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B-Riesz Transforms Generated by Generalized Translate Operator on HM^p_{q,Δ_ν} Hardy-Morrey Spaces

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Abstract

We study the decomposition of Hardy-Morrey spaces via atoms and molecules, which have similar properties of $H^p_{\Delta_\nu}(\mathbb{R}^n_+)$ Hardy spaces. Then we introduce the HM^p_{q,Δ_ν} boundedness of B-Riesz transforms generated by a generalized translate operator that is associated to the Laplace Bessel operator for $0 < p \leq 1 < q \leq \infty$ with $p \neq q$ through atomic decomposition and molecular characterization.

1. Introduction

The notion of classical Hardy-Morrey space HM^p_q originates from Jia and Wang [1, 2]. Since then, this theory received continuous development and now is increasingly mature; see, for example [3]-[5].
It is well known that the classical Hardy-Morrey space generalizes both Morrey (M^p_q , $q > 1$) and Hardy (H^p , $p \leq 1$) spaces [6]. It plays important roles in several fields of harmonic analysis and PDEs. Also, these spaces are important because they have close relations with L^p spaces, Hardy spaces and BMO^{-1} spaces, and etc.
In recent years, studies in the classical theory of Hardy-Morrey spaces related to some operators have gained great interest and importance. Therefore, our study focused on these spaces. Similar results in other function spaces can be developed in this spaces. These results can be seen in decomposition of Hardy-Morrey spaces, decomposition of Hardy-Morrey spaces with weighted, and decomposition of weighted Hardy-Morrey spaces with variable exponent in [1],[3]-[5].
In this paper, our main purpose is to prove that some properties of Hardy-Morrey spaces, and Hardy-Morrey characterization of the operators depend on conditions via atoms can be obtained. For example, the boundedness of an singular integral operators can be often proved by estimating Ta when a is an atom. While it is generally not true that atoms are mapped into atoms, for many convolution type operators Ta is a function enjoying many of the properties of atoms. Such functions were called molecules. Moreover, classical Hardy spaces and Hardy-Morrey spaces have molecular characterizations that are completely analogous to their atomic characterizations.
We define Hardy-Morrey spaces called HM^p_{q,Δ_ν} Hardy-Morrey spaces which was similar with Hardy spaces associated to the following Laplace-Bessel differential operator [7]

$$\Delta_\nu := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}, \quad \nu > 0.$$

The main conclusion of this article is to prove that the B-Riesz transformation defined in (4.1) is a bounded operator from Hardy-Morrey spaces HM^p_{q,Δ_ν} to Hardy-Morrey spaces HM^p_{q,Δ_ν}. Here $R^{(k)}_\nu$,B-Riesz transform related to Laplace-Bessel

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differential operator Δ_ν. This operator has been studied by many mathematicians on weighted Lebesgue spaces (see [8]-[11]). Even though the boundedness of B-Riesz transform is well known for $1 < p < \infty$ on Lebesgue spaces, we cannot say for $0 < p < 1$ on Lebesgue spaces. But these transformations are bounded in Hardy spaces for $0 < p < 1$ (see [7]). Therefore, in this study, a new characterization of the B-Riesz transform obtained by generalized translation has been obtained for $0 < p \leq 1$ in Hardy-Morrey spaces HM^p_{q,Δ_ν}.
We investigate the Hardy-Morrey spaces characterizing boundedness properties of related Riesz transforms called B-Riesz transforms. These operators give us the most popular examples of Calderon-Zygmund singular integral operators. Also these transforms are related to generalized translate operator. Furthermore, they present some applications especially in the area of partial differential equations. To characterize the boundedness of these transforms, we apply the atomic decomposition. By using this decomposition we give the molecular characterizations for HM^p_{q,Δ_ν} Hardy-Morrey spaces. We follow the ideas in [7] to obtain the boundedness of high order B-Riesz transforms on HM^p_{q,Δ_ν} Hardy-Morrey spaces at the end of Section 4 as an application of our main result. For this reason, we pass by other characterizations of HM^p_{q,Δ_ν} Hardy-Morrey spaces.
The remainder of this paper is structured as follows. The HM^p_{q,Δ_ν} Hardy-Morrey spaces are introduced, also their atomic decompositions are given in Section 2. In Section 3, we will give appropriate definition of molecule is given. We will show that each such molecule has an atomic decomposition. As an application, we present the B-Riesz transforms and give its boundedness properties on HM^p_{q,Δ_ν} Hardy-Morrey spaces extending the results in [7].
Throughout this paper, we denote dyadic cubes with Q or J. Moreover, C indicates constant depending on n, ν, p, q.

2. Preliminaries

Let $\mathbb{R}^n$ be the n dimensional Euclidean space and $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $E(x,t) = \{y \in \mathbb{R}^n_+ \ ; \ |x-y| < t\}$ and $E(x,t)^c = \mathbb{R}^n_+ \backslash E(x,t)$. Let us take a measurable set E on $\mathbb{R}^n_+$, we can define

$$|E|_\nu = \int_E x_n^\nu dx,$$

where $\nu > 0$. Denoting $|E(0,r)|_\nu = \omega(n,\nu) r^{n+\nu}$, where

$$\omega(n,\nu) = \int_{E(0,1)} x_n^\nu dx = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{n+\nu-2}{2}\right)}.$$

The generalized translate operator T^y is defined by

$$T^y f(x) = c_\nu \int_0^\pi \ldots \int_0^\pi f\left(x' - y', (x_n, y_n)_\theta\right) \, d\nu(\theta), \tag{2.1}$$

where $c_\nu = \dfrac{\pi^{-\frac{1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$, $(x_n, y_n)_\theta = \sqrt{x_n^2 - 2x_n y_n \cos\theta + y_n^2}$, $d\nu(\theta) = \sin^{\nu-1}\theta \;\, d\theta$ [9, 10, 12, 13]. Note that the generalized translate operator is closely connected with Δ_ν-Laplace-Bessel differential operator denoted by

$$\Delta_\nu = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + B_{x_n}, \qquad B_{x_n} = \frac{\partial^2}{\partial x_n^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}, \quad \nu > 0.$$

The B_{x_n}-convolution operator related to T^y is defined by

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y) T^y g(x) y_n^\gamma dy.$$

Let $L^p_\nu = L^p_\nu(\mathbb{R}^n_+)$ be the space of measurable functions with a finite norm

$$\|f\|_{L^p_\nu} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\nu dx\right)^{1/p}$$

is denoted by $L^p_\nu \equiv L^p_\nu(\mathbb{R}^n_+)$, $1 \leq p < \infty$. We denote by $\mathscr{S}'_+ = \mathscr{S}'_+\left(\mathbb{R}^n_+\right)$ the topological dual of $\mathscr{S}_+$ is the collection of all tempered distributions on $\mathbb{R}^n_+$.
First, let's start by giving the definition of Morrey space [14, 15].

Definition 2.1. *For p and q satisfying $0 < q \leq p < \infty$, the homogeneous Morrey spaces M^p_q are defined as*

$$M^p_q = \left\{f \in L^q_{loc} : \|f\|_{M^p_q} = \sup_{x \in \mathbb{R}^n, R > 0} |B(x,R)|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(B(x,R))} < \infty\right\},$$

where $B(x,R)$ is the closed ball of $\mathbb{R}^n$ with center x and radius R.

Let $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. The set

$$Q_{jk} = \left\{ x \in \mathbb{R}^n : 2^{-j}k_i \leq x_i \leq 2^{-j}k_{i+1},\ i = 1,2,\dots n \right\},$$

is called a dyadic cube. We remark that

$$||f||_{M_q^p} \approx \sup_{J:dyadic} |J|^{\frac{1}{p}-\frac{1}{q}} ||f||_{L^q(J)}.$$

We now introduce the Hardy-Morrey spaces that we work mainly on and give their decomposition results.
The HM^p_{q,Δ_ν} Hardy-Morrey spaces is given by the set of all distributions $f \in \mathscr{S}_+ \setminus \mathscr{P}$ with the quasi-norm

$$||f||_{HM^p_{q,\Delta_\nu}} = \left|\left| \sup_{t>0} |\phi_t \otimes f| \right|\right|_{M^p_{q,\nu}}$$

is finite. Here $\phi \in \mathscr{S}_+(\mathbb{R}^n_+)$ satisfies $\int \phi(x)x_n^\nu dx = 1$. Also, $\mathscr{P}$ indicates the set of polynomials.
For the Hardy-Morrey space, if $1 < p \leq q \leq \infty$, it is obvious that $HM^p_{q,\Delta_\nu} = M^p_{q,\Delta_\nu}$ since the Hardy-Littlewood maximal operator associated with the Laplace-Bessel differential operator Δ_ν is bounded on $M^p_{q,\nu}$. Moreover, the HM^p_{q,Δ_ν} Hardy-Morrey spaces cover Hardy spaces for $0 < p \leq 1$. In general, $H^p_{\Delta_\nu} = HM^p_{p,\Delta_\nu} \subset HM^p_{q,\Delta_\nu}$ for $p \leq q \leq \infty$ and $HM^p_{1,\Delta_\nu} \neq M^p_{1,\Delta_\nu}$. Here , the Hardy spaces $H^p_{\Delta_\nu}$ are defined by

$$H^p_{\Delta_\nu} = \left\{ ||f||_{H^p_{\Delta_\nu}} = \left\| \sup_{t>0} \phi_t \otimes f \right\|_{L_p} < \infty \right\}$$

[2].
Now, let us start with to give the definition of (p,q,s)-atoms.

Definition 2.2. *Let $0 < p \leq 1 < q \leq \infty$ with $p \neq q$ and $s \in \mathbb{N} \cup \{0\}$. For a dyadic cube Q, a function a_Q is called a (p,q,s)-atom of HM^p_{q,Δ_ν} if the following properties are satisfied:*

- *(i) a_Q be supported on a cube Q, namely, supp $a_Q \subset Q$,*
- *(ii) $\|a_Q\|_{L_{q,\nu}} \leq |Q|_\nu^{\frac{1}{q}-\frac{1}{p}}$,*
- *(iii) $\int_{\mathbb{R}^n_+} a_Q(x)x^\alpha x_n^\nu dx = 0$ for all $s \geq [(n+k+\nu)\left(\frac{1}{p}-1\right)], 1 \leq k \leq n$, with $|\alpha| \leq s$.*

Also, we introduce atomic decomposition theorem in HM^p_{q,Δ_ν} space is as follows:

Theorem 2.3. *Let $0 < p \leq 1 < q \leq \infty$ with $p \neq q$, $\{a_Q : Q \text{ dyadic}\}$ be a collection of (p,q,s)-atoms and $\{\lambda_Q : Q \text{ dyadic}\}$ be a sequence of scalars with*

$$||\lambda||_{p,q} = \left\{ \sup_J \left(\frac{1}{|J|_\nu} \right)^{1-p/q} \sum_{Q \subset J} |Q|_\nu^{1-p/q} |\lambda_Q|^p \right\}^{1/p} < \infty.$$

Then the sum

$$f = \sum_Q \lambda_Q a_Q \tag{2.2}$$

converges in $\mathscr{S}'_+ \setminus \mathscr{P}$ and $f \in HM^p_{q,\Delta_\nu}$ with $||f||_{HM^p_{q,\Delta_\nu}} \leq C||\lambda||_{p,q}$, where $C = C(n,p,q,\nu)$. Conversely, $\forall f \in HM^p_{q,\Delta_\nu}$ has atomic decomposition (2.2) in $\mathscr{S}'_+ \setminus \mathscr{P}$. Here a_Q are (p,q,s)-atoms and $\lambda = \{\lambda_Q\}$ satisfies $||\lambda||_{p,q} \leq C||f||_{HM^p_{q,\Delta_\nu}}$, where $C > 0$ independent of f.

Proof. The proof of Theorem 2.3 can be found in [1, 16], so we omit it here. □

3. Molecular characterizations of HM^p_{q,Δ_ν}

Next, we continue to give the notion of molecule related to HM^p_{q,Δ_ν}. The following definition for molecule is modified from the corresponding definition of molecule from [2].

Definition 3.1. *Let $0 < p \leq 1 < q \leq \infty$ with $p \neq q$, $s = [(n+k+\nu)\left(\frac{1}{p}-1\right)]$ and $\varepsilon > (n+k+\nu)\left(\frac{1}{p}-\frac{1}{2}\right), 1 \leq k \leq n$. A measurable function $m_Q(x)$ is called a (p,q,s,ε)-molecule for a dyadic cube Q if and only if*

- *(i) $\left(\int_{\mathbb{R}^n_+} |m_Q(x)|^2(1+|x-x_Q|_\nu/\ell_Q)^{2s}x_n^\nu dx\right)^{1/2} \leq |Q|_\nu^{1/2-1/p}$, (this means that ℓ_Q is large)*
- *(ii) $\int_{\mathbb{R}^n_+} m_Q(x)x^\alpha x_n^\nu dx = 0, |\alpha| \leq s$.*

Similar to the atomic decomposition of HM^p_{q,Δ_ν} Hardy-Morrey space, the decomposition in terms of molecule is given as follows:

Theorem 3.2. *Let $0 < p \leq 1 < q \leq \infty$ with $p \neq q$ and $\varepsilon > (n+k+\nu)\left(\frac{1}{p} - \frac{1}{2}\right)$. There is exists a sequence of scalars $\{\lambda_Q : Q \text{ dyadic}\}$, a collection of (p,q,s,ε)-molecules $\{m_Q : Q \text{ dyadic}\}$ for HM^p_{q,Δ_ν}, the series*

$$f = \sum_Q \lambda_Q m_Q \tag{3.1}$$

converges in $\mathscr{S}'_+ \setminus \mathscr{P}$ and $f \in HM^p_{q,\Delta_\nu}$ with

$$||f||_{HM^p_{q,\Delta_\nu}} \leq C||\lambda||_{p,q},$$

where $C > 0$ independent of f.

Proof. The proof of this theorem has a similar technique to those of [2, 17, 18]. Let us start with consider the sets

$$E_0 = \{x \in \mathbb{R}^n_+ : |x| \leq \sigma\}$$

$$E_j = \{x \in \mathbb{R}^n_+ : 2^{j-1}\sigma \leq |x| < 2^j\sigma\}, \quad j = 1,2,\ldots,$$

where $\sigma^{(n+k+\nu)\left(\frac{1}{p}-\frac{1}{2}\right)} = ||\lambda||^{-1}_{p,2}$. Set $m_j = m\chi_{E_j}$, where χ_{E_j} is the characteristic function of E_j. For all $j = 1,2,\ldots$, α a multi-index such that $|\alpha| \leq s$, let φ^α_j be the function on E_j (the restriction to E_j of a polynomial of degree at most s). If $P_j = \varphi_j\chi_j$ then

$$\int_{\mathbb{R}^n_+} (m_j - P_j)x^\alpha x_n^\nu dx = 0, |\alpha| \leq s.$$

Since $m = \sum_{j=0}^\infty m_j = \sum(m_j - P_j) + \sum P_j$, to show both $\sum(m_j - P_j)$ and $\sum P_j$ in HM^p_{q,Δ_ν}, it suffices to verify that

(i) each $(m_j - P_j)$ is a multiple of a (p,q,s)-atom with coefficients sum appropriately,
(ii) the sum $\sum P_j$ can be written as an infinite liner combination of (p,∞,s)-atom with coefficients sum appropriately.

For a dyadic cube Q, we define $E_0 = Q$ and for all $j \geq 1$, $Q_j = 2^jQ$ and $E_j = Q_j - Q_{j-1}$. For $j \geq 0$, let $\{\varphi^\alpha_{E_j} : |\alpha| \leq s\}$ (or $\{\Phi^\alpha_{E_j} : |\alpha| \leq s\}$, respectively) be the Gram-Schmidt orthonormalization of monomials $\{x^\alpha : |\alpha| \leq s\}$(or the dual basis of monomials $\{\Phi^\alpha_{E_j} : |\alpha| \leq s\}$, respectively) on E_j according to the weight $1/|E_j|_\nu$. We consider the function $\varphi^\alpha_{E_j}$ to be defined on $\mathbb{R}^n$, having the value zero outside E_j. (namely, if $x \notin E_j$, then we set $\Phi^\alpha_{E_j}(x) = 0$.) By homogeneity and the uniqueness of Gram-Schmidt orthogonalization process (see [18]), we obtain the following estimate

$$|\varphi^\alpha_{E_j}(x)| \leq C, \text{ for } x \in E_j, \tag{3.2}$$

and

$$|\Phi^\alpha_{E_j}(x)| \leq C(2^j\sigma)^{-|\alpha|}, \tag{3.3}$$

where C depends on s. Let m_Q be a molecule function. We set $m_{E_j}(x) = m_Q(x)\chi_{E_j}(x)$ and

$$P_{E_j}(x) = P_{E_j}(m_Q)(x) = \sum_{|\alpha|\leq s} a^\alpha_{E_j}\varphi^\alpha_{E_j}(x) = \sum_{|\alpha|\leq s} m^\alpha_{E_j}\Phi^\alpha_{E_j}(x), \tag{3.4}$$

where

$$a^\alpha_{E_j} = \int m_{E_j}(x)\varphi^\alpha_{E_j}(x)x_n^\nu \frac{dx}{|E_j|_\nu}, m^\alpha_{E_j} = \int m_{E_j}(x)x^\alpha x_n^\nu \frac{dx}{|E_j|_\nu}.$$

From [19], we obtain

$$\begin{aligned} \int (m_{E_j} - P_{E_j})x^\alpha x_n^\nu dx &= 0, \text{for all } |\alpha| \leq s, \\ ||m_{E_j} - P_{E_j}||_{L^2_\nu(E_j)} &\leq C||m_{E_j}||_{L^2_\nu(E_j)} \end{aligned} \tag{3.5}$$

We may write a decomposition of the molecule $m_Q(x)$ as folllows

$$m_Q(x) = \sum_{j=0}^\infty (m_{E_j} - P_{E_j})(x) + \sum_{j=0}^\infty P_{E_j}(x). \tag{3.6}$$

By the equality (3.4), and the cancellation properties of the molecule, we get

$$\sum_{j=0}^{\infty} P_{E_j}(x)=\sum_{j=0}^{\infty}\sum_{|\alpha|\leq s}\left(\frac{\Phi^{\alpha}_{E_{j+1}}}{|E_{j+1}|_{\nu}}-\frac{\Phi^{\alpha}_{E_j}}{|E_j|_{\nu}}\right)E_{Q_{j,\alpha}}, \tag{3.7}$$

here

$$E_{Q_{0,\alpha}}=\sum_{j\geq 0}\int m_{E_j}(x)x^{\alpha}x_n^{\nu}dx=\int m(x)x^{\alpha}x_n^{\nu}dx=0,$$

$$E_{Q_{j,\alpha}}=\sum_{i=j}^{\infty}\int m_{E_j}(x)x^{\alpha}x_n^{\nu}dx=\int_{|x|\geq 2^j\sigma} m_Q(x)x^{\alpha}x_n^{\nu}dx,\ \text{ for all }\ j\geq 1.$$

By using (3.6) and (3.7), we may write

$$m_Q(x)=\sum_{j=0}^{\infty} t_{Q_j}a_{Q_j}(x)+\sum_{j\geq 0}\sum_{|\alpha|\leq s}\delta_{Q_{j,\alpha}}b_{Q_{j,\alpha}}(x), \tag{3.8}$$

where for each $j\geq 0$

$$t_{Q_j}=||m_{E_j}-P_{E_j}||_{L^2_{\nu}(E_j)}|Q_j|_{\nu}^{\frac{1}{p}-\frac{1}{2}},\ \ a_{Q_j}(x)=\frac{(m_{E_j}-P_{E_j})(x)}{||m_{E_j}-P_{E_j}||_{L^2_{\nu}(E_j)}}|Q_j|_{\nu}^{\frac{1}{p}-\frac{1}{2}},$$

and

$$\lambda_{Q_{j,\alpha}}=E_{Q_{j,\alpha}}|Q_j|_{\nu}^{\frac{1}{p}-1}(2^j\sigma)^{-|\alpha|},\ b_{Q_{j,\alpha}}(x)=\left(\frac{\Phi^{\alpha}_{E_{j+1}}}{|E_{j+1}|_{\nu}}-\frac{\Phi^{\alpha}_{E_j}}{|E_j|_{\nu}}\right)|Q_j|_{\nu}^{1-\frac{1}{p}}(2^j\sigma)^{|\alpha|}.$$

From the inequalities (3.2), (3.3) and (3.5), it can be easily seen that a_{Q_j} and $b_{Q_{j,\alpha}}$ are supported in a cube Q_j and they are $(p,q,2)$-atoms and (p,q,∞)-atoms respectively. For simplicity, we now just consider the sum (3.1) is finite. Then by (3.8), we obtain

$$f=\sum_{Q,j}\lambda_Q t_{Q_j}a_{Q_j}(x)+\sum_{Q,j}\lambda_Q\sum_{|\alpha|\leq s}\delta_{Q_{j,\alpha}}b_{Q_{j,\alpha}}(x) \tag{3.9}$$

in $\mathscr{S}'(\mathbb{R}^n_+)$. Let J be a fixed dyadic cube. We consider the following equality

$$\sum_{Q_j\subset J}|\lambda_Q t_{Q_j}|^p|Q_j|_{\nu}^{1-p/q}=\sum_{Q\subset J}|\lambda_Q|^p\sum_{j:Q_j\subset J}|t_{Q_j}|^p|Q_j|_{\nu}^{1-p/q}.$$

By the Hölder's inequality, (3.5) and $\varepsilon>(n+k+\nu)\left(\frac{1}{p}-\frac{1}{2}\right)$, we find that

$$\sum_{j:Q_j\subset J}|t_{Q_j}|^p|Q_j|_{\nu}^{1-p/q}\leq C|Q|_{\nu}^{1-p/q}. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\left|\left|\sum_{Q,j}\lambda_Q t_{Q_j}\right|\right|_{HM^p_{q,\Delta_\nu}}\leq C||\lambda||_{p,q}. \tag{3.11}$$

From an argument similar to that used in above (3.9)-(3.11), it also follows that

$$\left|\left|\sum_{Q}\sum_{j\geq 0}\sum_{Q,j}\lambda_Q t_{Q_j}\right|\right|_{HM^p_{q,\Delta_\nu}}\leq C||\lambda||_{p,q}. \tag{3.12}$$

Combining the inequalities (3.11) and (3.12), we end of the proof if the sum (3.1) is finite. Also, this sum converges in the sense of distributions. □

With the above theorem, we are ready to give the following section which offers an important estimates for Hardy-Morrey spaces related to Laplace-Bessel operator used in the proof of our main result.

4. The B-Riesz transform on Hardy-Morrey spaces HM^p_{q,Δ_ν}

In this section, we restrict ourselves to the high order B-Riesz transforms and give its boundedness properties on Hardy-Morrey spaces. We recall the high order B-Riesz transform.

Definition 4.1. ([8, 9]) *Let $1 \le p < \infty$ and $f \in L^p_\nu$. B-Riesz transform of f with high order is defined*

$$\begin{aligned} R^{(k)}_\nu(f)(x) &= C_{k,\nu}\left[\, p.v\left(\frac{P_k(y)}{|y|^{n+k+\nu}} \otimes f\right)\right](x),\ 1 \le k \le n, \\ &= C_{k,\nu}\left[\, p.v\left(K \otimes f\right)\right](x) \\ &= C_{k,\nu} \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|} \frac{P_k(y)}{|y|^{n+k+\nu}} T^y f(x) y_n^\nu dy, \end{aligned} \tag{4.1}$$

where $C_{k,\nu} = 2^{\frac{n+\nu}{2}}\Gamma(\frac{n+k+\nu}{2})\left[\Gamma(\frac{k}{2})\right]^{-1}$ and $P_k(y) = P_k(y_1, y_2, \ldots, y_{n-1}, y_n^2)$ is a homogeneous polynomial of degree k which holds $\triangle_\nu P_k(y) = 0$ on $\mathbb{R}^n_+$. Also, the following two conditions are satisfied for this polynomial:

$$\int_{S_+} P_k(\theta)(\theta')^\nu d\theta = 0 \tag{4.2}$$

and

$$\sup_{\theta\in S_+} |P_k(\theta)| = M < \infty, \tag{4.3}$$

here $S_+ = \{y \in \mathbb{R}^n_+ : |y| = 1\}$ and $\theta = \frac{y}{|y|}$. Also, here T^y denotes the generalized translate operator given in (2.1).

Before establishing the B-Riesz transform characterization of $HM^p_{q,\Delta_\nu}(\mathbb{R}^n_+)$, we first introduce some background on this kernel of this transform.
Let $R^{(k)}_\nu f := K \otimes f$ be defined as in (4.1). There exists a bounded distribution function $K(x)$ with $|F_\nu[K(x)]| \le C$. We give the following equality

$$F_\nu[R^{(k)}_\nu f](x) = i^k P_k(x)|x|^{-k} F_\nu(f)(x)$$

for all $f \in L^2_\nu$. Here, for any $f \in \mathscr{S}(\mathbb{R}^n_+)$, we use $F_\nu f$ to denote its Fourier-Bessel transform, which is defined by setting

$$F_\nu f(x) \ = \int_{\mathbb{R}^n_+} f(y)\, e^{-i(x'y')}\, j_{\frac{\nu-1}{2}}(x_n y_n)\, y_n^\nu dy, \ \text{ for all } x \in \mathbb{R}^n_+,$$

where $(x'y') = x_1y_1 + \ldots + x_{n-1}y_{n-1}$, j_ν, $(\nu > -1/2)$ is Bessel function and $C_{n,\nu} = (2\pi)^{n-1}2^{\nu-1}\Gamma^2((\nu+1)/2) = \frac{2}{\pi}\omega(2,\nu)$.
This transform is also associated with Laplace-Bessel differential operator.
Moreover, $K(x)$ satisfies the following Hörmander's condition,

$$\int\limits_{|x|\ge A_1|y|} |T^y K(x) - K(x)| x_n^\nu dx \le A_2, \tag{4.4}$$

for some $A_1, A_2 < \infty$ (See more detail [20]). So, we conclude that property (4.4) and the L^2_ν-boundedness of $R^{(k)}_\nu f$ maps HM^p_{q,Δ_ν} to itself for $0 < p \le 1 < q \le \infty$ with $p \ne q$.
However, we make stronger assumption on kernel, that is $K \in C^\infty(\mathbb{R}^n_+ \setminus \{0\})$ satisfies for all $|\alpha| \le s$ and $x \ne 0$,

$$|D^\alpha_\nu T^y K(x)| \le AM|x|^{-n-k-\nu-|\alpha|}.$$

We also have the following L^p_ν and $H^p_{\Delta_\nu}$ boundedness of high order B-Riesz transform.

Theorem 4.2. ([8, 21]) *Let P_k be the characteristic of the singular integral (4.1) satisfiying the conditions (4.2) and (4.3). Then there exists a constant $C > 0$ such that for all $1 < p < \infty$ and $\nu > 0$*

$$\|R^{(k)}_\nu(f)\|_{L^p_\nu} \le CM\|f\|_{L^p_\nu},$$

where C is a constant independent of f and P_k is a homogeneous polynomial of degree k.

Theorem 4.3. ([7]) *Let $R^{(k)}_\nu f := K \otimes f$ and $0 < p \le 1$. Then there exists a constant $C^*_{n,p,\nu}$ such that for all $f \in H^p_{\Delta_\nu}$*

$$\|K \otimes f\|_{H^p_{\Delta_\nu,at}} \le C^*_{n,p,\nu}\|f\|_{H^p_{\Delta_\nu,at}} \qquad \nu > 0.$$

The following main theorem demonstrate B-Riesz characterization of HM^p_{q,Δ_ν} Hardy-Morrey spaces.

Theorem 4.4. *Let $0 < p \le 1 < q \le \infty$ with $p \ne q$. Then B-Riesz transform can be extended to the bounded transform on Hardy-Morrey spaces HM^p_{q,Δ_ν}.*

Proof. In order to prove this theorem, it is sufficient to show $R^{(k)}_\nu(f)$ is a (p,q,s,ε)-molecule whenever f is a (p,q,s)-atom. We prove this theorem by following the similar strategy used in [7]. Let us take the function supported in the upper half ball $B(0,1)$ with $\int \varphi(x) x^n_\nu dx$ on $\mathbb{R}^n_+$. We define $K^{(t)} = \varphi_t \otimes K$. Then the function $K^{(t)}$ satisfies the following inequalities

$$\sup_{t>0} F_\nu(K^{(t)})(x) \le C||F_\nu \varphi_t||_{L^\infty_\nu}$$

and

$$\sup_{t>0}(K^{(t)})(x) \le C_\varphi M|x|^{-n-k-\nu-|\alpha|}, \quad |\alpha| \le s\ .$$

For a dyadic cube Q, $m_Q(x)$ be a (p,q,s)-molecule and a_Q be a (p,q,s)-atom of HM^p_{q,Δ_ν}. Finally, the proof rests on the checking that $m_Q(x) = R^{(k)}_\nu(a_Q)(x)$ satisfies the moment and size condition. Namely,

(i) $\left(\int_{\mathbb{R}^n_+} |R^{(k)}_\nu a_Q(x)|^2 (1+|x-x_Q|_\nu/\sigma)^{2s} x^\nu_n dx\right)^{1/2} \le |Q|^{1/2-1/p}_\nu$,

(ii) $\int_{\mathbb{R}^n_+} R^{(k)}_\nu a_Q(x) x^\alpha x^\nu_n dx = 0, \ \ |\alpha| \le s.$

So, we omit the details and leave it to the reader. □

5. Conclusion

In this study, the decomposition of Hardy-Morrey spaces related to the Laplace-Bessel differential operator are introduced in terms of atoms and molecules. Also, we give the HM^p_{q,Δ_ν} boundedness of higher order B-Riesz transforms for $0 < q \le p < \infty$ by using this atomic decomposition and molecular characterization. We follow the similar approach for developing the atomic decomposition and molecular characterization as classical Hardy-Morrey spaces. The interesting of our result depends on the existence of the different differential operator.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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