [10, 11]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. A comprehensive study on applications of q-calculus in operator theory may be found in [1].

For a function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of function f is defined by (see [10, 11])

$$D_q f\,(z) = \frac{f\,(qz) - f\,(z)}{(q-1)\,z} \qquad (z \neq 0) , \tag{1.2}$$

and $D_q f(0) = f'(0)$. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where the q-number $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q}.$$

As $q \to 1^-$, $[n]_q \to n$. For a function $g(z) = z^n$, we get

$$D_q(z^n) = [n]_q z^{n-1}$$

and

$$\lim_{q\to 1^-} (D_q(z^n)) = nz^{n-1} = g'(z),$$

where g' is the ordinary derivative.

For $f \in \mathcal{S}$ given by (1.1), Fekete and Szegö [7] proved a noticeable result that

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} 3-4\lambda & , \quad \lambda \leq 0 \\ 1+2\exp\left(\frac{-2\lambda}{1-\lambda}\right) & , \quad 0 \leq \lambda \leq 1 \\ 4\lambda - 3 & , \quad \lambda \geq 1 \end{cases} \tag{1.3}$$

holds. The result is sharp in the sense that for each λ there is a function in the class under consideration for which equality holds.

The coefficient functional

$$\phi_\lambda(f) = a_3 - \lambda a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\lambda}{2}(f''(0))^2\right)$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_\lambda\left(e^{-i\theta} f\left(e^{i\theta}z\right)\right) = e^{2i\theta}\phi_\lambda(f) \quad (\theta \in \mathbb{R}).$$

Thus it is quite natural to ask about inequalities for ϕ_λ corresponding to subclasses of $\mathcal{S}$.

Definition 1.1 ([17])**.** Let the function p be said to belong to the class $k - \mathcal{P}_q$ and let k be any positive real number if

$$p(z) \prec \frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\tilde{p}_k(z)} \qquad (z \in \mathbb{U}),$$

where

$$\tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2} = \frac{1+\tau_k^2 z^2}{1-\left(\tau_k^2 - 1\right) z - \tau_k^2 z^2} \tag{1.4}$$

with

$$\tau_k = \frac{k-\sqrt{k^2+4}}{2}. \tag{1.5}$$

In a very recent paper, Shafiq *et al.* [17] introduced a new subclass of $\mathcal{A}$ which consists of q-starlike functions related to k-Fibonacci numbers as follows:

Definition 1.2 ([17])**.** Let k be any positive real number. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}(k,q)$ if and only if

$$\frac{z\, D_q f(z)}{f(z)} \prec \frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\tilde{p}_k(z)} \qquad (z \in \mathbb{U}),$$

where $\tilde{p}_k(z)$ is given by (1.4).

Remark 1.3. For $q \to 1^-$, the class $\mathcal{SL}(k,q)$ reduces to the class $\mathcal{SL}^k$ which consists of functions $f \in \mathcal{S}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z) \qquad (z \in \mathbb{U}). \tag{1.6}$$

This class was introduced by Yılmaz Özgür and Sokół [14].

Remark 1.4. For $q \to 1^-$ and $k = 1$, the class $\mathcal{SL}(k,q)$ reduces to the class $\mathcal{SL}$ which consists of functions $f \in \mathcal{S}$ defined by (1.1) satisfying

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) \qquad (z \in \mathbb{U}),$$

where

$$\tilde{p}(z) := \tilde{p}_1(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}, \qquad \tau := \tau_1 = \frac{1-\sqrt{5}}{2}.$$

This class was introduced by Sokół [18].

Definition 1.5 ([6])**.** For any positive real number k, the k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \qquad (n \in \mathbb{N})$$

with initial conditions

$$F_{k,0} = 0, \qquad F_{k,1} = 1.$$

Furthermore n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}, \tag{1.7}$$

where τ_k is given by (1.5).

Note that for $k = 1$, we obtain the classic Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}_0}$:

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{N}).$$

Yılmaz Özgür and Sokół [14] showed that the coefficients of the function $\tilde{p}_k(z)$ defined by (1.4) are connected with k-Fibonacci numbers. This connection is pointed out in the following theorem.

Theorem 1.6 ([14])**.** *Let $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ be the sequence of k-Fibonacci numbers defined in Definition 1.5. If*

$$\tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2} := 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n, \tag{1.8}$$

then we have

$$\tilde{p}_{k,1} = k\tau_k, \quad \tilde{p}_{k,2} = \left(k^2+2\right)\tau_k^2, \quad \tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\,\tau_k^n \qquad (n \in \mathbb{N}). \tag{1.9}$$

For more details about the classes $\mathcal{SL}$ and $\mathcal{SL}^k$, please refer to [3–5,8,9,19,20]. Recently, Shafiq *et al.* [17] investigated the Fekete-Szegö problem for functions belong to the class $\mathcal{SL}(k,q)$ and obtained the following result:

Theorem 1.7 ([17])**.** *Let the function $f \in \mathcal{A}$ given by (1.1) belong to the class $\mathcal{SL}(k,q)$. Then*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{\tau_k^2}{4q^2}\left[(1+q)^2(1+|\lambda|)\,k^2 + 4q\right]. \tag{1.10}$$

Remark 1.8. For $q \to 1^-$, we get [21, Theorem 2.3]; and for $q \to 1^-$ and $k = 1$, we get [15, Theorem 2.4].

The main purpose of this paper is to improve the results of the above-mentioned theorem (Theorem 1.7). For this, we need the following lemmas:

Lemma 1.9 ([12]). *If $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & , \quad \nu \leq 0 \\ 2 & , \quad 0 \leq \nu \leq 1 \\ 4\nu - 2 & , \quad \nu \geq 1 \end{cases} .$$

When $\nu < 0$ or $\nu > 1$, equality holds true if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then equality holds true if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, then the equality holds true if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $\nu = 1$, then the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when $\nu = 0$.

Although the above upper bound is sharp, in the case when $0 < \nu < 1$, it can be further improved as follows:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \qquad \left(0 < \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} < \nu \leq 1\right).$$

Lemma 1.10 ([13]). *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N}).$$

Lemma 1.11 ([16]). *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then for any complex number ν*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |2\nu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

2. Main results

Theorem 2.1. *A function f given by (1.1) belongs to the class $\mathcal{SL}(k,q)$ if and only if there exist a function h,*

$$h(z) \prec \frac{2\tilde{p}_k(z)}{(1+q) + (1-q)\tilde{p}_k(z)} \qquad (z \in \mathbb{U})$$

such that

$$f(z) = z\left(\exp \int_0^z \frac{h(t)-1}{t} d_q t\right)^{\frac{\ln q}{q-1}} \qquad (z \in \mathbb{U}). \tag{2.1}$$

Proof. Let $f \in \mathcal{SL}(k,q)$ and consider

$$h(z) = \frac{z\, D_q f(z)}{f(z)},$$

where h is analytic and $h(0)=1$ in $\mathbb{U}$. It follows that

$$\begin{aligned}\int_0^z \frac{h(t)-1}{t}d_qt &= \int_0^z \frac{t\,D_qf(t)-f(t)}{t\,f(t)}d_qt \\ &= \int_0^z \frac{D_qf(t)}{f(t)}d_qt-\int_0^z \frac{1}{t}d_qt \\ &= \left(\frac{q-1}{\ln q}\right)\log\left(f(z)\right)-\left(\frac{q-1}{\ln q}\right)\log\left(z\right) \\ &= \log\left(\frac{f(z)}{z}\right)^{\frac{q-1}{\ln q}},\end{aligned}$$

which implies that

$$z\left(\exp\int_0^z \frac{h(t)-1}{t}d_qt\right)^{\frac{\ln q}{q-1}}=f(z),$$

which is (2.1). Conversely, let (2.1) holds true, that is, there exists an analytic function h,

$$h(z)\prec\frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\,\tilde{p}_k(z)}$$

such that

$$\frac{f(z)}{z}=\left(\exp\int_0^z \frac{h(t)-1}{t}d_qt\right)^{\frac{\ln q}{q-1}}. \tag{2.2}$$

Then q-Logarithmic differentiation of (2.2) gives us

$$\frac{\ln q}{q-1}\left(\frac{D_qf(z)}{f(z)}\right)-\frac{\ln q}{q-1}\left(\frac{1}{z}\right)=\frac{\ln q}{q-1}\left(\frac{h(z)-1}{z}\right),$$

or, equivalently

$$\frac{z\,D_qf(z)}{f(z)}=h(z),$$

which implies that $f\in\mathcal{SL}(k,q)$. Thus the proof of the theorem is completed. □

Letting $q\to 1^-$ in Theorem 2.1, we get following consequence.

Corollary 2.2 ([14]). *A function f given by* (1.1) *belongs to the class $\mathcal{SL}^k$ if and only if there exist a function h,*

$$h(z)\prec\tilde{p}_k(z)\qquad(z\in\mathbb{U})$$

such that

$$f(z)=z\,\exp\int_0^z \frac{h(t)-1}{t}dt\qquad(z\in\mathbb{U}).$$

Letting $q\to 1^-$ and $k=1$ in Theorem 2.1, we get following consequence.

Corollary 2.3 ([5]). *A function f given by* (1.1) *belongs to the class $\mathcal{SL}$ if and only if there exist a function h,*

$$h(z)\prec\tilde{p}(z)\qquad(z\in\mathbb{U})$$

such that

$$f(z)=z\,\exp\int_0^z \frac{h(t)-1}{t}dt\qquad(z\in\mathbb{U}).$$

Now, we give the upper bound of the Fekete-Szegö functional $\left|a_3-\lambda a_2^2\right|$ of functions $f\in\mathcal{SL}(k,q)$ given by (1.1) when $\lambda\in\mathbb{R}$.

Theorem 2.4. *If the function f given by* (1.1) *is in the class* $\mathcal{SL}(k,q)$, *then we have*

$$\left|a_3-\lambda a_2^2\right|\leq\begin{cases}\tau_k^2\frac{(1+q)^2k^2+4q-\lambda(1+q)^2k^2}{4q^2} & , \quad \lambda\leq\frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k}\\ \frac{k|\tau_k|}{2q} & , \quad \frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k}\leq\lambda\leq\frac{\left((1+q)^2k^2+4q\right)\tau_k-2qk}{(1+q)^2k^2\tau_k}\\ \tau_k^2\frac{\lambda(1+q)^2k^2-(1+q)^2k^2-4q}{4q^2} & , \quad \lambda\geq\frac{\left((1+q)^2k^2+4q\right)\tau_k-2qk}{(1+q)^2k^2\tau_k}\end{cases}.$$

If $\frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k}\leq\lambda\leq\frac{(1+q)^2k^2+4q}{(1+q)^2k^2}$, *then*

$$\left|a_3-\lambda a_2^2\right|+\left(\lambda-\frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k}\right)|a_2|^2\leq\frac{k|\tau_k|}{2q}.$$

Furthermore, if $\frac{(1+q)^2k^2+4q}{(1+q)^2k^2}\leq\lambda\leq\frac{\left((1+q)^2k^2+4q\right)\tau_k-2kq}{(1+q)^2k^2\tau_k}$, *then*

$$\left|a_3-\lambda a_2^2\right|+\left(\frac{\left((1+q)^2k^2+4q\right)\tau_k-2qk}{(1+q)^2k^2\tau_k}-\lambda\right)|a_2|^2\leq\frac{k|\tau_k|}{2q}.$$

Each of these results is sharp.

Proof. If $f\in\mathcal{SL}(k,q)$, then it follows from Definition 1.2 that

$$\frac{z\,(D_qf)(z)}{f(z)}\prec\frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\,\tilde{p}_k(z)}=:\varphi_{k,q}(z)\quad(z\in\mathbb{U}),\tag{2.3}$$

where the function $\tilde{p}_k$ is given by (1.8). So by the principle of subordination, there exists a Schwarz function $\omega\in\Omega$ such that

$$\frac{z\,(D_qf)(z)}{f(z)}=\varphi_{k,q}(\omega(z)).$$

Therefore, the function

$$g(z):=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+\cdots\quad(z\in\mathbb{U})\tag{2.4}$$

is in the class $\mathcal{P}$. Now, defining the function $p(z)$ by

$$p(z)=\frac{z\,(D_qf)(z)}{f(z)}=1+p_1z+p_2z^2+\cdots,\tag{2.5}$$

it follows from (2.3) and (2.4) that

$$p(z)=\varphi_{k,q}\left(\frac{g(z)-1}{g(z)+1}\right).\tag{2.6}$$

Note that

$$\omega(z)=\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots$$

and so

$$\begin{aligned}\varphi_{k,q}(\omega(z)) &= 1+\frac{(1+q)\,\tilde{p}_{k,1}c_1}{4}z\\ &\quad+\left\{\frac{1+q}{4}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_{k,1}+\frac{1+q}{16}c_1^2\left[(q-1)\,\tilde{p}_{k,1}^2+2\tilde{p}_{k,2}\right]\right\}z^2+\cdots\end{aligned}\tag{2.7}$$

Thus, by using (2.4) in (2.6) and by considering the values $\tilde{p}_{k,j}$ $(j=1,2)$ given in (1.9), we obtain

$$p_1 = \frac{(1+q)\,k\tau_k}{4}c_1$$

and

$$p_2 = \frac{(1+q)\,k\tau_k}{4}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(1+q)\left[(1+q)\,k^2+4\right]\tau_k^2}{16}c_1^2.$$

On the other hand, a simple calculation shows that

$$\frac{z\,(D_q f)\,(z)}{f(z)} = 1 + qa_2 z + q\left[(1+q)\,a_3 - a_2^2\right]z^2 + \cdots,$$

which, in view of (2.5), yields

$$p_1 = qa_2 \qquad \text{and} \qquad p_2 = q\left[(1+q)\,a_3 - a_2^2\right]$$

or equivalently

$$a_2 = \frac{p_1}{q} \qquad a_3 = \frac{q\,p_2 + p_1^2}{q^2\,(1+q)}.$$

Thus, we obtain

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{1}{q\,(1+q)}\left[p_2 - \frac{(1+q)\,\lambda - 1}{q}p_1^2\right] \\ &= \frac{1}{q\,(1+q)}\left[\frac{(1+q)\,k\tau_k}{4}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(1+q)\left[(1+q)\,k^2+4\right]\tau_k^2}{16}c_1^2 \right. \\ &\quad \left. - \frac{(1+q)^2\left[(1+q)\,\lambda - 1\right]k^2\tau_k^2}{16q}c_1^2\right] \\ &= \frac{k\tau_k}{4q}\left(c_2 - \nu c_1^2\right), \end{aligned}$$

where

$$\nu = \frac{1}{2} - \frac{(1+q)^2\,k^2 + 4q - \lambda\,(1+q)^2\,k^2}{4qk}\tau_k.$$

The assertion of Theorem 2.4 now follows by an application of Lemma 1.9.

To show that the bounds asserted by Theorem 2.4 are sharp, we define the following functions:

$$K_{\varphi_{k,q,n}}(z) \qquad (n \in \mathbb{N}\backslash\{1\}),$$

with

$$K_{\varphi_{k,q,n}}(0) = 0 = K'_{\varphi_{k,q,n}}(0) - 1,$$

by

$$\frac{zK'_{\varphi_{k,q,n}}(z)}{K_{\varphi_{k,q,n}}(z)} = \varphi_{k,q}\left(z^{n-1}\right), \tag{2.8}$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \le \eta \le 1)$, with

$$F_\eta(0) = 0 = F'_\eta(0) - 1 \quad \text{and} \quad G_\eta(0) = 0 = G'_\eta(0) - 1,$$

by

$$\frac{zF'_\eta(z)}{F_\eta(z)} = \varphi_{k,q}\left(\frac{z\,(z+\eta)}{1+\eta z}\right)$$

and

$$\frac{zG'_\eta(z)}{G_\eta(z)} = \varphi_{k,q}\left(-\frac{z\,(z+\eta)}{1+\eta z}\right),$$

respectively. Then, clearly, the functions $K_{\varphi_{k,q,n}}, F_\eta, G_\eta \in \mathcal{SL}(k,q)$. We also write

$$K_{\varphi_{k,q}} = K_{\varphi_{k,q,2}}.$$

If $\lambda < \frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k}$ or $\lambda > \frac{\left((1+q)^2k^2+4q\right)\tau_k-2qk}{(1+q)^2k^2\tau_k}$, then the equality in Theorem 2.4 holds if and only if f is $K_{\varphi_{k,q}}$ or one of its rotations.

When $\frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k} < \lambda < \frac{\left((1+q)^2k^2+4q\right)\tau_k-2qk}{(1+q)^2k^2\tau_k}$, then the equality holds if and only if f is $K_{\varphi_{k,q,3}}$ or one of its rotations.

If $\lambda = \frac{\left((1+q)^2k^2+4q\right)\tau_k+2qk}{(1+q)^2k^2\tau_k}$, then the equality holds if and only if f is F_η or one of its rotations.

If $\lambda = \frac{\left((1+q)^2k^2+4q\right)\tau_k-2qk}{(1+q)^2k^2\tau_k}$, then the equality holds if and only if f is G_η or one of its rotations. □

For $q \to 1^-$, we have the following result.

Corollary 2.5 ([2]). *If the function f given by (1.1) is in the class $\mathcal{SL}^k$, then we have*

$$\left|a_3 - \lambda a_2^2\right| \le \begin{cases} \tau_k^2\left(k^2+1-\lambda k^2\right) & , \quad \lambda \le \frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k} \\ \frac{k|\tau_k|}{2} & , \quad \frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k} \le \lambda \le \frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k} \\ \tau_k^2\left(\lambda k^2-k^2-1\right) & , \quad \lambda \ge \frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k} \end{cases}.$$

If $\frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k} \le \lambda \le \frac{k^2+1}{k^2}$, *then*

$$\left|a_3 - \lambda a_2^2\right| + \left(\lambda - \frac{2\left(k^2+1\right)\tau_k+k}{2k^2\tau_k}\right)|a_2|^2 \le \frac{k|\tau_k|}{2}.$$

Furthermore, if $\frac{k^2+1}{k^2} \le \lambda \le \frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k}$, *then*

$$\left|a_3 - \lambda a_2^2\right| + \left(\frac{2\left(k^2+1\right)\tau_k-k}{2k^2\tau_k} - \lambda\right)|a_2|^2 \le \frac{k|\tau_k|}{2}.$$

Each of these results is sharp.

Now, we give the upper bound for the Fekete-Szegö functional $\left|a_3-\lambda a_2^2\right|$ of functions $f \in \mathcal{SL}(k,q)$ given by (1.1) when $\lambda \in \mathbb{C}$.

Theorem 2.6. *If the function f given by (1.1) is in the class $\mathcal{SL}(k,q)$, then we have*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{k|\tau_k|}{2q}\max\left\{1, \frac{\left|(1+q)^2k^2+4q-\lambda(1+q)^2k^2\right|}{2qk}|\tau_k|\right\}$$

for all $\lambda \in \mathbb{C}$. The result is sharp.

Proof. Let the function $f \in \mathcal{A}$ given by (1.1) be in the class $\mathcal{SL}(k,q)$. Define the function $p(z) = 1 + p_1z + p_2z^2 + \cdots$ by

$$\frac{z\,(D_qf)(z)}{f(z)} = p(z),$$

then we have

$$p(z) \prec \frac{2\tilde{p}_k(z)}{(1+q)+(1-q)\tilde{p}_k(z)},$$

where $\tilde{p}_k(z)$ is given by (1.4). As shown in the proof of Theorem 2.4, we obtain

$$a_2 = \frac{p_1}{q}, \qquad a_3 = \frac{q\,p_2+p_1^2}{q^2(1+q)}.$$

Therefore for any $\lambda \in \mathbb{C}$, we have

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{q(1+q)}\left|p_2 - \frac{(1+q)\lambda - 1}{q}p_1^2\right|.$$

Now, by Theorem 2.10, this equality implies that

$$\left|a_3 - \lambda a_2^2\right| \le \frac{k|\tau_k|}{2q}\max\left\{1, \frac{\left|(1+q)^2 k^2 + 4q - \lambda(1+q)^2 k^2\right|}{2qk}|\tau_k|\right\}.$$

This evidently completes the proof of theorem. □

Remark 2.7. It is worthy to note that Theorem 2.6 improves the results given in Theorem 1.7.

Corollary 2.8. *If the function f given by* (1.1) *is in the class $\mathcal{SL}(k,q)$, then we have*

$$\left|a_3 - a_2^2\right| \le \begin{cases} \frac{\tau_k^2}{q} & , \quad 0 < k \le \frac{2}{\sqrt{3}} \\ \frac{k|\tau_k|}{2q} & , \quad k \ge \frac{2}{\sqrt{3}} \end{cases}.$$

Theorem 2.9. *If the function f given by* (1.1) *is in the class $\mathcal{SL}(k,q)$, then we have*

$$|a_2| \le \frac{(1+q)k}{2q}|\tau_k| \tag{2.9}$$

and

$$|a_3| \le \frac{k|\tau_k|}{2q}\max\left\{1, \frac{(1+q)^2 k^2 + 4q}{2qk}|\tau_k|\right\}. \tag{2.10}$$

Proof. Let $f \in \mathcal{SL}(k,q)$. Therefore, as explained in the proof of Theorem 2.4, we obtain

$$a_2 = \frac{(1+q)k\tau_k}{4q}c_1 \tag{2.11}$$

and

$$a_3 = \frac{k\tau_k}{4q}\left(c_2 - \frac{2qk - \left[(1+q)^2 k^2 + 4q\right]\tau_k}{4qk}c_1^2\right). \tag{2.12}$$

From (2.11) and Lemma 1.10, we get (2.9). Also from (2.12) and Lemma 1.11, we obtain (2.10). □

Theorem 2.10. *If the function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ belongs to the class $k - \mathcal{P}_q$, then we have*

$$|p_1| \le \frac{(1+q)k}{2}|\tau_k| \tag{2.13}$$

and

$$|p_2| \le \frac{(1+q)k}{2}|\tau_k|\max\left\{1, \frac{(1+q)k^2 + 4}{2k}|\tau_k|\right\}. \tag{2.14}$$

The above estimates are sharp for the function $K_{\varphi_{k,q,2}}(z)$ and $K_{\varphi_{k,q,3}}$ given in (2.8).

Proof. Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $\tilde{p}_k(z) = 1 + \tilde{p}_{k,1} z + \tilde{p}_{k,2} z^2 + \cdots$. By the hypothesis, since

$$p(z) \prec \frac{2\tilde{p}_k(z)}{(1+q) + (1-q)\tilde{p}_k(z)} = \varphi_{k,q}(z),$$

the principle of subordination implies that there exists a function $\omega \in \Omega$ such that

$$p(z) = \varphi_{k,q}(\omega(z)) \quad (z \in \mathbb{U}).$$

Therefore, as explained in the proof of Theorem 2.4, we obtain

$$\begin{aligned}\varphi_{k,q}(\omega(z)) &= 1+\frac{(1+q)\,\tilde{p}_{k,1}c_1}{4}z \\ &+\left\{\frac{1+q}{4}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_{k,1}+\frac{1+q}{16}c_1^2\left[(q-1)\,\tilde{p}_{k,1}^2+2\tilde{p}_{k,2}\right]\right\}z^2+\cdots.\end{aligned}$$

So equating the coefficients of the functions $p(z)$ and $\varphi_{k,q}(\omega(z))$, and considering the values $\tilde{p}_{k,j}$ $(j=1,2)$ given in (1.9), we have

$$p_1=\frac{(1+q)\,k\tau_k}{4}c_1 \tag{2.15}$$

and

$$p_2=\frac{(1+q)\,k\tau_k}{4}\left(c_2-\frac{c_1^2}{2}\right)+\frac{(1+q)\left[(1+q)\,k^2+4\right]\tau_k^2}{16}c_1^2. \tag{2.16}$$

From (2.15) and Lemma 1.10, we get (2.13). Also from (2.16), we can write

$$|p_2|=\frac{(1+q)\,k\,|\tau_k|}{4}\left|c_2-\frac{2k-\left[(1+q)\,k^2+4\right]\tau_k}{4k}c_1^2\right|.$$

Therefore by using Lemma 1.11, we obtain (2.14). □

For $q\to1^-$, we have the following result.

Corollary 2.11 ([21]). *If* $p(z)=1+p_1z+p_2z^2+\cdots$ *and*

$$p(z)\prec\tilde{p}_k(z)\qquad(z\in\mathbb{U}),$$

then we have

$$|p_1|\le k\,|\tau_k|$$

and

$$|p_2|\le\left(k^2+2\right)\tau_k^2=\left(k^2+2\right)(k\tau_k+1).$$

The above estimates are sharp.

Theorem 2.12. *If the function* $p(z)=1+p_1z+p_2z^2+\cdots$ *belongs to the class* $k-\mathcal{P}_q$*, then we have*

$$\left|p_2-\gamma p_1^2\right|\le\frac{(1+q)\,k\,|\tau_k|}{2}\max\left\{1,\ \frac{\left|(1+q)\,k^2+4-\gamma(1+q)\,k^2\right|}{2k}\,|\tau_k|\right\}$$

for all $\gamma\in\mathbb{C}$.

Proof. On the other hand, by means of Lemma 1.11, we also have

$$\begin{aligned}\left|p_2-\gamma p_1^2\right| &= \frac{(1+q)\,k\,|\tau_k|}{4} \\ &\quad\times\left|c_2-\frac{2(1+q)\,k-\left[(1+q)^2k^2+4(1+q)-\gamma(1+q)^2k^2\right]\tau_k}{4(1+q)\,k}c_1^2\right| \\ &\le \frac{(1+q)\,k\,|\tau_k|}{2}\max\left\{1,\ \frac{\left|(1+q)^2k^2+4(1+q)-\gamma(1+q)^2k^2\right|}{2(1+q)\,k}\,|\tau_k|\right\}\end{aligned}$$

for all $\gamma\in\mathbb{C}$. □

For $q\to1^-$, we have the following result.

Corollary 2.13 ([2]). *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and*

$$p(z) \prec \tilde{p}_k(z) \qquad (z \in \mathbb{U}),$$

then we have

$$\left| p_2 - \gamma p_1^2 \right| \leq k \left| \tau_k \right| \max \left\{ 1, \ \frac{\left| k^2 + 2 - \gamma k^2 \right|}{k} \left| \tau_k \right| \right\}$$

for all $\gamma \in \mathbb{C}$. *The above estimates are sharp.*

Corollary 2.14. *If the function* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *belongs to the class* $k - \mathcal{P}_q$, *then we have*

$$\left| p_2 - p_1^2 \right| \leq \begin{cases} (1+q)\,\tau_k^2 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \frac{1+q}{2} k \left| \tau_k \right| & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}.$$

3. Conclusion and future work

In this study, we consider following two subclasses of functions:

$$\mathcal{SL}(k,q) = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n : \frac{z\, D_q f(z)}{f(z)} \prec \frac{2\tilde{p}_k(z)}{(1+q) + (1-q)\,\tilde{p}_k(z)} \quad (z \in \mathbb{U}) \right\},$$

$$k - \mathcal{P}_q = \left\{ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n : p(z) \prec \frac{2\tilde{p}_k(z)}{(1+q) + (1-q)\,\tilde{p}_k(z)} \quad (z \in \mathbb{U}) \right\}.$$

For functions $f \in \mathcal{SL}(k,q)$, we obtain sharp bounds for the Fekete-Szegö functional $\phi_\lambda(f) = a_3 - \lambda a_2^2$. Also we give upper bounds for the initial coefficients a_2 and a_3. In the general case, the coefficient bound for $|a_n|$ is open problem.

Furthermore, for functions $p \in k - \mathcal{P}_q$, we obtain sharp bounds for the $|p_2 - \gamma p_1^2|$, $|p_1|$ and $|p_2|$. In the general case, the coefficient bound for $|p_n|$ is open problem.

This study could inspire light on further research.

References

[1] A. Aral, V. Gupta and R.P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York, USA, 2013.

[2] S. Bulut, *Fekete-Szegö problem for starlike functions connected with k-Fibonacci numbers*, Math. Slovaca **71** (4), 823–830, 2021.

[3] J. Dziok, R.K. Raina and J. Sokół, *On α-convex functions related to shell-like functions connected with Fibonacci numbers*, Appl. Math. Comput. **218**, 996–1002, 2011.

[4] J. Dziok, R.K. Raina and J. Sokół, *Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers*, Comput. Math. Appl. **61**, 2605–2613, 2011.

[5] J. Dziok, R.K. Raina and J. Sokół, *On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers*, Math. Comput. Model. **57**, 1203–1211, 2013.

[6] S. Falcón and A. Plaza, *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos Solitons Fractals **33** (1), 38–49, 2007.

[7] M. Fekete and G. Szegö, *Eine bemerkung über ungerade schlichte funktionen*, J. Lond. Math. Soc. **8**, 85–89, 1933.

[8] H.Ö. Güney, S. İlhan and J. Sokół, *An upper bound for third Hankel determinant of starlike functions connected with k-Fibonacci numbers*, Bol. Soc. Mat. Mex. (3) **25** (1), 117–129, 2019.

[9] H.Ö. Güney, J. Sokółand S. İlhan, *Second Hankel determinant problem for some analytic function classes with connected k-Fibonacci numbers*, Acta Univ. Apulensis Math. Inform. **54**, 161–174, 2018.

[10] F.H. Jackson, *On q-definite integrals*, Quarterly J. Pure Appl. Math. **41**, 193–203, 1910.

[11] F.H. Jackson, *On q-functions and a certain difference operator*, Trans. R. Soc. Edinburgh **46**, 253–281, 1908.

[12] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[13] C. Pommerenke, *Univalent Functions*; Vanderhoeck and Ruprecht: Göttingen, Germany, 1975.

[14] N. Yılmaz Özgür and J. Sokół, *On starlike functions connected with k-Fibonacci numbers*, Bull. Malays. Math. Sci. Soc. **38** (1), 249–258, 2015.

[15] R.K. Raina and J. Sokół, *Fekete-Szegö problem for some starlike functions related to shell-like curves*, Math. Slovaca **66** (1), 135–140, 2016.

[16] V. Ravichandran, A. Gangadharan and M. Darus, *Fekete-Szegö inequality for certain class of Bazilevic functions*, Far East J. Math. Sci. **15**, 171–180, 2004.

[17] M. Shafiq, H.M. Srivastava, N. Khan, Q.Z. Ahmad, M. Darus and S. Kiran, *An upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with k-Fibonacci numbers*, Symmetry **12**:1043, 1–17, 2020.

[18] J. Sokół, *On starlike functions connected with Fibonacci numbers*, Folia Scient. Univ. Tech. Resoviensis **175** (23), 111–116, 1999.

[19] J. Sokół, S. İlhan and H.Ö. Güney, *An upper bound for third Hankel determinant of starlike functions related to shell-like curves connected with Fibonacci numbers*, J. Math. Appl. **41**, 195–206, 2018.

[20] J. Sokół, S. İlhan and H.Ö. Güney, *Second Hankel determinant problem for several classes of analytic functions related to shell-like curves connected with Fibonacci numbers*, TWMS J. App.& Eng. Math. **8** (1a), 220–229, 2018.

[21] J. Sokół, R.K. Raina and N. Yılmaz Özgür, *Applications of k-Fibonacci numbers for the starlike analytic functions*, Hacet. J. Math. Stat. **44** (1), 121–127, 2015.