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FACTOR RELATIONS BETWEEN SOME SUMMABILITY METHODS

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Abstract. In the present paper, using the result of Bennett [1] on characterization of factorable matrices, we give necessary and sufficient conditions in order that $\Sigma\lambda_n x_n$ is summable $|R,p_n|_s$ whenever $\Sigma\mu_n x_n$ is summable $|C,0|_k$, and $\Sigma\lambda_n x_n$ is summable $|C,0|_s$ whenever $\Sigma\mu_n x_n$ is summable $|R,p_n|_r$, . where $1<k\leq s<\infty$. Therefore we also extend some known results.

1. Introduction

Consider an infinite series Σx_n with partial sum s_n, and by (σ_n^α), we denote the n-th Cesàro means of order α with $\alpha>-1$ of the sequence (s_n). The series Σx_n is said to be summable $|C,\alpha|_k$, $k\geq 1$, if $\left(n^{1-1/k}\left(\sigma_n^\alpha-\sigma_{n-1}^\alpha\right)\right)\in \ell_k$ (see [7]), where ℓ_k is the set of all sequences consisting k- absolutely convergent series. Note that the summability $|C,0|_k$ reduces to $\left(n^{1-1/k}x_n\right)\in\ell_k$. Let (p_n) be a sequence of positive real numbers with $P_n=p_0+p_1+\cdots+p_n\to\infty$ as $n\to n$. The sequence-to-sequence transformation

$$u_n=\frac{1}{P_n}\sum_{n=0}^{n}p_n s_n. \tag{1.1}$$

defines the sequence (u_n) of the (R,p_n) Riesz means of the sequence (s_n), generated by the sequence of numbers (p_n). The series Σx_n is said to be summable $|R,p_n|_k$, $k\geq 1$, if $\left(n^{1-1/k}\left(u_n-u_{n-1}\right)\right)\in\ell_k$ (see [19]).

A summability method Y is said to be include another summability method X, if every series summable by X is also summable by Y. If the methods include each other, then, these methods are called equivalent. Hereof, the inclusion relations of the absolute summability methods of single series were studied by various authors (see, for example, [2-24]).

The following result was established by Bor [2].

Theorem 1.1. Let $1<k<\infty$ and

$$\sum_{n=v}^{\infty}\frac{n^{k-1}p_n^k}{P_n^kP_{n-1}}=O\left(\frac{v^{k-1}p_{v-1}^k}{P_{v-1}^k}\right). \tag{1.2}$$

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If exists $d > 1$ such that

$$\frac{P_{n+1}}{P_n} \geq d \text{ for all } n \geq 1, \tag{1.3}$$

then, the summability methods $|R, p_n|_k$ and $|C, 0|_k$ are equivalent.

Also, in [16], this result was extended as follows.

Theorem 1.2. Let $1 < k \leq s < \infty$. Then, the necessary and sufficient condition in order that the summability method $|R, p_n|_s$ includes the summability method $|C, 0|_k$ is

$$\left\{ \sum_{v=1}^{m} \frac{P_{v-1}^{k^*}}{v} \right\}^{1/k^*} \left\{ \sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} p_n}{P_n P_{n-1}} \right)^s \right\}^{1/s} = O(1),$$

where k^* denotes the conjugate of the index $k > 1$, i.e., $1/k + 1/k^* = 1$.

Theorem 1.3. Let $1 < k \leq s < \infty$. Then, the necessary and sufficient condition in order that the summability method $|C, 0|_s$ includes the summability method $|R, p_n|_k$ is

$$\left\{ \sum_{v=m-1}^{m} \frac{1}{v} \left| \frac{P_{v-1} P_v}{p_v} \right|^{k^*} \right\}^{1/k^*} \left\{ \sum_{n=m}^{m+1} \frac{n^{s-1}}{P_n^s} \right\}^{1/s} = O(1).$$

2. The main Result

This paper gives necessary and sufficient conditions in order that $\Sigma \lambda_n x_n$ is summable $|C, 0|_s$ whenever $\Sigma \mu_n x_n$, is summable $|R, p_n|_k$, and also $\Sigma \lambda_n x_n$ is summable $|R, p_n|_s$ whenever $\Sigma \mu_n x_n$, is summable $|C, 0|_k$, where$1 < r \leq s < \infty$, which generalizes the above results.

A factorable matrix T is defined by

$$t_{nv} = \begin{cases} b_n a_v, 0 \leq v \leq n, \\ 0, \qquad v > n. \end{cases}$$

where (b_n) and (a_n) are sequences of real or complex numbers.

Now we prove the following theorems.

Theorem 2.1. Let $1 < k \leq s < \infty$ and $\lambda = (\lambda_n)$ be a sequence of numbers. Further, let $\mu = (\mu_n)$ be a sequence of non-zero numbers. Then, necessary and sufficient condition in order that $\Sigma \lambda_n x_n$ is summable $|R, p_n|_s$ whenever $\Sigma \mu_n x_n$ is summable $|C, 0|_k$ is

$$\left\{ \sum_{v=1}^{m} \frac{P_{v-1}^{k^*}}{v} \left| \frac{\lambda_v}{\mu_v} \right|^{k^*} \right\}^{1/k^*} \left\{ \sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} p_n}{P_n P_{n-1}} \right)^s \right\}^{1/s} = O(1). \tag{2.1}$$

Theorem 2.2. Let $1 < k \leq s < \infty$, λ and μ be as in Theorem 2.1. Then, necessary and sufficient condition in order that $\Sigma \lambda_n x_n$ is summable $|C, 0|_s$ whenever $\Sigma \mu_n x_n$ is summable $|R, p_n|_k$ is

$$\left\{ \sum_{v=m-1}^{m} \frac{1}{v} \left(\frac{P_{v-1} P_v}{p_v} \right)^{k^*} \right\}^{1/k^*} \left\{ \sum_{n=m}^{m+1} \left| \frac{n^{1/s^*} \lambda_n}{P_n \mu_n} \right|^s \right\}^{1/s} = O(1). \tag{2.2}$$

It may be noticed that Theorem 2.1 and Theorem 2.2. are, in the special case $\mu_n = \lambda_n = 1$ for all $n \geq 0$, reduced to Theorem 1.2. and Theorem 1.3, respectively.

Also, if $p_n = 1$ for all $n \geq 0$, then the summability$|R, p_n|_k$ coincides with the summability $|C, 1|_k$. Further, $P_n = n + 1$ and

$$\sum_{n=m}^{\infty} \frac{p_n}{P_{n-1}P_n^s} = \sum_{n=m}^{\infty} \frac{1}{n(n+1)^s} = O\left(\frac{1}{m^s}\right).$$

Hence, the following results is immediately obtained.

Corollary 2.3. Let $1 < k \leq s < \infty$, λ and μ be as in Theorem 2.1. Then, necessary and sufficient condition in order that $\Sigma x_n \lambda_n$ is summable $|C, 1|_s$ whenever $\Sigma x_n \mu_n$ is summable $|C, 0|_k$ is

$$\sum_{v=1}^{m} v^{k^*-1} \left|\frac{\lambda_v}{\mu_v}\right|^{k^*} = O(m^{k^*}).$$

Corollary 2.4. Let $1 < k \leq s < \infty$, λ and μ be as in Theorem 2.1. Then, necessary and sufficient condition in order that $\Sigma x_n \lambda_n$ is summable $|C, 0|_s$ whenever $\Sigma x_n \mu_n$ is summable $|C, 1|_k$ is

$$\sum_{n=m}^{m+1} \frac{1}{P_n^s} \left|\frac{\lambda_n}{\mu_n}\right|^{s} = O(m^{1-2s-s/k}).$$

Proof of Theorem 2.1. We first note a result of Bennett [1] that a factorable matrix T defines a bounded linear operator $L_T : \ell_k \to \ell_s$ such that $L_T(x) = T(x)$ for all $x \in \ell_k$ if and only if

$$\left(\sum_{v=0}^{m} |a_v|^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} |b_n|^{s}\right)^{1/s} = O(1), \tag{2.3}$$

where $k^*$is the conjugate of indices k. Let σ_n^0 and u_n be Cesàro $(C, 0)$ and Riesz means (R, p_n) of the series $\Sigma \mu_n x_n$ and $\Sigma \lambda_n x_n$, respectively. Then, by (1.1),

$$\sigma_n^0 = \sum_{v=0}^{n} \mu_v x_v$$

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} \lambda_r x_r$$

and so $\Delta u_0 = \lambda_0 x_0$,

$$\Delta u_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v x_v, \text{ for } n \geq 1.$$

Now, say $t_n' = n^{1/s^*} \Delta u_n$ and $\sigma_n^{0\prime} = n^{1/k^*} \mu_n x_n$ for $n \geq 1$. Then, it easily seen that

$$\begin{aligned} t_n' &= \frac{n^{1/s^*} p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_v}{v^{1/k^*} \mu_v} \sigma_v^{0\prime} \\ &= \sum_{v=1}^{\infty} t_{nv} \sigma_v^{0\prime} \end{aligned}$$

where the matrix $T=(t_{nv})$ is given by

$$t_{nv}=\begin{cases}\frac{n^{1/s^*}p_nP_{v-1}\lambda_v}{P_nP_{n-1}v^{1/k^*}\mu_v}, & 1\leq v\leq n,\\ 0, & v>n.\end{cases}$$

This means that $\Sigma x_n\lambda_n$ is summable $|R,p_n|_s$ whenever $\Sigma x_n\mu_n$ is summable $|C,0|_k$ if and only if $(t'_n)\in\ell_s$ for all $\left(\sigma_n^{0\prime}\right)\in\ell_k$, or, $T:\ell_k\to\ell_s$ is a bounded linear operator. Thus, by applying (2.3) to the matrix T, we have (2.1).

Proof of Theorem 2.2. Let u_n and σ_n^0 be means of Riesz (R,p_n) and Cesàro $(C,0)$ of the series $\Sigma\mu_nx_n$ and $\Sigma\lambda_nx_n$, respectively. Then, as above, $\Delta\sigma_n^0=\lambda_nx_n$, and also $\Delta u_0=\mu_0x_0$,

$$\Delta u_n=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\mu_vx_v,\text{for } n\geq 1 \tag{2.4}$$

By inversion of (2.4), it can be stated that, for $n\geq 1$,

$$x_n=\frac{1}{\mu_nP_{n-1}}\left(\frac{P_{n-1}P_n}{p_n}\Delta u_n-\frac{P_{n-1}P_{n-2}}{p_{n-1}}\Delta u_{n-1}\right)$$

Say $t'_n=n^{1/k^*}\Delta u_n$ and $\sigma_n^{0\prime}=n^{1/s^*}\lambda_nx_n$ for $n\geq 1$. Then, it can be written that

$$\begin{aligned}\sigma_n^{0\prime} &= \frac{n^{1/s^*}\lambda_n}{\mu_nP_{n-1}}\left(\frac{P_{n-1}P_nt'_n}{n^{1/k^*}p_n}-\frac{P_{n-1}P_{n-2}t'_{n-1}}{(n-1)^{1/k^*}p_{n-1}}\right)\\ &= \sum_{v=1}^{\infty}d_{nv}t'_v\end{aligned}$$

where the matrix $D=(d_{nv})$ is defined by

$$d_{nv}=\begin{cases}\frac{n^{1/s^*}\lambda_n}{\mu_nP_{n-1}}\left(-\frac{P_{n-1}P_{n-2}}{(n-1)^{1/k^*}p_{n-1}}\right), & v=n-1\\ \frac{n^{1/s^*}\lambda_n}{\mu_nP_{n-1}}\left(\frac{P_{n-1}P_n}{n^{1/k^*}p_n}\right), & v=n\\ 0, & v>n.\end{cases}$$

This gives that $\Sigma x_n\lambda_n$ is summable $|C,0|_s$ whenever $\Sigma x_n\mu_n$ is summable $|R,p_n|_k$ if and only if $\left(\sigma_n^{0\prime}\right)\in\ell_s$ for all $(t'_n)\in\ell_k$, or, $D:\ell_k\to\ell_s$ is a bounded linear operator. Thus, by applying (2.3) to the matrix D, we get (2.2).

This completes the proof.

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